Available online at www.sciencedirect.com
SCIENCE DIRECT®
J. Differential Equations 220 (2006) 46–67
Journal of Differential Equations
www.elsevier.com/locate/jde

On non-autonomous H^∞ control with infinite horizon

F. Colonius[a,1], R. Fabbri[a,2], R. Johnson[b,*,2]

[a] *Universität Augsburg, Germany*
[b] *Dipartimento di Sistemi e Informatica, Università di Firenze, Via di Santa Marta, 3 50139, Firenze, Italy*

Received 9 September 2004
Available online 10 February 2005

Abstract

We study the linear H^∞ control problem in the infinite-horizon case when the coefficients are time varying and bounded. We pass in a standard way from a Riccati equation to a linear Hamiltonian system of ordinary differential equations, which we study using exponential dichotomies and rotation numbers. In particular, we use the dichotomy concept to define the critical attenuation value.

MSC: 93B36; 37B55

Keywords: H^∞-control; Non-autonomous differential equations

1. Introduction

Speaking in general terms, the main problem of H^∞-control theory might be formulated as follows. Suppose a plant is subjected to a disturbance $w = w(t)$ whose detailed behavior is not known and which is only restricted by, say, L^2-boundedness.

* Corresponding author. Fax: +39 055 4796363.
E-mail address: johnson@dsi.unifi.it (R. Johnson).
[1] Partially supported by a VIGONI project.
[2] Supported by the G.N.A.M.P.A. and M.I.U.R. (Italy) and a VIGONI project.

0022-0396/$ - see front matter © 2005 Elsevier Inc. All rights reserved.
doi:10.1016/j.jde.2004.12.009

One wants to determine a control u of feedback type which stabilizes the plant in such a way as to minimize a performance index when the disturbance is "worst possible".

A substantial theory of H^∞ control for linear, time-invariant systems was developed in the 1980s; see, e.g., [30,9] for information about this vast subject. As is well-known, this theory was for the most part formulated in the frequency domain, where the main problem is translated into that of minimizing the operator norm of a certain transfer function acting on Hardy-type H^∞ spaces. More recently, attention has been given to the worst-case control of non-linear plants [15]. In this situation, it is no longer natural to work in the frequency domain, and it has been found convenient to develop the theory in the time domain. In spite of this fact, one still speaks of "H^∞-control" because of the success of the theory worked out for linear autonomous systems.

Our goal in this paper is to study a *linear* control problem of H^∞ type, but in the case of time-varying coefficients and infinite horizon. We are motivated by the fact that, though H^∞-control problems have been posed and studied for linear systems when the coefficients vary with time, most known (to us) results have been proved in the finite-horizon case [5]. We will see that, in the infinite-horizon case, it is convenient to introduce a certain linear Hamiltonian system of ordinary differential equations; these are called the Caratheodory equations in [5]. This system can be studied to good effect by making use of the concepts of *exponential dichotomy* and *rotation number*.

Let us be a bit more specific about the problem we will study. Consider the differential system

$$\begin{aligned} x' &= A(t)x + B(t)u + D(t)w, \\ x(0) &= x_0, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is a state vector, $u \in \mathbb{R}^m$ is a control vector, and $w \in \mathbb{R}^l$ represents a general disturbance. The functions A, B, and D take values in the sets of matrices of dimensions $n \times n$, $n \times m$, and $n \times l$, respectively. They will be assumed to be uniformly bounded and uniformly continuous. We make the usual mental switch from the concept of "worst case" control to that of "minimal attenuation" control. Namely, for each $\gamma > 0$, introduce the functional

$$L_\gamma(u, w) = \int_0^\infty \{\langle Q(t)x(t), x(t)\rangle + \langle u(t), u(t)\rangle - \gamma^2\langle w(t), w(t)\rangle\}\, dt.$$

The function Q takes values in the set of $n \times n$ real symmetric matrices; it is assumed to be uniformly bounded, uniformly continuous, and positive semi-definite: $Q(t) \geqslant 0$ for all $t \in \mathbb{R}$. The disturbance w lies in $L^2([0,\infty), \mathbb{R}^l)$. For each $\gamma > 0$, one looks for a linear feedback controller $u = -B^t(t)m_\gamma(t)x$, defined by a function $m_\gamma(\cdot)$ with values in the set of symmetric, positive-definite $n \times n$ matrices, such that (i) if $w = 0$, then the feedback system obtained from (1) is stable and (ii) the following dissipation inequality holds:

$$L_\gamma(u, w) \leqslant \langle m_\gamma(0)x_0, x_0\rangle. \tag{2}$$

One wishes to determine the minimal attenuation value $\gamma^* = \inf\{\gamma > 0 \mid$ there is a linear feedback controller as above for which (i) and (ii) are satisfied$\}$.

It is well-known that the problem of minimal attenuation control is related to a certain differential game, at least if A is a Hurwitz matrix function. Namely, for each $\gamma > 0$, set $v_1 = \min_u \max_w L_\gamma(u, w)$ and $v_2 = \max_w \min_u L_\gamma(u, w)$. If the upper value v_1 exists and equals the lower value v_2, then one says that the game determined by (1) and L_γ has value $v_1 = v_2$. See [5] for an excellent analysis of the relation between differential games and H^∞-control theory.

Motivated by the game-theoretic interpretation of our H^∞-control problem, we will study the matrix Riccati equation

$$m' + A^t m + mA - m[BB^t - \gamma^{-2}DD^t]m + Q = 0, \tag{3}$$

where the superscript "t" indicates the matrix transpose. If for some $\gamma > 0$ the Riccati equation (3) admits a solution $m(t)$ which is bounded on all of $\mathbb{R}$, then (modulo certain details) the H^∞-control problem admits a solution, namely, $u = -B^t(t)m(t)x$. Thus, the study of the bounded (i.e., *non-conjugate*) solutions of (3) is the key to understanding which values of γ give rise to a stabilizing control for (1), for which (2) holds.

The non-conjugate solutions of (3) are best studied by introducing the corresponding system of linear, non-autonomous Hamiltonian differential equations

$$z' = \begin{pmatrix} A(t) & -[B(t)B^t(t) - \gamma^{-2}D(t)D^t(t)] \\ -Q(t) & -A^t(t) \end{pmatrix} z, \tag{4}$$

where $z = \binom{x}{y} \in \mathbb{R}^{2n}$. We will discuss the basic facts concerning *exponential dichotomies* and *rotation numbers* for these equations. These facts will guide us in giving a precise definition of the minimal attenuation value γ^* which is appropriate in the case of time-varying coefficients and infinite horizon.

We will then impose certain controllability conditions together with a mild recurrence condition on the coefficients A, B, D, Q, and introduce the number $\gamma_l = \inf\{\gamma > 0 \mid$ Eq. (4) has zero rotation number$\}$. This number is interesting for two reasons: first, Eq. (4) admits an exponential dichotomy when $\gamma > \gamma_l$; second, $\gamma^* \geqslant \gamma_l$. It turns out that the two possibilities $\gamma^* > \gamma_l$ and $\gamma^* = \gamma_l$ are of a qualitatively different nature. We will see that, if $\gamma^* = \gamma_l$, then the notion of *weak disconjugacy* is of help in understanding whether or not there exists a stabilizing feedback control for which (2) is true.

Some of our discussion of Eq. (4) uses facts drawn from paper [13], where a non-autonomous version of the Yakubovich Frequency Theorem [28,29] was worked out. There is however an important technical difference between the structure of Eq. (4) and that of the Hamiltonian system studied in [13] namely, that the lower-left hand corner $-Q(t)$ of the matrix function in (4) is semidefinite, and not the upper right-hand corner as is the case for the system considered in [13].

The paper is organized as follows. In Section 2 we formulate our H^∞-control problem using the language of non-autonomous differential systems. In Section 3 we state and prove our results concerning the critical attenuation value γ^* and its relation to γ_l. In particular, we prove the existence of a stabilizing feedback control for which (2) holds when $\gamma > \gamma^*$.

We finish this introduction by fixing some notation and discussing some basic concepts.

First, the symbol $\langle , \rangle$ denotes the Euclidean inner product on a given Euclidean space $\mathbb{R}^d$, and $|\cdot|$ denotes the corresponding norm on $\mathbb{R}^d$.

Second, consider a control system

$$x' = A(t)x + B(t)u \qquad (x \in \mathbb{R}^n, u \in \mathbb{R}^m), \tag{$*$}$$

where A and B are continuous matrix functions of the appropriate dimension. The systems $(*)$ are said to be *null controllable* if to each $x_0 \in \mathbb{R}^n$ there correspond a number $T > 0$ and an integrable control $u : [0, T] \to \mathbb{R}^m$ such that, if $x(t)$ is the solution of $(*)$ with $x(0) = x_0$, then $x(T) = 0$.

Third, let Ξ be a metric space. A real *flow* on Ξ is defined by a 1-parameter group $\{\tau_t \mid t \in \mathbb{R}\}$ of homeomorphisms of Ξ: that is (i) $\tau_0(\xi) = \xi$ for all $\xi \in \Xi$, (ii) $\tau_t \circ \tau_s = \tau_{t+s}$ for all $t, s \in \mathbb{R}$, and (iii) the map $\tau : \Xi \times \mathbb{R} \to \Xi : (\xi, t) \to \tau_t(\xi)$ is continuous. A flow $(\Xi, \{\tau_t\})$ is called *minimal* or *Birkhoff recurrent* if Ξ is compact and if each orbit $\{\tau_t(\xi) \mid t \in \mathbb{R}\}$ is dense in Ξ [10].

Fourth, if Ξ is compact and $(\Xi, \{\tau_t\})$ is a flow, then a regular Borel probability measure μ on Ξ is called *invariant* if $\mu(\tau_t(B)) = \mu(B)$ for each Borel set $B \subset \Xi$ and each $t \in \mathbb{R}$. It is called *ergodic* if, in addition to being invariant, it satisfies the following indecomposibility condition. Let Δ denote the symmetric difference of sets: Suppose that $B \subset \Xi$ is a Borel set such that $\mu(\tau_t(B)\Delta B) = 0$ for all $t \in \mathbb{R}$, then either $\mu(B) = 0$ or $\mu(B) = 1$. We will often require that Ξ be the *topological support* Supp μ of a given ergodic measure μ; this means that $\mu(V) > 0$ for each open set $V \subset \Xi$.

Fifth, let $n \geqslant 1$ and let $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ be the standard $2n \times 2n$ antisymmetric matrix; here I_n is the $n \times n$ identity matrix. Recall that an n-dimensional vector subspace $\lambda \subset \mathbb{R}^{2n}$ is called a *Lagrange subspace* if $\langle x, Jy \rangle = 0$ for all $x, y \in \lambda$. Let Λ be the set of all Lagrange subspaces of $\mathbb{R}^{2n}$. Then Λ carries the structure of a $\frac{n(n+1)}{2}$-dimensional real analytic manifold. Let $\{e_1, \dots, e_{2n}\}$ be the canonical basis in $\mathbb{R}^{2n}$. One checks that $\lambda_h = Span\{e_1, \dots, e_n\}$ and $\lambda_v = Span\{e_{2n+1}, \dots, e_{2n}\}$ are elements of Λ, called the *horizontal*, resp. *vertical* Lagrange subspace.

Next, abuse notation and let $\{e_1, \dots, e_n\}$ be the canonical basis in $\mathbb{R}^n$. If $\lambda \in \Lambda$ is a Lagrange subspace of $\mathbb{R}^{2n}$, which is transversal to λ_v (i.e., $\lambda \cap \lambda_v = \{0\}$), then there is an $n \times n$ symmetric real matrix m such that $\lambda = Span\{\binom{e_1}{m(e_1)}, \dots, \binom{e_n}{m(e_n)}\}$. If in addition λ is transverse to λ_h, then $det\, m \neq 0$. One thinks of m as parametrizing λ. The *vertical* Maslov cycle C_v is by definition $\{\lambda \in \Lambda \mid \lambda \text{ is } \textit{not} \text{ transversal to } \lambda_v\}$, while the *horizontal* Maslov cycle $C_h = \{\lambda \in \Lambda \mid \lambda \text{ is not transversal to } \lambda_h\}$.

2. Formulation of the problem

Our point of departure is the linear, non-autonomous differential system (1):

$$x' = A(t)x + B(t)u + D(t)w,$$
$$x(0) = x_0,$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $w \in \mathbb{R}^l$. The matrix-valued functions A, B, and D have dimensions $n \times n$, $n \times m$, and $n \times l$, respectively. Let Q be a matrix-valued function of dimensions $n \times n$ whose values are symmetric and positive semi-definite: $Q(t) \geqslant 0$ for all $t \in \mathbb{R}$. All the functions A, B, D, and Q are assumed to be uniformly bounded and uniformly continuous.

For each positive real number γ, let L_γ be the functional

$$L_\gamma(u, w) = \int_0^\infty \{\langle Q(t)x(t), x(t)\rangle + |u(t)|^2 - \gamma^2 |w(t)|^2\}\, dt.$$

It is understood that $u \in L^2([0, \infty), \mathbb{R}^m)$, and that $x(\cdot)$ is the solution of (1) corresponding to the given functions u, w. We will usually write simply "L^2" for $L^2([0, \infty), \mathbb{R}^d)$ whenever the dimension d and the interval $[0, \infty)$ are determined by the context. We look for values of γ for which there is a linear feedback control $u = -B^t(t)m_\gamma(t)x$ which stabilizes (1) when $w = 0$, and for which the dissipation inequality (2) holds.

We will make systematic use of the Bebutov or translation flow [4], which is defined on certain spaces of vector- and matrix-valued functions of $t \in \mathbb{R}$. (This explains why we consider functions A, B, D, and Q which are defined on all of $\mathbb{R}$, even though our H^∞-control problem is defined on the half-line $[0, \infty)$.) We need some notation to describe the manifestation of the Bebutov flow which we will use. First, let $\mathbb{M}_{r,s}$ be the set of $r \times s$ real matrices ($1 \leqslant r, s < \infty$). Let $\mathcal{G}_{r,s} = \{G : \mathbb{R} \to \mathbb{M}_{r,s} \mid G \text{ is uniformly bounded and uniformly continuous}\}$. Give $\mathcal{G}_{r,s}$ the topology of uniform convergence on compact subsets of $\mathbb{R}$. The Bebutov flow $\{\tau_t \mid t \in \mathbb{R}\}$ on $\mathcal{G}_{r,s}$ is defined as follows: if $G \in \mathcal{G}_{r,s}$, then $\tau_t(G)(\cdot) = G(\cdot + t)$ ($t \in \mathbb{R}$). It is easily seen that $(\mathcal{G}_{r,s}, \{\tau_t\})$ is indeed a flow. It is also easy to see that, if $G \in \mathcal{G}_{r,s}$, then the orbit closure $cls\{\tau_t(G) \mid t \in \mathbb{R}\}$ is compact.

Next, let $\mathcal{G} = \mathcal{G}_{n,n} \times \mathcal{G}_{n,m} \times \mathcal{G}_{n,l} \times \mathcal{G}_{n,n}$, so that $\xi_0 = (A, B, D, Q)$ is a point in $\mathcal{G}$. There is a Bebutov flow $\{\tau_t\}$ on $\mathcal{G}$. Let $\Xi = cls\{\tau_t(\xi_0) \mid t \in \mathbb{R}\}$, so that Ξ is a compact, translation-invariant subset of $\mathcal{G}$. Clearly, each element $\xi \in \Xi$ is a four-tuple $(A_\xi, B_\xi, D_\xi, Q_\xi)$ of uniformly bounded, uniformly continuous, matrix-valued functions of t. Observe that $Q_\xi(t) \geqslant 0$ for each $\xi \in \Xi$, $t \in \mathbb{R}$. Define $\mathcal{A} : \mathcal{G} \to \mathbb{R} : (a, b, d, q) \to a(0)$. Then $A_\xi(t) = \mathcal{A}(\tau_t(\xi))$; that is, $A_\xi(\cdot)$ is obtained by evaluating the continuous function $\mathcal{A}$ along the orbit through ξ. Similarly, B_ξ, D_ξ, and Q_ξ are obtained by evaluating continuous functions $\mathcal{B}$, $\mathcal{D}$, and $\mathcal{Q} : \mathcal{G} \to \mathbb{R}$ along the orbit through ξ.

For each $\xi \in \Xi$ and $\gamma > 0$, consider the differential system

$$x' = A_\xi(t)x + B_\xi(t)u + D_\xi(t)w, \tag{1_ξ}$$

$$x(0) = x_0,$$

together with the functional

$$L_{\gamma,\xi}(u, w) = \int_0^\infty \{\langle Q_\xi(t)x(t), x(t)\rangle + |u(t)|^2 - \gamma^2|w(t)|^2\}\, dt. \tag{2_ξ}$$

Motivated by the connection between H^∞-control theory and the theory of two-player, zero-sum differential games [5], we introduce the Riccati equation

$$m' + A_\xi^t m + mA_\xi - m[B_\xi B_\xi^t - \gamma^{-2} D_\xi D_\xi^t]m + Q_\xi = 0 \tag{3_ξ}$$

together with the related family of Hamiltonian differential systems

$$z' = \begin{pmatrix} A_\xi & -[B_\xi B_\xi^t - \gamma^{-2} D_\xi D_\xi^t] \\ -Q_\xi & -A_\xi^t \end{pmatrix} z, \quad z = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{2n}. \tag{4_ξ}$$

The relation between (3_ξ) and (4_ξ) can be expressed as follows. Let $z_1(t), \ldots, z_n(t)$ be n linearly independent solutions of (4_ξ). Write the $2n \times n$ matrix $(z_1(t), \ldots, z_n(t))$ whose columns are $z_1(t), \ldots, z_n(t)$ in the form $\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}$, where $X(t)$ and $Y(t)$ are $n \times n$ matrix-valued functions. If $X(t)$ is invertible on some open interval $I \subset \mathbb{R}$, then $m(t) = Y(t)X(t)^{-1}$ is a solution of the Riccati equation (3_ξ) on I.

We will be particularly interested in conditions guaranteeing that Eqs. (4_ξ) admit an *exponential dichotomy* over Ξ. We recall the definition of this concept [7,25]. Let $\mathcal{P}$ be the family of all linear projections $P : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. For each $\xi \in \Xi$, let $\Phi_\xi(t)$ be the fundamental matrix solution of (4_ξ).

Definition 2.1. The family of differential equations (4_ξ) admits an *exponential dichotomy* over Ξ if there are positive constants k, β together with a continuous function $P : \Xi \to \mathcal{P} : \xi \to P_\xi$ such that the following estimates hold:

$$\|\Phi_\xi(t)P_\xi\Phi_\xi(s)^{-1}\| \leqslant ke^{-\beta(t-s)} \quad t \geqslant s,$$

$$\|\Phi_\xi(t)(I - P_\xi)\Phi_\xi(s)^{-1}\| \leqslant ke^{\beta(t-s)} \quad t \leqslant s.$$

We will also encounter the concept of *weak disconjugacy* for the single equation (4) and for the family of Eqs. (4_ξ). We will actually use a variant of the definition of weak disconjugacy given in [14] (which in turn is a variant of the classical definition of *disconjugacy*; see, e.g., [6]).

Definition 2.2. (a) Say that Eq. (4) is *weakly disconjugate* on $[0, \infty)$ if there exists $T > 0$ such that, whenever $z(t) = \binom{x(t)}{y(t)}$ is a non-trivial solution of (4) such that $y(0) = 0$, then $y(t) \neq 0$ for all $t \geqslant T$. The family $\{(4_\xi) \mid \xi \in \Xi\}$ is said to be weakly disconjugate if each single equation (4_ξ) is weakly disconjugate.

(b) Let $z_1(t), \ldots, z_n(t)$ be linearly independent solutions of (4). Let $\lambda(t) = Span\{z_1(t), \ldots, z_n(t)\} \subset \mathbb{R}^{2n}$ $(t \geqslant 0)$, and write the $2n \times n$ matrix $(z_1(t), \ldots, z_n(t))$ in the form $\binom{X(t)}{Y(t)}$. Say that $\binom{X(t)}{Y(t)}$ is a *principal solution* of (4) if (i) $\lambda(t)$ is a Lagrange subspace for some (hence all) $t \geqslant 0$; (ii) $det\, Y(t) \neq 0$ for all $t \geqslant 0$; (iii) $\lim_{t\to\infty} S(t)^{-1} = 0$ where $S(t) = \int_0^t Y(s)^{-1} Q(s) Y(s)^{-1*}\, ds$.

We will systematically apply the concept of *rotation number* α for the family (4_ξ) [16,23,11,12]. The rotation number is defined with respect to a fixed ergodic measure μ on Ξ. We recall one of the equivalent definitions of this quantity. First, recall that the vertical Maslov cycle $C_v \subset \Lambda$ is 2-sided in Λ [1]. Moreover, the complement $\Lambda \setminus C_v$ is simply connected; in fact it is homeomorphic to $\mathbb{R}^{\frac{n(n+1)}{2}}$. These facts permit one to define an oriented intersection index $i(c)$ of each continuous closed curve $c : [0, T] \to \Lambda$ with the cycle C_v whenever $c(0)$ and $c(T)$ lie off C_v. See [1,2] for the construction of this intersection index.

We use the intersection index to define the rotation number α as follows. Let $\xi \in \Xi$, $T > 0$, and let $\lambda \in \Lambda$ be a Lagrange plane which is transverse to λ_v. Let $c_T(t) = \Phi_\xi(t) \cdot \lambda$, so that $c_T : [0, T] \to \Lambda$ is a continuous closed curve in Λ. If $c_T(T) \in C_v$, we "bump it off" in some systematic way, then let n_T be the intersection index of c_T with C_v. We define

$$\alpha(\mu) = \lim_{T\to\infty} \pi \frac{n_T}{T}.$$

It turns out that the limit on the right-hand side is well-defined in the following sense. There is a Borel subset $\Xi_0 \subset \Xi$ with $\mu(\Xi_0) = 1$, such that, if $\xi \in \Xi_0$, then the limit on the right-hand side is defined and is independent of the choice of $\xi \in \Xi_0$ and $\lambda \in \Lambda \setminus C_v$ [16,23,11].

One can also define a rotation number $\tilde{\alpha}(\mu)$ by substituting C_h for C_v in the above construction; it is no surprise that $\alpha(\mu) = \tilde{\alpha}(\mu)$.

The rotation number can be related to the concepts of exponential dichotomy and weak disconjugacy for family (4_ξ) in the case when Ξ is equal to the topological support $\operatorname{Supp} \mu$ of the ergodic measure μ. Let us first describe the connection with the weak disconjugacy concept, summarizing and adapting the results of [14].

Let us suppose that the following controllability condition is satisfied:

Hypothesis 2.3. Each minimal subset $M \subset \Xi$ contains a point p such that the control system

$$y' = -A_p^t(t) y + Q_p(t) v$$

is null-controllable.

As proved in [18], this hypothesis actually implies a uniform controllability condition:

Proposition 2.4. *Let $\Psi_\xi(t)$ be the fundamental matrix solution of the equation $y' = -A^t_\xi(t)y$. Then there exist positive constants T and δ, which do not depend on $\xi \in \Xi$, such that*

$$\int_0^T |Q_\xi(s)\Psi^t_\xi(s)^{-1}|\, ds \geqslant \delta I.$$

The following result is proved in [14, Theorem 2.5].

Theorem 2.5. *Suppose that Supp $\mu = \Xi$ and that Hypothesis 2.3 is valid. Then the following statements hold*:
(a) *Eqs.* (4_ξ) *are all weakly disconjugate if and only if* $\alpha(\mu) = 0$.
(b) *If* $\alpha(\mu) = 0$, *then each Eq.* (4_ξ) *admits a unique principal solution.*

Next we discuss the relation between the rotation number α and the exponential dichotomy concept. We recall a condition of Atkinson type [3] which is useful in this context.

Hypothesis 2.6. Let $\Gamma : \Xi \to \mathbb{M}_{2n,2n}$ be a continuous function whose values are symmetric and positive semi-definite: $\Gamma(\xi) \geqslant 0$ for all $\xi \in \Xi$. Write $\Gamma_\xi(t) = \Gamma(\tau_t(\xi))(\xi \in \Xi, t \in \mathbb{R})$. Say that Eq. (4_ξ) satisfy an *Atkinson condition* with respect to Γ if each minimal subset $M \subset \Xi$ contains a point p such that

$$\int_{-\infty}^{\infty} |\Gamma_p(t)\Phi_p(t)|^2\, dt > 0.$$

The Atkinson Hypothesis 2.6 is closely related to a null controllability hypothesis on the family of control systems

$$z' = \begin{pmatrix} -A^t_\xi & Q_\xi \\ B_\xi B^t_\xi & A_\xi \end{pmatrix} z + \Gamma_\xi v,$$

where now $v = \binom{v_1}{v_2}$ is a control vector in $\mathbb{R}^{2n}$. Using this connection, it is proved in [18] that Hypothesis 2.6 actually implies that there exist positive constants T, δ, which do not depend on $\xi \in \Xi$, such that

$$\int_0^T |\Gamma_\xi(t)\Phi_\xi(t)|^2\, dt \geqslant \delta I$$

for all $\xi \in \Xi$.

We now state

Theorem 2.7. *Consider the Atkinson-type spectral problem*

$$z' = \left[\begin{pmatrix} A_\xi & -B_\xi B_\xi^t \\ -Q_\xi & -A_\xi^t \end{pmatrix} + \eta J^{-1} \Gamma_\xi \right] z \quad (\xi \in \Xi), \tag{5_ξ}$$

where $\eta \in \mathbb{C}$ is a parameter. For each $\eta \in \mathbb{R}$, let $\alpha = \alpha(\eta)$ be the rotation number of the above family of Hamiltonian systems with respect to μ. Suppose that the Atkinson Hypothesis 2.6 holds. Suppose that $\alpha(\eta)$ is constant on some open interval $I \subset \mathbb{R}$. Then for each $\eta \in I$, the family admits an exponential dichotomy on Ξ.

This theorem is proved in [17].

3. Analysis

For each $\xi \in \Xi$, $x_0 \in \mathbb{R}^n$, and $\gamma > 0$ we consider the differential system (1_ξ)

$$x' = A_\xi(t)x + B_\xi(t)u + D_\xi(t)w$$

together with the functional

$$L_{\gamma,\xi}(u, w) = \int_0^\infty \{\langle Q_\xi(t)x(t), x(t)\rangle + |u(t)|^2 - \gamma^2 |w(t)|^2\}\, dt.$$

We look for values of γ for which there is a linear feedback control $u = -B_\xi(t)^t m_{\gamma,\xi}(t)x$ which stabilizes the system

$$x' = A_\xi(t)x + B_\xi(t)u$$

and for which the dissipation inequality (2_ξ)

$$L_{\gamma,\xi}(u, w) \leqslant \langle m_{\gamma,\xi}(0)x_0, x_0\rangle$$

holds ($w \in L^2, \xi \in \Xi, x_0 \in \mathbb{R}^n$). To simplify the notation we will usually write L_ξ for $L_{\gamma,\xi}$ and m_ξ for $m_{\gamma,\xi}$.

We impose a second controllability hypothesis

Hypothesis 3.1. Each minimal subset $M \subset \Xi$ contains a point p such that the control system

$$x' = A_p(t)x + B_p(t)u$$

is null controllable.

The same result of [18] which allows to pass from Hypothesis 2.3 to Proposition 2.4 yields the next result; we again write $\Psi_\xi(t)$ for the fundamental matrix solution of $y' = -A^t_\xi(t)y$.

Proposition 3.2. *There exist positive constants T, δ, which do not depend on $\xi \in \Xi$, such that*

$$\int_0^T |B_\xi(t)\Psi_\xi(t)|^2\,dt \geqslant \delta I$$

for all $\xi \in \Xi$.

Lemma 3.3. *Let $\xi \in \Xi$. The control system $x' = A_\xi x + B_\xi u$ is null controllable if and only if the system*

$$x' = A_\xi x + B_\xi B^t_\xi u$$

is null controllable.

Proof. The simple arguments required to prove this statement can be found in the proof of [13, Lemma 3.3]. □

Now we return to the Riccati equation (3_ξ) and set $\gamma = \infty$, i.e., $\gamma^{-2} = 0$: we obtain

$$m' + A^t_\xi m + mA_\xi - mB_\xi B^t_\xi m + Q_\xi = 0. \tag{6_ξ}$$

Associated with the Riccati equation is the Hamiltonian system

$$z' = \begin{pmatrix} A_\xi & -B_\xi B^t_\xi \\ -Q_\xi & -A^t_\xi \end{pmatrix} z. \tag{7_ξ}$$

We will use Hypotheses 2.3 and 3.1 to analyze family (7_ξ).

Proposition 3.4. *Suppose that Hypotheses 2.3 and 3.1 are valid. Then the family of differential systems (7_ξ) admits an exponential dichotomy over Ξ.*

Proof. The argument necessary to prove this statement is given in [18, Lemma 4.4 and 4.5]. For completeness we sketch the details.

Let $M \subset \Xi$ be a minimal subset, and let μ be an ergodic measure supported on M. Let $\alpha(\mu)$ be the rotation number of family (7_ξ) with respect to μ.

Introduce a real parameter η as follows:

$$z' = \left[\begin{pmatrix} A_\xi & -B_\xi B^t_\xi \\ -Q_\xi & -A^t_\xi \end{pmatrix} + \eta J^{-1}\begin{pmatrix} Q_\xi & 0 \\ 0 & B_\xi B^t_\xi \end{pmatrix}\right] z.$$

These equations coincide with Eq. (5_ξ) if

$$\Gamma_\xi(t) = \begin{pmatrix} Q_\xi(t) & 0 \\ 0 & B_\xi(t)B_\xi^t(t) \end{pmatrix}$$

and in this proof we will refer to the above family as Eq. (5_ξ).

It is easy to see that the rotation number $\alpha = \alpha(\mu, \eta)$ of Eq. (5_ξ) equals zero if $\eta \in (-\frac{1}{2}, \frac{1}{2})$ [18, Lemma 4.4]. Using Hypotheses 2.3 and 3.1, one checks that the Atkinson condition 2.6 is satisfied [18, Corollary 4.3]. By Theorem 2.7, Eq. (5_ξ) admit an exponential dichotomy over M for $\eta \in (-\frac{1}{2}, \frac{1}{2})$, in particular for $\eta = 0$. Therefore, Eq. (7_ξ) admit an exponential dichotomy over M.

Let P_ξ be the dichotomy projection for $\xi \in M$. Then $\lambda(\xi) = Im\, P_\xi$ is a Lagrange subspace of $\mathbb{R}^{2n}$ [23]. In particular, $dim\, \lambda(\xi) = n$. We conclude that Eq. (7_ξ) have an exponential dichotomy over M for each minimal subset $M \subset \Xi$, and that the dimension of $Im\, P_\xi$ equals n for all $\xi \in M$, whenever $M \subset \Xi$ is minimal.

One next argues as in the proof of [18, Lemma 4.5] to show that, for each $\xi \in \Xi$, Eq. (7_ξ) admits no non-trivial solution $z(t)$ which is bounded on all of $\mathbb{R}$. Using a result of Sacker–Sell [26]; see also Selgrade [24], we conclude that Eq. (5_ξ) have an exponential dichotomy over all of Ξ. This completes the proof of Proposition 3.4. $\square$

Let $\xi \in \Xi$, and let P_ξ be the dichotomy projection for Eq. (7_ξ). Let $z_0 = \binom{x_0}{y_0} \in \lambda(\xi) = Im\, P_\xi$, and let $z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ be the solution of (7_ξ) such that $z(0) = z_0$. Then $z(t) \to 0$ exponentially as $t \to \infty$; in fact there are positive constants K, β, which do not depend on $\xi \in \Xi$, such that $|z(t)| \leqslant Ke^{-\beta t}|z_0|$ for all $t \geqslant 0$. It turns out [18, Lemma 4.6] that $\lambda(\xi)$ is transverse to both the vertical Lagrange plane λ_v and to the horizontal Lagrange subspace λ_h. The fact that $\lambda(\xi)$ is transverse to λ_v implies that there is a unique $n \times n$ symmetric real matrix $m(\xi)$ with the following property: if $e_1, \ldots, e_n$ is the canonical basis in $\mathbb{R}^n$, then a basis of $\lambda(\xi)$ is given by

$$\binom{e_1}{m(\xi)e_1}, \ldots, \binom{e_n}{m(\xi)e_n}.$$

The mapping $\xi \to m(\xi) : \Xi \to \mathbb{M}_{n,n}$ is continuous. It further turns out that $m(\xi)$ is positive definite for all $\xi \in \Xi$. All this is of course no surprise in view of basic facts concerning the linear regulator problem.

Now let γ decrease from $\gamma = \infty$. We can apply the standard perturbation theory for exponential dichotomies [7,26] to conclude that, if γ is sufficiently large, then Eq. (4_ξ) admit an exponential dichotomy over Ξ. Let P_ξ be the dichotomy projection for Eq. (4_ξ), and let $\lambda(\xi) = Im\, P_\xi$ (we again fail to indicate explicitly the dependence on γ). Since the dichotomy projections are continuous in γ [7], we can affirm that, for γ sufficiently large, the Lagrange subspace $\lambda(\xi)$ is transverse both to λ_v and to λ_h.

Definition 3.5. We define the critical attenuation value γ^* for family (4_ξ) to be

$$\gamma^* = \inf\{\bar{\gamma} \mid \text{for all } \gamma \geqslant \bar{\gamma},\ \text{Eq. } (4_\xi) \text{ admit an exponential dichotomy over } \Xi, \text{ and moreover } \lambda(\xi) \text{ is transverse to } \lambda_v \text{ for all } \xi \in \Xi\}.$$

This definition of γ^* is appropriate because it permits one to use the powerful roughness properties of exponential dichotomies to deal with natural robustness questions. We will deal with robustness questions in a later paper. It seems to be less convenient to define γ^* in terms of the existence of bounded solutions of the Riccati equation for the non-autonomous infinite horizon H^∞ control problem.

Let us now show that, if $\gamma > \gamma^*$, then our H^∞-control problem admits a solution.

Theorem 3.6. *Consider the family of H^∞ control problems defined by Eqs.* (1_ξ) *and the functional* $L_{\gamma,\xi}$ $(\xi \in \Xi)$. *Suppose that the controllability Hypotheses* 2.3 *and* 3.1 *are valid. Let γ^* be the critical attenuation value for family* (4_ξ). *Suppose that $\gamma > \gamma^*$. Then for each $\xi \in \Xi$, there is a linear feedback control $u = -B^t_\xi(t)m_\xi(t)x$ such that the system*

$$x' = [A_\xi(t) - B_\xi(t)B^t_\xi(t)m_\xi(t)]x \tag{8_ξ}$$

is uniformly exponentially stable. Moreover, for all $x_0 \in \mathbb{R}^n$ and all $w \in L^2$ one has

$$L_{\gamma,\xi}(u, w) \leqslant \langle m_\xi(0)x_0, x_0\rangle.$$

The matrix $m_\xi(t)$ is positive definite for all $\xi \in \Xi$ and $t \in \mathbb{R}$.

Proof. By assumption, the Lagrange subspace $\lambda(\xi)$ is transverse to λ_v for all $\xi \in \Xi$; hence $\lambda(\xi)$ is parametrized by a real $n \times n$ symmetric matrix $m(\xi)$. The function $\xi \to m(\xi) : \Xi \to \mathbb{M}_{n,n}$ is continuous and hence bounded.

For the next few lines it will be convenient to explicitly indicate the dependence of the quantities λ and m on γ. Our goal in these lines is to show that $m_\gamma(\xi)$ is positive definite for all $\gamma > \gamma^*$ and all $\xi \in \Xi$. By continuity in γ of the dichotomy projections $P_\xi = P_{\gamma,\xi}$, we have that $m_\gamma(\xi) > 0$ for all $\xi \in \Xi$ if γ is sufficiently large.

Suppose for contradiction that there exist $\gamma_1 > \gamma^*$ and $\xi_1 \in \Xi$ such that $m_{\gamma_1}(\xi_1)$ is not positive definite. There is then no loss in generality in assuming that $det\, m_{\gamma_1}(\xi_1) = 0$ and in assuming that $\gamma_1 = \max\{\gamma > \gamma^* \mid \text{there exists } \xi \in \Xi \text{ such that } m_\gamma(\xi) \text{ is not positive definite}\}$. A moment's thought shows that one must have $\lambda_{\gamma_1}(\xi_1) \in C_h$, the horizontal Maslov cycle. Moreover, if $\gamma > \gamma_1$, then $\lambda_\gamma(\xi) \notin C_h$ for all $\xi \in \Xi$.

Next note that, if μ is any ergodic measure on Ξ, then the rotation number $\alpha(\mu)$ of family (4_ξ) is zero. This follows from the definition of γ^* and the definition of the rotation number. We can now argue as in the proof of Proposition 3 of [14]: using the controllability Hypothesis 2.3 on the control systems $y' = -A_p^* y + Q_p v$, we conclude that it is not the case that $\lambda_{\gamma_1}(\tau_t(\xi_1))$ lies on C_h for all $t \in \mathbb{R}$.

We claim that there exist times $t_1 < 0$ and $t_2 > 0$ such that $\lambda_{\gamma_1}(\tau_{t_1}(\xi_1)) \notin C_h$ and $\lambda_{\gamma_1}(\tau_{t_2}(\xi_1)) \notin C_h$. For if, for example, $\lambda_{\gamma_1}(\tau_t(\xi_1)) \in C_h$ for all $t \geqslant 0$, then each point $\hat{\xi}$ in the ω-limit set $\omega(\xi_1) \subset \Xi$ has the property that $\lambda_{\gamma_1}(\tau_t(\hat{\xi})) \in C_h$ for all $t \in \mathbb{R}$. If $t \leqslant 0$, we substitute the α-limit set $\alpha(\xi_1)$ for $\omega(\xi_1)$ in this argument. Next, let c be a closed curve in Λ obtained by sliding $\lambda_{\gamma_1}(t_2)$ through the simply connected set $\Lambda \setminus C_h$ to $\lambda_{\gamma_1}(t_1)$. Arguing as in the proof of Lemma 4 in [14], we see that the intersection index $i(c)$ of this curve c with respect to C_h is strictly positive.

On the other hand, let $\varepsilon > 0$ and let $\gamma = \gamma_1 + \varepsilon$. The curve $c_\varepsilon : [t_1, t_2] \to \Lambda : c_\varepsilon(t) = \lambda_\gamma(\tau_t(\xi_1))$ lies entirely in $\Lambda \setminus C_h$; hence, if it is closed up by sliding its endpoints $c_\varepsilon(t_1)$ and $c_\varepsilon(t_2)$ together in $\Lambda \setminus C_h$, one obtains a closed curve (again called c_ε) whose intersection number $i(c_\varepsilon)$ with C_h is zero.

However, if ε is sufficiently small, the curves c and c_ε are homotopic; hence, their intersection indices are zero [1]. We have arrived at a contradiction. We conclude that $m_\gamma(\xi)$ is indeed positive definite whenever $\gamma > \gamma^*$ and $\xi \in \Xi$.

Now fix $\gamma > \gamma^*$ and write $m_\xi(t) = m(\tau_t(\xi))$, where we do not explicitly indicate the dependence of m_ξ on γ. Then there is a constant K' such that $|m_\xi(t)| \leqslant K'$ and $|m_\xi(t)^{-1}| \leqslant K'$ for all $\xi \in \Xi$ and $t \in \mathbb{R}$.

Let $w \in L^2$, let $u : [0, \infty) \to \mathbb{R}^m$ be a continuous function, and let $x(t)$ be the corresponding solution of Eq. (1_ξ). We apply the classical completing-the-square argument to L_ξ. Namely, let $T > 0$ and integrate the expression $\frac{d}{dt}\langle m_\xi(t)x(t), x(t)\rangle$ from 0 to T; after some rearranging one gets

$$\begin{aligned}
&\int_0^T \{\langle Q_\xi(t)x(t), x(t)\rangle + |u(t)|^2 - \gamma^2 |w(t)|^2\}\, dt \\
&\quad = -\langle m_\xi(T)x(T), x(T)\rangle + \langle m_\xi(0)x_0, x_0\rangle \\
&\qquad -\gamma^2 \int_0^T |w(t) - \gamma^{-2} D_\xi^t(t) m_\xi(t) x(t)|^2\, dt \\
&\qquad + \int_0^T |u(t) + B_\xi^t(t) m_\xi(t) x(t)|^2\, dt. \qquad (9)
\end{aligned}$$

Next, we introduce the feedback control $u = -B_\xi^t m_\xi x$ and set $w = 0$ to obtain

$$\begin{aligned}
&\int_0^T \{\langle Q_\xi(t)x(t), x(t)\rangle + |u(t)|^2 + \gamma^{-2} |D_\xi^t(t) m_\xi(t) x(t)|^2\}\, dt \\
&\quad = -\langle m_\xi(T)x(T), x(T)\rangle + \langle m_\xi(0)x_0, x_0\rangle. \qquad (10)
\end{aligned}$$

Here $x(t)$ is the solution of the linear system (8_ξ)

$$x' = [A_\xi(t) - B_\xi(t)B_\xi(t)^t m_\xi(t)]x$$

which satisfies $x(0) = x_0$.

Our goal is to show that family (8_ξ) is uniformly exponentially stable. Explicitly, we seek fixed positive constants K_1, β_1, such that, if $\xi \in \Xi$, $x_0 \in \mathbb{R}^n$, and $x(t)$ is the corresponding solution of (8_ξ), then

$$|x(t)| \leqslant K_1 e^{-\beta_1 t}|x_0| \quad (t \geqslant 0).$$

The first step is to apply Lemma 4 of [26]: according to this result it is sufficient to show that, if $\xi \in \Xi$ and $x_0 \in \mathbb{R}^n$, then $x(t) \to 0$ as $t \to \infty$. So we show that each solution $x(t)$ of each Eq. (8_ξ) decays to zero as $t \to \infty$. To do this, it is convenient to introduce the linear skew-product flow $\{\hat{\tau}_t \mid t \in \mathbb{R}\}$ on $\Xi \times \mathbb{R}^n$ defined by $\hat{\tau}_t(\xi, x_0, t) = (\tau_t(\xi), x(t))$ $(\xi \in \Xi, x_0 \in \mathbb{R}^n, t \in \mathbb{R})$. See, e.g., [25] for basic facts about such linear skew-product flows.

Fix $\xi \in \Xi$ and $x_0 \in \mathbb{R}^n$ together with the solution $x(t)$ of (8_ξ) which satisfies $x(0) = x_0$. Since $\langle m_\xi(t)x(t), x(t)\rangle \leqslant \langle m_\xi(0)x_0, x_0\rangle$ and since $|m_\xi(t)^{-1}| \leqslant K'$ for all $t \geqslant 0$, we see that $x(t)$ is bounded on $[0, \infty)$. Since in addition $|m_\xi(t)| \leqslant K'$, we see that $\lim_{t\to\infty} x(t) = 0$ if and only if $\lim_{t\to\infty}\langle m_\xi(t)x(t), x(t)\rangle = 0$.

Suppose for contradiction that we can find a point $(\xi_1, x_1) \in \Xi \times \mathbb{R}^n$ such that the corresponding quantity $\langle m_{\xi_1}(t)x_1(t), x_1(t)\rangle$ does not tend to zero as $t \to \infty$. Since this quantity is monotone non-increasing, we can find a number $\varepsilon > 0$ such that $|x_1(t)| \geqslant \varepsilon$ for all $t \geqslant 0$. Let $\Omega \subset \Xi \times \mathbb{R}^n$ be the ω-limit set of (ξ_1, x_1) with respect to the flow $\{\hat{\tau}_t\}$. Then Ω is compact, invariant under $\{\hat{\tau}_t\}$, and, moreover, if $(\xi, x_0) \in \Omega$ then $|x_0| \geqslant \varepsilon$.

Let M be a minimal subset of Ω and let $(\xi, x_0) \in M$. By the minimality of M, the function $g : \mathbb{R} \to \mathbb{R} : g(t) = \langle m_\xi(t)x(t), x(t)\rangle$ is Birkhoff recurrent. On the other hand, g is also non-increasing. It follows that g is a constant function, so that $\langle m_\xi(t)x(t), x(t)\rangle = \langle m_\xi(0)x_0, x_0\rangle$ for all $t \in \mathbb{R}$.

Now, using (10), we conclude that (i) $B_\xi^t(t)m_\xi(t)x(t) = 0$ for all $t \geqslant 0$, and (ii) $Q_\xi(t)x(t) = 0$ for all $t \geqslant 0$. Using (i), we see that $x' = A_\xi(t)x$. Thus we have $x(t) = \Psi_\xi^t(t)^{-1}x_0$ where $\Psi_\xi(t)$ is the fundamental matrix solution of $y' = -A_\xi^t y$. But then (ii) contradicts the uniform controllability property expressed in Proposition 2.4. We have arrived at a contradiction, and so can conclude that Eq. (8_ξ) are indeed uniformly exponentially stable.

The dissipation relation

$$L_{\gamma,\xi}(u, w) \leqslant \langle m_\xi(0)x_0, x_0\rangle$$

follows directly from Eq. (9). This completes the proof of Theorem 3.6. □

Remark 3.7. If $\gamma > \gamma^*$ and if the family $x' = A_\xi(t)x$ is uniformly exponentially stable (of Hurwitz type), then one can show that the differential game defined by (1_ξ) and L_ξ admits the value $\langle m_\xi(0)x_0, x_0\rangle$ for each $\xi \in \Xi$.

We introduce a controllability condition involving the matrix functions D_ξ.

Hypothesis 3.8. The Atkinson condition 2.6 holds for Eq. (4_ξ) with

$$\Gamma_\xi(t) = \begin{pmatrix} 0 & 0 \\ 0 & D_\xi(t)D_\xi^t(t) \end{pmatrix}.$$

Explicitly, each minimal subset $M \subset \Xi$ contains a point p such that, if $\Phi_p(t)$ is the fundamental matrix solution of $z' = \begin{pmatrix} A_p & -B_pB_p^t \\ -Q_p & -A_p^t \end{pmatrix} z$, then

$$\int_{-\infty}^{\infty} \left| \begin{pmatrix} 0 & 0 \\ 0 & D_p(t)D_p^t(t) \end{pmatrix} \Phi_p(t) \right|^2 dt > 0.$$

Remark 3.9. (a) The Atkinson Hypothesis 3.8 implies the uniform null controllability of the family of control systems

$$z' = \begin{pmatrix} -A_\xi^t & Q_\xi \\ B_\xi B_\xi^t & A_\xi \end{pmatrix} z + \begin{pmatrix} 0 & 0 \\ 0 & D_\xi D_\xi^t \end{pmatrix} v,$$

where now $v = \binom{v_1}{v_2}$ and $v_1, v_2 \in \mathbb{R}^n$. That is, if Hypothesis 3.8 holds, then there are positive constants T, δ, which do not depend on $\xi \in \Xi$, such that

$$\int_0^T \left| \begin{pmatrix} 0 & 0 \\ 0 & D_\xi(t)D_\xi^t(t) \end{pmatrix} \Phi_\xi(t) \right|^2 dt \geqslant \delta I$$

for all $\xi \in \Xi$.

(b) Hypothesis 3.8 is somewhat stronger than that of the uniform null controllability of the family of control systems

$$x' = A_\xi x + D_\xi w.$$

(c) Suppose that $det\, D_\xi D_\xi^t \neq 0$ for all $\xi \in \Xi$. Then Hypothesis 3.8 is valid. To see this, let $M \subset \Xi$ be a minimal subset, and let $p \in M$. Let $x_0,\ y_0 \in \mathbb{R}^n$. By the null controllability of the system $x' = -A_p^t x + Q_p v$, we can determine $T > 0$ and a control function $v_1 : [0, T] \to \mathbb{R}^n$ such that, if $x(t)$ satisfies $x' = -A_p^t + Q_p v_1$ and $x(0) = x_0$, then $x(T) = 0$. One can choose v_1 in such a way that v_1 is of class C^1, $v_1(0) = y_0$, and $v_1(T) = 0$. Next set $y(t) = v_1(t)$ for $0 \leqslant t \leqslant T$; then define v_2

by $y' - A_p y - B_p B_p^t x = D_p D_p^t v_2$. Then the control $v = \binom{v_1}{v_2}$ steers $\binom{x_0}{y_0}$ to zero in time T for the control system (9_p).

We now define a number γ_l which is significant in the study of Eq. (4_ξ).

Definition 3.10. Set $\gamma_l = \inf\{\bar{\gamma} \mid$ Eq. (4_ξ) admit an exponential dichotomy over Ξ for all $\gamma > \bar{\gamma}\}$.

Theorem 3.11. *Consider the family of H^∞-control problems defined by Eqs.* (1_ξ) *and the functional* $L_{\gamma,\xi}$ $(\xi \in \Xi)$. *Suppose that the controllability Hypotheses* 2.3, 3.1, *and* 3.8 *are all valid. Let γ^* be the critical attenuation value for family* (4_ξ), *and let γ_l be as in Definition* 3.10. *Then $\gamma_l > 0$ and $\gamma^* \geqslant \gamma_l$.*

Proof. We need only to prove that $\gamma_l > 0$. To do this, we need the elements of the Atkinson spectral theory of the equations

$$z' = \left[\begin{pmatrix} A_\xi & -B_\xi B_\xi^t \\ -Q_\xi & -A_\xi^t \end{pmatrix} + \eta J^{-1}\begin{pmatrix} 0 & 0 \\ 0 & D_\xi D_\xi^t \end{pmatrix}\right] z. \tag{11_ξ}$$

We only outline the necessary facts and arguments, using [12,19] and the literature cited therein as a reference.

Let η be a complex number with $Im\,\eta \neq 0$. For each $\xi \in \Xi$, introduce the Titchmarsh–Weyl–Kodaira matrices $M^\pm(\xi,\eta)$: these are $n \times n$, symmetric, complex matrices. The matrix functions $\eta \to M^\pm(\xi,\eta)$ are holomorphic on $\mathbb{C} \setminus \mathbb{R}$ for each $\xi \in \Xi$. It follows from the Atkinson Hypothesis 3.8 that $sign\,\frac{Im\,M^\pm(\xi,\eta)}{Im\,\eta} = \pm 1$ $(\xi \in \Xi, \eta \in \mathbb{C} \setminus \mathbb{R})$. One can show that $M^\pm_{\xi,\eta}(t) = M^\pm(\tau_t(\xi),\eta)$ satisfies the Riccati equation (3_ξ) with η in place of γ^{-2}. In fact, Eq. (11_ξ) admit an exponential dichotomy over Ξ when $Im\,\eta \neq 0$; $M^+(\xi,\eta)$ parametrizes the dichotomy projection $P_{\xi,\eta}$, while $M^-(\xi,\eta)$ parametrizes $I - P_{\xi,\eta}$.

Now let $J \subset \mathbb{R}$ be an open interval, and suppose that Eq. (11_ξ) admit an exponential dichotomy over Ξ for each $\eta \in J$. Introduce the diagonal Green's function

$$G(\xi,\eta) = \begin{pmatrix} (M^- - M^+)^{-1} & \frac{1}{2}(M^- - M^+)^{-1}(M^- + M^+) \\ \frac{1}{2}(M^- + M^+)(M^- - M^+)^{-1} & M^+(M^- - M^+)^{-1}M^- \end{pmatrix}.$$

Then the function $\eta \to G(\xi,\eta)$ extends holomorphically through J for each $\xi \in \Xi$.

For each $\xi \in \Xi$, there is a $2n \times 2n$, symmetric "spectral matrix" (actually, matrix-valued measure) Q_ξ such that, for any η with $Im\,\eta > 0$:

$$\frac{Im\,G(\xi,\eta)}{Im\,\eta} = \int_{-\infty}^{\infty} \frac{dQ_\xi(t)}{|t-\lambda|^2}.$$

The spectral matrix is constructed by considering $\lim_{\varepsilon\to 0^+} G(\xi, t+i\varepsilon)$ $(t \in R)$. It follows from this construction and the holomorphic extension property that

$$\int_J dQ_\xi(t) = 0$$

for all $\xi \in \Xi$.

Next, one can argue as in the proof of Proposition 3.4 to show that, if $\eta \leqslant 0$, then Eq. (11_ξ) admit an exponential dichotomy over Ξ. By the roughness properties of exponential dichotomies, there exists $\varepsilon > 0$ such that Eq. (11_ξ) admits an exponential dichotomy for $-\infty < \eta \leqslant \varepsilon$. Thus $\int_{-\infty}^{\varepsilon} dQ_\xi(t) = 0$ for all $\xi \in \Xi$. However, $\int_{-\infty}^{\infty} dQ_\xi(t) \neq 0$ because $Im\, G(\xi, \eta) > 0$ if $Im\, \eta > 0$ $(\xi \in \Xi)$. We conclude that there exists $\eta_0 > \varepsilon$ for which Eq. (11_ξ) do not admit an exponential dichotomy. This implies that $\gamma_l \geqslant \eta_0^{-2} > 0$ and completes the proof of Theorem 3.11. $\square$

We now distinguish two possibilities: $\gamma^* > \gamma_l$ and $\gamma^* \leqslant \gamma_l$. We discuss the situation when $\gamma^* > \gamma_l$ in an informal way. In this case, there exists $\xi^* \in \Xi$ such that $\lambda_{\gamma^*}(\xi^*) \in C_v$. One can show that, if $\gamma \leqslant \gamma^*$, and if $\xi \in \Xi$ is a point whose forward semiorbit $\{\tau_t(\xi) \mid t \geqslant 0\}$ is dense in Ξ, then there is no linear feedback control for which the dissipation inequality (2_ξ) is valid. In particular, if the forward semiorbit of the quadruple $\xi_0 = (A, B, C, D)$ is dense in Ξ, then the dissipation inequality (2) does not hold for any $\gamma \leqslant \gamma^*$.

The case when $\gamma^* = \gamma_l$ is more interesting. To analyze it in a clean way, we suppose that Ξ is the topological support $\mathrm{Supp}\,\mu$ of a fixed ergodic measure μ on Ξ. This condition holds in particular if the quadruple $\xi_0 = (A, B, D, Q)$ is Birkhoff recurrent with respect to the Bebutov flow. Hence, it holds if all these functions are Bohr almost periodic.

The attenuation problem may or may not be solvable at $\gamma = \gamma^*$ when $\gamma^* = \gamma_l$. A simple example in which it is solvable is $(A, B, C, D) = (-1, 1, 1, 1)$ with $\gamma^* = \frac{\sqrt{2}}{2}$. One can determine periodic functions (A, B, C, D) for which $\gamma^* = \gamma_l$ and the attenuation problem is not solvable at γ^*.

We are going to give an example in which $\gamma^* = \gamma_l$ and the attentuation problem admits multiple solutions at $\gamma = \gamma^*$, for μ-a.a. $\xi \in \Xi$. That is, for μ-a.a. $\xi \in \Xi$, there are distinct feedback controls $u_1 = -B_\xi^t(t) m_1(t) x$ and $u_2 = -B_\xi^t m_2(t) x$ which stabilize Eqs. (1_ξ) when $w = 0$, such that $\int_0^\infty \{\langle Q_\xi(t)x(t), x(t)\rangle + |u_i(t)|^2\}\, dt \leqslant \langle m_i(0)x_0, x_0\rangle + \gamma^{*2} \int_0^\infty |w(t)|^2\, dt$ for all $w \in L^2$, $x_0 \in \mathbb{R}^n$. This phenomenon does not occur if the coefficient functions A, B, D, Q are all periodic with the same period.

Before giving the example we discuss the theoretical background. Let $\alpha(\mu; \gamma) = \alpha(\gamma)$ be the rotation number of Eq. (4_ξ) with respect to μ. We use the main theorem of [17] to conclude that $\gamma_l = \inf\{\gamma > 0 \mid \alpha(\gamma) = 0\}$. By the continuity properties of the rotation number, we have $\alpha(\gamma^*) = 0$. This condition implies that, at $\gamma = \gamma^*$, all the Eq. (4_ξ) are weakly disconjugate [14].

For each $\xi \in \Xi$, let $\lambda(\xi) \in \Lambda$ denote the initial value (viewed as a Lagrange plane) of the principal solution of (4_ξ). Arguing as in the proof of [14, Theorem 2], we see that $\lambda(\xi)$ is transverse to λ_h for all $\xi \in \Xi$. Even more, each element (ξ, λ) of $\Sigma = cls\,\{(\xi, \lambda(\xi)) \mid \xi \in \Xi\} \subset \Xi \times \Lambda$ has the property that λ is transversal to λ_h.

Let us now assume that the map $\xi \to \lambda(\xi)$ is discontinuous μ-a.e., and that, for each $(\xi, \lambda) \in \Sigma$, λ is transversal to λ_v. These conditions will be realized in our example. Then for μ-a.a. $\xi \in \Xi$, the fiber $\{\lambda \in \Lambda \mid (\xi, \lambda) \in \Sigma\}$ contains at least two points. Fix such a point $\xi \in \Xi$, and let λ_1, λ_2 be distinct points in fiber of Σ at ξ. Let $m_1(0)$ resp. $m_2(0)$ be the parameters of λ_1 resp. λ_2, and let $m_1(t)$ resp. $m_2(t)$ be the corresponding solutions of the Riccati equation (3_ξ). There is a positive constant K' such that $|m_i(t)| \leqslant K'$ and $|m_i(t)^{-1} \leqslant K'$ for all $t \in \mathbb{R}$. So the controls $u_1 = -B_\xi^t m_1(t)x$ and $u_2 = -B_\xi^t m_2(t)x$ stabilize (1_ξ) when $w = 0$ and satisfy (2_ξ) for all $w \in L^2$ and $x_0 \in \mathbb{R}^n$, at least if, say, $Q_\xi(t)$ or $D_\xi(t)D_\xi^t$ is strictly positive definite for all $t \geqslant 0$.

Example 3.12. The construction below uses a technique due to Millionščikov [21]; see also Vinograd [27]. Let $n = 1$. Consider a family of ordinary differential equations

$$z' = \begin{pmatrix} A(t) & \gamma^{-2}D^2(t) - B^2(t) \\ -Q(t) & -A(t) \end{pmatrix} z \quad (\gamma > 0),$$

where $z \in \mathbb{R}^2$ and A, B, D, Q are real-valued functions. These equations have form (4). We will determine the functions A, B, D, Q in such a way that $\gamma^* = \gamma_l = 1$ and so that the set Σ has the required properties.

Let T be a positive number. Set $A_0(t) = -1$, $Q_0(t) = \sqrt{3}$, $\Delta_0(t) = -\sqrt{3}$ $(0 \leqslant t \leqslant T)$, then set

$$G_0(t) = \begin{pmatrix} A_0(t) & \Delta_0(t) \\ -Q_0(t) & -A_0(t) \end{pmatrix} = \begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \quad (0 \leqslant t \leqslant T).$$

Abusing notation, write $G_0 = \begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$, and note that G_0 has eigenvalues ± 2 with eigenvectors $v_+ = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$, $v_- = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}$. The polar angle of v_+ resp. v_- is $\theta_+ = \frac{2\pi}{3}$ resp. $\theta_- = \frac{\pi}{6}$, so v_+ and v_- are orthogonal. Let $R : [\theta_-, \theta_+] \to [0, \frac{\pi}{2})$ be the rotation effectuated in time T by $e^{G_0 T}$ on non-zero vectors $v \in \mathbb{R}^2$ whose polar angles θ_v lie in $[\theta_-, \theta_+]$. That is, set $v(T) = e^{G_0 T} \cdot v$, and set $R(\theta_v) = \theta_{v(T)} - \theta_v$. Clearly $R(\theta_-) = R(\theta_+) = 0$. When $T > 0$, R assumes its maximal value in a unique point $\theta^T \in (\theta_-, \frac{5\pi}{12})$, and $\theta^T \to \frac{5\pi}{12} = \frac{\pi}{6} + \frac{\pi}{4}$ as $T \to 0$. Choose and fix a number $T > 0$ such that, if $0 \neq v^T \in \mathbb{R}^2$ has polar angle θ^T, then $e^{G_0 T} \cdot v^T$ lies in the open first quadrant $\{v \in \mathbb{R}^2 : v \neq 0, 0 < \theta_v < \frac{\pi}{2}\}$.

Next let $T_0 > T$, and set $A_0(t) = 0$, $Q_0(t) = 1$, $\Delta_0(t) = 1$ $(T < t \leqslant T_0)$. Set

$$G_0(t) = \begin{pmatrix} A_0(t) & \Delta_0(t) \\ -Q_0(t) & -A_0(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (T < t \leqslant T_0).$$

Abusing notation as before, write $G_0 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and note that $e^{G_0(t-T)}$ effects a rotation of $t - T$ radians in the clockwise sense in $\mathbb{R}^2$.

Now let $\Phi_0(t)$ be the fundamental matrix solution of

$$z' = G_0(t)z \quad (0 \leqslant t \leqslant T_0).$$

If $0 < T_0 - T$ is sufficiently small, then $\Phi_0(t)$ has the following properties: First, $\Phi_0(T_0)$ admits two normalized eigenvectors u_+ and u_- satisfying $0 < \theta_{u_-} < \theta_{u_+} < \frac{\pi}{2}$; in particular, $u_\pm$ lie in the open first quadrant. Second, $\Phi_0(T_0)u_- = \beta_- u_-$ and $\Phi_0(T_0)u_+ = \beta_+ u_+$ where $0 < \beta_- < 1 < \beta_+ = \frac{1}{\beta_-}$. Third, $\Phi_0(t)u_+$ lies in the open first quadrant for all $t \in [0, T_0]$; it follows that $\Phi_0(t)u_-$ also lies in the open first quadrant for all $t \in [0, T_0]$.

We now modify $G_0(\cdot)$ in such a way as to obtain a continuous, 2×2 matrix-valued function—again called $G_0(\cdot)$—such that $G_0(0) = G_0(T_0)$ and such that the properties of the preceding paragraph hold for the fundamental matrix solution $\Phi_0(t)$ of the modified system

$$z' = G_0(t)z. \tag{12}$$

The modification can be carried out so that the trace $tr\, G_0(t) = 0$ for all $t \in [0, T_0]$, and so that, if

$$G_0(t) = \begin{pmatrix} A_0(t) & \Delta_0(t) \\ -Q_0(t) & -A_0(t) \end{pmatrix},$$

then the following properties hold: $A_0(t) \leqslant 0$ for all $t \in [0, T_0]$; $A_0(t) \leqslant -1$ for all $t \in [0, T]$; $Q_0(t) \geqslant 1$ for all $t \in [0, T_0]$. Extend $G_0(t)$ to the entire real axis so as to obtain a continuous, T_0-periodic, 2×2 matrix-valued function. System (12) admits an exponential dichotomy over $\mathbb{R}$ (see [7]).

Now we apply the construction of Millionščikov [21], beginning with the T_0-periodic systems (12). We will not enter into the details of the construction, but will only describe what it produces. Thus set $\beta^0_\pm = \beta_\pm$, and define $\eta = \frac{(1-\beta_-)}{2}$. One obtains sequences $\{T_k \mid k = 0, 1, 2, \ldots\}$ and $\{\beta^k_- \mid k = 0, 1, 2, \ldots\}$ of positive numbers and a sequence $\{G_k = \begin{pmatrix} A_k & \Delta_k \\ -Q_k & -A_k \end{pmatrix} \mid k = 0, 1, 2, \ldots\}$ of continuous, T_k-periodic, 2×2 matrix-valued functions such that the following conditions hold:

(i) $T_k = j_k T_{k-1}$ for a positive integer j_k $(k = 1, 2, \ldots)$.

(ii) $|G_{k+1}(t) - G_k(t)| < \frac{1}{2^{k+1}}$ and $A_{k+1}(t) = A_k(t)$ for all $t \in [0, T_{k+1}]$ and all $k = 0, 1, \ldots$.

(iii) The fundamental matrix solution $\Phi_k(t)$ of $z' = G_k(t)z$ has the property that $\Phi_k(T_k)$ admits normalized eigenvectors $u^k_\pm$ lying in the open first quadrant. Moreover, u^k_- has polar angle less than that of u^k_+, and $u^{k+1}_\pm$ lie between $u^k_\pm$ in the natural sense $(k = 0, 1, \ldots)$.

(iv) The angle between u_+^k and u_-^k is less than $\frac{1}{k}$ $(k = 1, 2, \ldots)$.

(v) $\Phi_k(T_k)\, u_-^k = \beta_-^k u_-^k$ and $\Phi_k(T_k)\, u_+^k = \beta_+^k u_+^k$ where $\beta_-^k \leqslant 1 - \eta < 1 < \beta_+^k = \frac{1}{\beta_-^k}$ $(k = 0, 1, 2, \ldots)$.

Let $G(t) = \lim_{k\to\infty} G_k(t)$. By point (ii) the limit is uniform on $\mathbb{R}$, and hence $G(\cdot)$ is a Bohr almost periodic function. In fact it is the so-called limit periodic function because of point (i). Write

$$G(t) = \begin{pmatrix} A(t) & \Delta(t) \\ -Q(t) & -A(t) \end{pmatrix}.$$

It follows from point (ii) and the properties of Q_0, A_0 that $Q(t) \geqslant \frac{1}{2}$ for all $t \in \mathbb{R}$, and that $\lim_{T\to\infty} \frac{1}{T}\int_0^T A(s)\, ds < 0$. This last statement implies that the linear system $x' = A(t)x$ is of Hurwitz type.

Let $d = \inf_{t\in\mathbb{R}} \Delta(t)$, and define $D(t)$ by $D^2(t) = \Delta(t) - d + |d| + 1$. Then $\Delta(t) = D^2(t) - B^2(t)$ where $B(t) = \sqrt{|d| - d + 1}$ for all $t \in \mathbb{R}$. We see that the system $z' = G(t)z = \begin{pmatrix} A & D^2 - B^2 \\ -Q & -A \end{pmatrix} z$ has form (4) with parameter value $\gamma = 1$.

Now let Ξ be the closure of the set of translates of the function G in the space $\mathcal{G}_{2,2}$ introduced in Section 2. Then Ξ is invariant with respect to the Bebutov flow on $\mathcal{G}_{2,2}$. Since $G(\cdot)$ is Bohr almost periodic, the flow $(\Xi, \{\tau_t\})$ is minimal and admits a unique ergodic measure μ; moreover, $Supp\, \mu = \Xi$ [10].

Consider the family

$$z' = \begin{pmatrix} A_\xi & \gamma^{-2} D_\xi^2 - B_\xi^2 \\ -Q_\xi & -A_\xi \end{pmatrix} z, \tag{13_ξ}$$

where the notation is that of Section 2. Let $\alpha = \alpha(\gamma)$ be the rotation number of (13_ξ) with respect to the ergodic measure μ. As γ^{-2} increases, the rotation number $\alpha(\cdot)$ cannot increase. On the other hand, if $\gamma^{-2} = 0$, one can show that $\alpha = 0$. One can also use point (iii) of the construction of G to show that the rotation number of each approximating system $z' = G_k(t)z$ is zero; so by continuity properties of the rotation number [11] one has $\alpha(1) = 0$. It follows that $\alpha(\gamma) = 0$ for all $\gamma \geqslant 1$. Now, one can verify that the Atkinson Hypothesis 3.8 holds for family (13_ξ). Comparing these facts with the main result of [17], we see that Eq. (13_ξ) admit an exponential dichotomy over Ξ for all $\gamma > 1$.

Now, family (13_ξ) does not have an exponential dichotomy at $\gamma = 1$, for if it did then standard perturbation results for exponential dichotomies [7,26] would imply that point (iv) in the construction of G could not hold. It is worth noting that, by the Atkinson Hypothesis 3.8 and [17], we must have $\alpha(\gamma) \neq 0$ if $\gamma < 1$. This means that $\gamma_l = 1$.

Since $n = 1$, we can identify Λ with the projective space of lines through the origin in $\mathbb{R}^2$, which we in turn identify with the set of unit vectors $v \in \mathbb{R}^2$ whose polar angles θ_v lie in $[0, \pi)$. With this identification, one can use point (iii) in the construction of G together with arguments of [20] or [14] to show that the set Σ lies in the product of Ξ

with the open first quadrant. Further, one can show that, if $\gamma > 1$ and $\xi \in \Xi$, then the image of the dichotomy projection $P_{\gamma,\xi}$ is a line through the origin in $\mathbb{R}^2$ containing a unit vector v in the open first quadrant. All this shows that $\gamma^* = 1 = \gamma_l$.

Finally, using point (v) of the construction of G, one can show that there is a set $\Xi_0 \subset \Xi$ with $\mu(\Xi_0) = 1$, such that, if $\xi \in \Xi_0$, then the fiber $\Sigma \cap (\{\xi\} \times \mathbb{P}^1_{\mathbb{R}})$ contains at least two points λ_1 and λ_2. Then λ_1 and λ_2 are lines through the origin in $\mathbb{R}^2$ which pass through the open first quadrant, so $\lambda_i = span\begin{pmatrix} 1 \\ m_i(0) \end{pmatrix}$ for positive numbers $m_i(0)$, $i = 1, 2$. One can now check that the controls $u_i(t) = -B_\xi(t)^t m_i(t)x(t)$ stabilize (1_ξ) when $w = 0$ and satisfy (2_ξ) for all $w \in L^2$ and $x_0 \in \mathbb{R}$.

Remark 3.13. Let us set $\gamma = 1$ in the above example, so that γ equals the minimal attenuation value γ^*. Note that $x' = A_\xi(t)x$ is of Hurwitz type for each $\xi \in \Xi$. Therefore, the control system

$$x' = A_\xi(t)x + B_\xi(t)u + D_\xi(t)w$$

together with the functional

$$L_\xi(u, w) = \int_0^\infty \{\langle Q_\xi(t)x(t), x(t)\rangle + |u(t)|^2 - |w(t)|^2\}\, dt$$

defines a differential game for each $\xi \in \Xi$. In this game, w is chosen by the maximizing player and u is chosen by the minimizing player; see [5].

Now, for μ-a.a. $\xi \in \Xi$, there is a point $\lambda_0 \in \Sigma \cap (\{\xi\} \times \mathbb{P}^1_{\mathbb{R}})$ with the property that each non-zero solution $z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ of (13_ξ) with $z_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \lambda_0$ decays exponentially as $t \to \infty$. This is a consequence of property (v) of our construction and is a general characteristic of linear differential systems of Millionščikov type. It follows that, for each such ξ, the corresponding differential game has a value, namely

$$\begin{aligned} v &= \min_u \max_w L_\xi(u, w) = \max_w \min_u L_\xi(u, w) \\ &= \langle m_0 x_0, x_0\rangle = m_0 x_0^2, \end{aligned}$$

where $m_0 \in \mathbb{R}$ is defined by the condition $\lambda_0 = Span\begin{pmatrix} 1 \\ m_0 \end{pmatrix}$.

Of course this phenomenon cannot occur for time invariant or periodic H^∞-control systems.

References

[1] V.I. Arnold, On a characteristic class entering in a quantum condition, Funk. Anal. Appl. 1 (1967) 1–14.

[2] V.I. Arnold, The Sturm theorems and symplectic geometry, Funk. Anal. Appl. 19 (1985) 251–259.

[3] F.V. Atkinson, Discrete and Continuous Boundary Value Problems, Academic Press Inc., New York, 1964.
[4] M. Bebutov, On dynamical systems in the space of continuous functions, Boll. Moskov. Univ. Mat. (1941) 1–52.
[5] T. Basar, P. Bernard, H^{∞}-Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach, 2nd ed., Birkhäuser, Boston, 1995.
[6] W.A. Coppel, Disconjugacy, Lecture Notes in Mathematics, vol. 220, Springer, Heidelberg, 1971, .
[7] W.A. Coppel, Dichotomies in Stability Theory, Lecture Notes in Mathematics, vol. 629, Springer, Heidelberg, 1978.
[8] F. Colonius, W. Kliemann, The Dynamics of Control, Birkhäuser, Basel, 2000.
[9] P. Dorato (Ed.), Robust Control, IEEE Press, New York, 1987.
[10] R. Ellis, Lectures on Topological Dynamics, Benjamin, New York, 1969.
[11] R. Fabbri, R. Johnson, C. Núñez, The rotation number for non-autonomous linear Hamiltonian systems I: basic properties, Zeit. Angew. Math. Phys. 54 (2003) 484–502.
[12] R. Fabbri, R. Johnson, C. Núñez, The rotation number for non-autonomous linear Hamiltonian systems II: the Floquet coefficient, Zeit. Angew. Math. Phys. 54 (2003) 652–676.
[13] R. Fabbri, R. Johnson, C. Núñez, On the Yakubovich Frequency Theorem for linear non-autonomous control processes, Discrete Contin. Dynam. Systems 9 (2003) 677–704.
[14] R. Fabbri, R. Johnson, C. Núñez, Disconjugacy and the rotation number for linear, non-autonomous Hamiltonian systems, Ann. Mat. Pura Appl. (2005), to appear.
[15] J. Helton, M. James, Extending H^{∞} control to nonlinear systems, Frontiers in Applied Mathematics, SIAM, Philadelphia, 1999.
[16] R. Johnson, m-functions and Floquet exponents for linear differential systems, Ann. Mat. Pura Appl. 147 (1987) 211–248.
[17] R. Johnson, M. Nerurkar, Exponential dichotomy and rotation number for linear Hamiltonian systems, J. Differential Equations 108 (1994) 201–216.
[18] R. Johnson, M. Nerurkar, Stabilization and random linear regulator problem for random linear control processes, J. Math. Anal. Appl. 197 (1996) 608–629.
[19] R. Johnson, S. Novo, R. Obaya, Ergodic properties and Weyl M-functions for linear Hamiltonian systems, Proc. Roy. Soc. Edinburgh 130A (2000) 1045–1079.
[20] R. Johnson, S. Novo, R. Obaya, An ergodic and topological approach to disconjugate linear Hamiltonian systems, Illinois J. Math. 45 (2001) 1045–1079.
[21] V. Millionščikov, Proof of the existence of irregular systems of linear differential equations with almost periodic coefficients, Differentsial'nye Uraveniya 5 (1969) 1979–1983.
[22] V. Nemytskii, V. Stepanov, Qualitative Theory of Differential Equations, Princeton University Press, Princeton, NJ, 1960.
[23] S. Novo, C. Núñez, R. Obaya, Ergodic properties and rotation number for linear Hamiltonian systems, J. Differential Equations 148 (1998) 148–185.
[24] J. Selgrade, Isolated invariant sets for flows on vector bundles, Trans. Amer. Math. Soc. 203 (1975) 350–390.
[25] R.J. Sacker, G.R. Sell, Existence of dichotomies and invariant splittings for linear differential systems I, J. Differential Equations 15 (1974) 429–458.
[26] R.J. Sacker, G.R. Sell, A spectral theory for linear differential systems, J. Differential Equations 27 (1978) 320–358.
[27] R. Vinograd, A problem suggested by N.P. Erugin, Differentsial'nye Uraveniya 11 (1975) 632–638.
[28] V.A. Yakubovich, Linear-quadratic optimization problem and the frequency theorem for periodic systems I, Siberian Math. J. 27 (4) (1986) 614–630.
[29] V.A. Yakubovich, Linear-quadratic optimization problem and the frequency theorem for periodic systems II, Siberian Math. J. 31 (6) (1990) 1027–1039.
[30] G. Zames, Feedback and optimal sensitivity: model reference transformation, multiplicative seminorms, and approximate inverses, IEEE Trans. Autom. Control AC-26 (1981) 301–320.