JOURNAL OF DIFFERENTIAL EQUATIONS 51, 213–231 (1984)

A Stefan–Signorini Problem*

AVNER FRIEDMAN

Northwestern University, Evanston, Illinois 60201

AND

LI-SHANG JIANG

Peking University, Peking, China

Received April 21, 1982

INTRODUCTION

Consider a one-dimensional slab of ice occupying an interval $0 \leqslant x \leqslant L$. The initial temperature of the ice is $\leqslant 0$. Heat enters from the left at a rate $q(t)$. As the temperature at $x = 0$ increases to 0°C the ice begins to melt. We assume that the resulting water is immediately removed. The ice stops melting when the temperature at its left endpoint becomes strictly negative; due to the flow of heat $q(t)$ melting will resume after a while, the resulting water is again immediately removed, etc.

This physical problem was studied by Landau [4] and Lotkin [5], who obtained some numerical results.

In this paper we shall formulate the above model as a Stefan problem with Signorini boundary conditions at the moving boundary. We shall establish existence and uniqueness theorems and study regularity and some geometric features of the free boundary.

In Section we state an existence theorem; the proof is given in Sections 2 and 3. In Section 4 we prove a uniqueness theorem. In Section 5 we estimate the number of vertical segments of the free boundary. Finally, in Section 6 it is shown that the free boundary is in general not in $C^{1+\alpha}$ for $\alpha > \frac{1}{2}$; it is always in $C^{3/2}$.

* This work is partially supported by National Science Foundation Grant 740 6375 A01.

0022-0396/84 $3.00

505/51/2-5

1. The Stefan–Signorini Model

We denote the temperature of the ice by $-u(x, t)$ and the free boundary by $x = h(t)$. The *Stefan–Signorini problem* consists of finding functions $u(x, t)$, $h(t)$ satisfying:

$$-u_{xx} + u_t = 0 \qquad \text{if } h(t) < x < L, 0 < t < T, \tag{1.1}$$

$$u_x(L, t) = 0, \qquad \text{if} \quad 0 < t < T, \tag{1.2}$$

$$u(x, 0) = u_0(x) \qquad \text{if} \quad 0 < x < L, \tag{1.3}$$

$$\begin{gathered} u(h(t), t) \geqslant 0, \qquad -u_x(h(t), t) + g(t) \geqslant 0, \\ u(h(t), t)[-u_x(h(t), t) + g(t)] = 0 \qquad \text{for} \quad 0 < t < T, \end{gathered} \tag{1.4}$$

and

$$h'(t) = -u_x(h(t), t) + g(t) \qquad \text{for} \quad 0 < t < T \tag{1.5}$$

where $h(0) = 0, 0 < h(t) < L$; further, either

$$T = \infty \qquad \text{or} \qquad h(T-0) = L. \tag{1.6}$$

We shall always assume that

$$\begin{gathered} u_0(x) \geqslant 0, \qquad g(t) \geqslant 0, \qquad u_0'(L) = 0, \\ u_0 \in C^2[0, L], \qquad g \in C^1[0, T], \qquad \text{and} \qquad -u_0'(0) + g(0) \geqslant 0, \\ u_0(0)(-u_0'(0) + g(0)) = 0. \end{gathered} \tag{1.7}$$

THEOREM 1.1. *There exists a solution of the Stefan–Signorini problem.*

The solution is classical in the sense that $h \in C^1[0, T]$ and u_x is continuous in $h(t) \leqslant x \leqslant L$, $0 \leqslant t < T$.

By integrating the heat equation (1.1) over $h(t) < x < L$, $0 < t < \tilde{T}$, and letting $\tilde{T} \to T$ we obtain the conservation of energy law:

$$L + \int_0^L u_0(x)\, dx = \int_0^T g(t)\, dt. \tag{1.8}$$

Consequently, if

$$\int_0^\infty g(t)\, dt = \infty$$

then T must be finite.

In Section 2 we study the Signorini problem (1.1)–(1.4) for a fixed function $h(t)$; this will be used, in Section 3, to prove Theorem 1.1.

2. A Signorini Problem

Consider the Signorini problem (1.1)–(1.4) for a fixed function $h(t)$ satisfying:

$$h \in C^1[0, T], \qquad h(0) = 0, \; h'(t) \geqslant 0, \quad h(T) < L. \tag{2.1}$$

Lemma 2.1. *The problem* (1.1)–(1.4) *has a unique classical solution* u; *further*

$$0 \leqslant u_x(h(t), t) + g \leqslant C_0, \qquad |u_x| \leqslant C_0, \tag{2.2}$$

$$|u_{xx}| + |u_t| \leqslant C, \tag{2.3}$$

$$|u_x(x, t_1) - u_x(x, t_2)| \leqslant C\,|t_1 - t_2|^{1/2} \tag{2.4}$$

where C_0, C *are constants*: C_0 *depends only on* $\sup g$ *and the* C^1 *norm of* u_0, *and* C *depends on the* C^1 *norm of* g, *the* C^2 *norm of* u_0, *the* C^1 *norm of* h *and any positive lower bound on* $L - h(T)$.

Proof. Set $l = u_0(0)$, $l_1 = -u_0'(0) + g(0)$. Then $l \geqslant 0$, $l_1 \geqslant 0$, $ll_1 = 0$. We can choose a family of C^2 functions $\beta_\varepsilon(t)$ in $C^2(-\infty, \infty)$ $(0 < \varepsilon < 1)$ satisfying:

$$\beta_\varepsilon \leqslant 0, \qquad \beta_\varepsilon' \geqslant 0, \tag{2.5}$$

$$\beta_\varepsilon(0) = -l_1 \quad \text{if} \quad l = 0, \qquad \beta_\varepsilon(l) = 0 \quad \text{if } l > 0, \tag{2.6}$$

and

$$\beta_\varepsilon(t) = \begin{cases} 0 & \text{if} \quad t > (1 + l_1)\,\varepsilon \\ \dfrac{t + \varepsilon}{\varepsilon} & \text{if} \quad t < -l_2\,\varepsilon \end{cases} \tag{2.7}$$

for some positive constant l_2 depending on l, l_1.

Consider the penalized problem

$$-u_{xx} + u_t = 0 \qquad \text{for } h(t) < x < L,\; 0 < t < T, \tag{2.8}$$

$$u_x(L, t) = 0 \qquad \text{for} \quad 0 < t < T, \tag{2.9}$$

$$u(x, 0) = u_0(x) \qquad \text{for} \quad 0 < x < L, \tag{2.10}$$

$$-u_x(h(t), t) + \beta_\varepsilon(u(h(t), t)) + g(t) = 0 \quad \text{for } 0 < t < T. \tag{2.11}$$

By well-known results [1] this system has a unique classical solution u_ε; notice that the consistency conditions of the initial and boundary data are satisfied at $(0, 0)$ (by (2.6)) and at $(L, 0)$ (since $u_0'(L) = 0$).

We shall derive the estimates

$$-C_0\varepsilon \leqslant u_\varepsilon(x, t) \leqslant C_0, \tag{2.12}$$

$$0 \leqslant -u_{\varepsilon,x}(h(t), t) + g(t) \leqslant C_0, \tag{2.13}$$

$$|u_{\varepsilon,x}(x, t)| \leqslant C_0, \tag{2.14}$$

$$|u_{\varepsilon,t}(x, t)| + |u_{\varepsilon,xx}(x, t)| \leqslant C, \tag{2.15}$$

$$|u_{\varepsilon,x}(x, t_1) - u_{\varepsilon,x}(x, t_2)| \leqslant C\,|t_1 - t_2|^{1/2} \tag{2.16}$$

where C_0, C are as in the statement of Lemma 2.1.

If u_ε attains negative minimum smaller than $-l_2\varepsilon$ at some point $(h(t_0), t_0)$ $(t_0 \in (0, T])$ then $u_{\varepsilon,x}(h(t_0), t_0) > 0$; hence

$$\beta_\varepsilon(u_\varepsilon(h(t_0)), t_0) + g(t_0) > 0.$$

From (2.7) we then obtain

$$\frac{u_\varepsilon(h(t_0), t_0) + \varepsilon}{\varepsilon} + g(t_0) \geqslant 0,$$

that is, $u_\varepsilon(h(t_0), t_0) \geqslant -C_0\varepsilon$. Since u_ε cannot take minimum on $\{x = L\}$ and since $u_0 \geqslant 0$, the first estimate of (2.12) follows.

Next, if u_ε takes a positive maximum larger than $\max u_0$ at some point $(h(t_0), t_0)$ $(t_0 \in (0, T])$ then $u_{\varepsilon,x} < 0$ at that point, so that

$$\beta_\varepsilon(u_\varepsilon(h(t_0), t_0)) + g(t_0) < 0.$$

Since $g(t_0) \geqslant 0$, we deduce that $\beta_\varepsilon(u_\varepsilon(h(t_0), t_0)) < 0$, so that by (2.7)

$$u_\varepsilon(h(t_0), t_0) \leqslant (1 + l_1)\,\varepsilon.$$

This completes the proof of the second estimate in (2.12).

Next (2.13) follows from (2.12), (2.11), and (2.14) is a consequence of the maximum principle applied to $u_{\varepsilon,x}$.

In order to derive (2.15), (2.16) we may assume that $h \in C^2[0, T]$, otherwise we approximate h by C^2 functions h_j and, after deriving (2.15), (2.16) for the corresponding $u_\varepsilon = u_{\varepsilon,j}$, let $j \to \infty$.

Consider the function $v = u_{\varepsilon,t}$. It satisfies:

$$-v_{xx} + v_t = 0,\; v_x(L, t) = 0,\; v(x, 0) = u_0''(x),$$

and, by differentiating (2.11) with respect to t,

$$-v_x + [\beta_\varepsilon'(u_\varepsilon(h(t), t)) - h'(t)]\, v + \beta_\varepsilon'(u_\varepsilon(h(t), t))\, h'(t)\, u_{\varepsilon,x}(h(t), t) + g'(t) = 0 \qquad \text{for} \quad 0 < t < T.$$

Consider the function w defined by

$$v = we^{\alpha(x-(L-\delta))^2+\gamma t}$$

where

$$\delta = \tfrac{1}{2}(L - h(T)).$$

We compute

$$-w_{xx} + w_t - 4\alpha[x - (L - \delta)]\, w_x + \{\gamma - 2\alpha[1 + 2\alpha(x - (L - \delta))^2]\}\, w = 0,$$

$$w_x(L, t) + 2\alpha\delta w(L, t) = 0,$$

$$-w_x + [\beta_\varepsilon'(\cdots) - h'(t) + 2\alpha(L - h(t) - \delta)]\, w = -[g'(t) + \beta_\varepsilon'(\cdots)\, h'(t)\, u_{\varepsilon,x}]\, e^{-\alpha(x-(L-\delta))^2-\gamma t} \qquad \text{on} \quad x = h(t), \tag{2.17}$$

$$w(x, 0) = u_0''(x)\, e^{-\alpha(x-(L-\delta))^2}.$$

Taking

$$\alpha = (1/2\delta)(1 + \max|h'|),\ \gamma = 2\alpha(1 + 2\alpha L^2),$$

we find that

$$\{\gamma - 2\alpha[1 + 2\alpha(x - (L - \delta))^2]\} \geqslant 0, \tag{2.18}$$

$$[\beta_\varepsilon'(\cdots) - h'(t) + 2\alpha(L - h(t) - \delta)] \geqslant 1 + \beta_\varepsilon'(\cdots). \tag{2.19}$$

We can modify the function $u_0(x)$ near $x = 0$ and $x = L$ into functions $u_{0j}(x)$ with $u_{0j}(0) > 0$ such that for the corresponding $u_{\varepsilon,t}$ the initial and boundary conditions satisfy the consistency conditions at $(0, 0)$, $(L, 0)$ and

$$u_{0j} \to u_0 \qquad \text{uniformly in } 0 \leqslant x \leqslant L,$$

$$|u_{0j}|_{C^2[0,L]} \leqslant 1 + |u_0|_{C^2[0,L]}.$$

But then the corresponding $w = w_j$ is continuous at $(0, 0)$, $(0, L)$ (we use here the assumption that $h \in C^2[0, T]$). Recalling also (2.18), we see that the maximum principle can be applied to w_j. Since $w_j \to w$ as $j \to \infty$, we may assume that the maximum principle applies also to w.

Notice that $\pm w$ cannot take maximum on $x = L$. If $\pm w$ takes maximum on $x = h(t)$, then $\pm w_x \leqslant 0$ and then, by (2.17), (2.19) and (2.13),

$$\max |w| \leqslant \frac{\max |g'| + \beta'_\varepsilon(\cdots)\, C}{1 + \beta'_\varepsilon(\cdots)} \leqslant C.$$

Finally, if $\pm w$ takes maximum on $t = 0$ then $\max |w| \leqslant \max |u''_0|$. Recalling the definition of w we deduce that $|u_{\varepsilon,t}| \leqslant C$; hence also $|u_{\varepsilon,xx}| \leqslant C$, and (2.15) is proved.

To prove (2.16) let $0 < t_1 < t_2 < T$, $\tau = (t_2 - t_1)^{1/2}$. Then

$$\begin{aligned}\int_x^{x+\tau} [u_{\varepsilon,x}(y, t_1) - u_{\varepsilon,x}(y, t_2)]\, dy &= \int_x^{x+\tau} dy \int_{t_1}^{t_2} u_{\varepsilon,xt}(y, t)\, dt \\ &= \int_{t_1}^{t_2} dt \int_x^{x+\tau} u_{\varepsilon,xxx}(y, t)\, dy = \int_{t_1}^{t_2} [u_{\varepsilon,xx}(x+\tau, t) - u_{\varepsilon,xx}(x, t)]\, dt \\ &= 0(1)(t_2 - t_1).\end{aligned}$$

Hence there exists a point $\xi \in (x, x+\tau)$ such that

$$|u_{\varepsilon,x}(\xi, t_1) - u_{\varepsilon,x}(\xi, t_2)| \leqslant C\, |t_1 - t_2|^{1/2}.$$

We can now estimate

$$\begin{aligned}&|u_{\varepsilon,x}(x, t_1) - u_{\varepsilon,x}(x, t_2)| \\ &\quad \leqslant |u_{\varepsilon,x}(x, t_1) - u_{\varepsilon,x}(\xi, t_1)| + |u_{\varepsilon,x}(x, t_2) - u_{\varepsilon,x}(\xi, t_2)| + C\, |t_1 - t_2|^{1/2} \\ &\quad \leqslant (C + \sup |u_{\varepsilon,xx}|)\, |t_1 - t_2|^{1/2},\end{aligned}$$

which proves (2.16).

In view of the estimates (2.12)–(2.16) there is a sequence $\varepsilon \to 0$ for which

$$u_\varepsilon \to u, \qquad u_{\varepsilon,x} \to u_x \qquad \text{uniformly,}$$

$$u_{\varepsilon,xx} \to u_{xx} \qquad \text{in the } L^\infty\text{-weak star topology,}$$

and $u \geqslant 0$ (by (2.12)). All the assertions of the theorem regarding the solution of (1.1)–(1.4) are satisfied; the only point that requires an argument is the last condition in (1.4), namely, if $u(h(t_0), t_0) > 0$ then

$$-u_x(h(t_0), t_0) + g(t_0) = 0. \tag{2.20}$$

But since $u_\varepsilon(h(t_0), t_0) \geqslant c > 0$ for some $c > 0$ and all ε small enough, we have $\beta_\varepsilon(u_\varepsilon(h(t_0), t_0)) = 0$ and thus, by (2.11),

$$-u_{\varepsilon,x}(h(t_0), t_0) + g(t_0) = 0,$$

which gives (2.20).

To prove uniqueness we rewrite (1.1)–(1.4) as a variational inequality:

$$\iint_\Omega u_x(v-u)_x + \int_0^t (v-u)(h(t),t))\, g(t)\, dt$$

$$+\iint_\Omega u_t(v-u) \geqslant 0 \qquad \forall v \in H^1(\Omega) \cap C^\circ(\bar\Omega), \quad v(h(t),t) \geqslant 0 \quad (2.21)$$

where $\Omega = \{(x,t);\ h(t) < x < L,\ 0 < t < T\}$. If u_1 is another solution then take $v = u_1$ in (2.21) and take $v = u$ in the corresponding variational inequality for u_1. Adding the inequalities we easily find that

$$\iint_\Omega ((u-u_1)_x)^2 + \frac{1}{2}\int_0^t (u-u_1)^2(h(t),t)\, h'(t)\, dt \leqslant 0,$$

which implies that $u \equiv u_1$.

3. Proof of Theorem 1.1.

Denote by Σ_A the class of functions $h(t)$ in $C^1[0,T]$ satisfying:

$$h(0) = 0, \qquad 0 \leqslant h'(t) \leqslant A$$

where A is a fixed positive number and T is small enough so that

$$h(T) \leqslant AT < L.$$

If u is a solution of (1.1)–(1.4) then by integrating the heat equation (1.1) we find that

$$\int_0^t [g(t) - u_x(h(t),t)]\, dt$$

$$= \int_0^t g(t)\, dt + \int_{h(t)}^L u(x,t)\, dx - \int_0^L u_0(x)\, dx + \int_0^t u(h(t),t)\, h'(t)\, dt. \qquad (3.1)$$

For any $h \in \Sigma_A$ we solve the Signorini problem (1.1)–(1.4) and set $Sh = \tilde h$, where

$$\tilde h(t) = \int_0^t g(t) + \int_{h(t)}^L u(x,t)\, dx - \int_0^L u_0(x)\, dx + \int_0^t u(h(t),t)\, h'(t)\, dt. \quad (3.2)$$

If we can show that S has a fixed point in Σ_A then from (3.1), (3.2) it follows that

$$h(t) = \int_0^t (g(t) - u_x(h(t), t))\,dt,$$

i.e., (1.5) is satisfied. Thus u, h will then form a solution of (1.1)–(1.5).

We shall consider Σ_A as a compact subset of $C[0, T]$.

To show that S maps Σ_A into Σ_A we observe that $\tilde{h}$ is again in $C^1[0, T]$ and $\tilde{h}(0) = 0$. Further,

$$\begin{aligned}\tilde{h}'(t) &= g(t) + \int_{h(t)}^L u_{xx}(x, t)\,dx \\ &= g(t) - u_x(h(t), t)) \geqslant 0.\end{aligned}$$

Finally, by (2.2),

$$\tilde{h}'(t) \leqslant C_0 \tag{3.3}$$

where C_0 is a positive constant independent of h. Hence if we take $A = C_0$ then S maps Σ_A into itself.

We next show that S is a continuous mapping. We can write

$$\begin{aligned}\tilde{h}_1(t) - \tilde{h}_2(t) = &\int_{h_1(t)}^L u_1(x, t)\,dx - \int_{h_2(t)}^L u_2(x, t)\,dt \\ &+ \int_0^t [u_1(h_1(t), t)\,h_1'(t) - u_2(h_2(t), t)\,h_2'(t)]\,dt.\end{aligned}$$

Setting

$$\tilde{u}_i(x, t) = \begin{cases} u_i(x, t) & \text{if } h_i(t) < x < L, \\ u_i(h_i(t), t) & \text{if } x < h_i(t), \end{cases}$$

$$h_0(t) = \min\{h_1(t), h_2(t)\},$$

and writing

$$\begin{aligned}&\int_0^t u_i(h_i(t), t)\,h_i'(t) \\ &\quad = \int_0^t u_i(h_i(t), t)\,h_0'(t)\,dt + \int_0^t u_i(h_i(t), t)\,\frac{d}{dt}(h_i(t) - h_0(t))\,dt \\ &\quad = -\int_0^t \left[\frac{d}{dt} u_i(h_i(t), t)\right][h_i(t) - h_0(t)]\,dt \\ &\qquad + u_i(h_i(t), t)(h_i(t) - h_0(t)) \\ &\qquad + \int_0^t u_i(h_i(t), t)\,h_0'(t)\,dt,\end{aligned}$$

we find that

$$\tilde{h}_1(t) - \tilde{h}_2(t) = \int_{h_0(t)}^{L} (\tilde{u}_1(x,t) - \tilde{u}_2(x,t))\,dx$$
$$- \int_0^t (h_1(t) - h_0(t))[u_{1,x}h_1' + u_{1,t}]_{x=h_1(t)}\,dt$$
$$+ \int_0^t [u_1(h_1(t),t) - u_2(h_2(t),t)]\,h_0'(t)\,dt$$
$$+ \int_0^t (h_2(t) - h_0(t))[u_{2,x}h_2' + u_{2,t}]_{x=h_2(t)}\,dt. \tag{3.4}$$

Set

$$\Omega_i = \{(x,t);\, h_i(t) < x < L,\, 0 < t < T\}, \qquad \Omega = \Omega_1 \cup \Omega_2.$$

From the variational inequality for u_i in Ω_i we get

$$\iint_\Omega u_{i,x}(v - \tilde{u}_i)_x + \int_0^t [v - \tilde{u}_i]_{x=h_i(t)}\,g(t)\,dt$$
$$+ \iint_\Omega \tilde{u}_{i,t}(v - \tilde{u}_i) - \iint_{\Omega\setminus\Omega_i} \tilde{u}_{i,t}(v - \tilde{u}_i) \geqslant 0.$$

Taking $v = \tilde{u}_2$ for $i = 1$ and $v = \tilde{u}_1$ for $i = 2$ and adding, we obtain

$$0 \geqslant \iint_\Omega [(\tilde{u}_1 - \tilde{u}_2)_x]^2 + \int_0^t \{[\tilde{u}_1 - \tilde{u}_2]_{x=h_1(t)} - [\tilde{u}_1 - \tilde{u}_2]_{x=h_2(t)}\}\,g(t)\,dt$$
$$+ \iint_\Omega (\tilde{u}_1 - \tilde{u}_2)\frac{\partial}{\partial t}(\tilde{u}_1 - \tilde{u}_2) - \iint_{\Omega\setminus\Omega_1} (\tilde{u}_1 - \tilde{u}_2)\,\tilde{u}_{1,t}$$
$$+ \iint_{\Omega\setminus\Omega_2} (\tilde{u}_1 - \tilde{u}_2)\,\tilde{u}_{2,t} \equiv \sum_{i=1}^{5} J_i. \tag{3.5}$$

Clearly

$$J_3 = \frac{1}{2}\int_0^t \frac{d}{dt}\left[\int_{h_0(t)}^{L} (\tilde{u}_1 - \tilde{u}_2)^2\,dx\right]dt + \frac{1}{2}\int_0^t [(\tilde{u}_1 - \tilde{u}_2)^2]_{x=h_0(t)}\,h_0'(t)\,dt$$
$$= \frac{1}{2}\int_{h_0(t)}^{L} (\tilde{u}_1 - \tilde{u}_2)^2\,dx + \frac{1}{2}\int_0^t [(\tilde{u}_1 - \tilde{u}_2)^2]_{x=h_0(t)}\,h_0'(t)\,dt. \tag{3.6}$$

Next

$$|J_2| \leqslant \int_0^t g(t)\,dt \left| \int_{h_1(t)}^{h_2(t)} \frac{\partial}{\partial x}(\tilde{u}_1 - \tilde{u}_2)\,dx \right| \leqslant C \max|h_1 - h_2|, \qquad \text{by (2.2).} \quad (3.7)$$

Since in $\Omega \backslash \Omega_1$

$$u_{1,t} = \frac{d}{dt} u_1(h_1(t), t)) = u_{1,x} h_1' + u_{1,t},$$

we also have, by (2.3),

$$|J_4| \leqslant C \text{ meas } (\Omega \backslash \Omega_1) \leqslant C \max|h_1 - h_2|. \quad (3.8)$$

Similarly

$$|J_5| \leqslant C \max|h_1 - h_2|. \quad (3.9)$$

Substituting the estimates (3.6)–(3.9) into (3.5), we obtain

$$\iint_\Omega ((\tilde{u}_1 - \tilde{u}_2)_x)^2 + \frac{1}{2}\int_{h_0(t)}^L (\tilde{u}_1 - \tilde{u}_2)^2(x,t)\,dx$$
$$+ \frac{1}{2}\int_0^t [(\tilde{u}_1 - \tilde{u}_2)^2]_{x=h_0(t)} h_0'(t)\,dt \leqslant C\max|h_1 - h_2|. \quad (3.10)$$

This estimate will be needed for proving the continuity of S.

Returning to (3.4) we can write

$$|\tilde{h}_1(t) - \tilde{h}_2(t)| \leqslant C \left\{ \int_{h_0(t)}^L |\tilde{u}_1(x,t) - \tilde{u}_2(x,t)|^2\,dx \right\}^{1/2} + C\max|h_1 - h_2|$$
$$+ \int_0^t |\tilde{u}_1(h_1(t), t) - \tilde{u}_1(h_0(t), t)|\, h_0'(t)\,dt$$
$$+ \int_0^t |\tilde{u}_2(h_2(t), t) - \tilde{u}_2(h_0(t), t)|\, h_0'(t)\,dt$$
$$+ \int_0^t |\tilde{u}_1(h_0(t), t) - \tilde{u}_2(h_0(t), t)|\, h_0'(t)\,dt$$
$$= \sum_{i=1}^5 \tilde{J}_i.$$

Clearly

$$|\tilde{J}_3| \leqslant \int_0^t |\tilde{u}_{1,x}|\,|h_1 - h_2|\, h_0'\,dt \leqslant C\max|h_1 - h_2|.$$

Similarly

$$|\tilde{J}_4| \leqslant C \max |h_1 - h_2|.$$

Next, by (3.10),

$$\tilde{J}_1 \leqslant C\{\max |h_1 - h_2|\}^{1/2}$$

and

$$\tilde{J}_5 \leqslant C \left\{\int_0^t |\tilde{u}_1(h_0(t), t) - \tilde{u}_2(h_0(t), t)|^2 \, h_0'(t)\, dt\right\}^{1/2} \leqslant C\{\max |h_1 - h_2|\}^{1/2}.$$

Consequently

$$|\tilde{h}_1(t) - \tilde{h}_2(t)| \leqslant C\{\max |h_1 - h_2|\}^{1/2};$$

that is,

$$\|Sh_1 - Sh_2\| \leqslant C \|h_1 - h_2\|^{1/2} \tag{3.11}$$

with the $C[0, T]$ norm.

We have thus proved that S maps a convex compact subset Σ_A of $C[0, T]$ into itself and it is continuous. By Schauder's fixed point theorem it follows that S has a fixed point in Σ_A. This gives a solution of (1.1)–(1.5) in $0 < t < T$.

By (2.3)

$$|u_{xx}(x, T)| \leqslant C \tag{3.12}$$

where C depends only on $\sup |u_0''|$, the C^1 norm of g, $\sup |h'|$ and a lower bound on $L - h(T)$. Since $|h'(t)| \leqslant C_0$ (cf. (3.3)), C then depends only on a lower bound on $L - h(T)$ and on the data u_0, g.

We can now proceed step by step to construct a solution in $T \leqslant t \leqslant T_1$, $T_1 \leqslant t \leqslant T_2$, etc. By the previous observation, the constants in the estimates (3.3) at T, T_1, T_2, etc., all remain uniformly bounded as long as $L - h(T_j) > \delta$, and $\inf(T_j - T_{j-1})$ remains uniformly positive; here δ is any given positive number. It follows that a solution of (1.1)–(1.5) can be constructed for all $t < T$, where either $T = \infty$ and $h(t) < L$ for all $t < \infty$, or $T < \infty$ and $h(t) \to L$ as $t \to T$. This establishes (1.6), and the proof of Theorem 1.1 is complete.

Remark 3.1. From (2.4), (1.5) it follows that

$$h(t) \in C^{3/2}[0, T). \tag{3.13}$$

Hence also u_{xx}, u_t belong to $C^{\alpha, \alpha/2}_{x,t}$ for any $\alpha < \frac{1}{2}$ (up to the boundary $x = h(t)$).

4. Uniqueness

Definition. Let u, h be a solution of (1.1)–(1.5). Suppose there exists a sequence $0 = t_0 \leqslant t_1 < t_2 < \cdots < t_n \to T$ such that

$$u(h(t), t) > 0 \qquad \text{in} \quad t_{2i-1} < t < t_{2i}, \tag{4.1}$$

$$u(h(t), t) = 0 \qquad \text{in} \quad t_{2i} < t < t_{2i+1}; \tag{4.2}$$

then we say that h has a *discrete set* of *switchings*, $\{t_j\}$.

Notice that (4.1) implies

$$h'(t) = 0 \qquad \text{if} \quad t_{2i-1} \leqslant t \leqslant t_{2i}$$

whereas (4.2) together with (1.5) form the Stefan conditions on the free boundary. The Stefan conditions imply that $h(t)$ is C^∞ and, if $g(t)$ is analytic, $h(t)$ is also analytic [2].

Theorem 4.1. *There exists at most one solution of the Stefan–Signorini problem having a discrete set of switchings.*

Existence of such a solution will be proved in Section 5.

Proof. Suppose u_1, h_1 and u_2, h_2 are two such solutions. Then, for a small enough $\varepsilon > 0$, only the following cases may occur:

$$h_1(t) = h_2(t) = 0 \qquad \text{if} \quad 0 < t < \varepsilon, \tag{4.3}$$

$$u_i(h_i(t), t) = 0 \qquad \text{if} \quad 0 < t < \varepsilon \ (i = 1, 2), \tag{4.4}$$

$$h_1(t) = 0, \qquad u_2(h_2(t), t) = 0 \qquad \text{if} \quad 0 < t < \varepsilon, \tag{4.5}$$

$$h_2(t) = 0, \qquad u_1(h_1(t), t) = 0 \qquad \text{if} \quad 0 < t < \varepsilon. \tag{4.6}$$

In case (4.3) u_1 and u_2 are solutions of the same second initial-boundary value problem and, therefore, $u_1 \equiv u_2$. In case (4.4), $u_1 = u_2$ by uniqueness for the Stefan problem.

In case (4.5), $h_2(t) \geqslant 0$ and by the maximum $u_1 \geqslant u_2$. If we integrate the heat equations for u_1 and u_2 and compare the results, we obtain

$$-h_2(t) = \int_0^L u_1(x, t)\, dx - \int_{h_2(t)}^L u_2(x, t)\, dx.$$

Since the right-hand side is $\geqslant 0$, we conclude that $h_2(t) \equiv 0$. The case (4.6) is treated similarly.

5. Finite Number of Switchings

In this section we assume:

there exists a finite number m of disjoint closed intervals J_l such that

$$\begin{aligned} &-u_0'(x)+g(0)\leqslant 0, \qquad -u_0'(x)+g(0)\not\equiv 0 \quad \text{in } J_l, \quad \text{and} \\ &-u_0'(x)+g(0)>0 \qquad \text{in } [0,L]\setminus\bigcup_l J_l, \end{aligned} \tag{5.1}$$

$$g(t)>0, \qquad g'(t)\geqslant 0 \qquad \text{if} \quad 0<t<T. \tag{5.2}$$

Theorem 5.1. *If (5.1), (5.2) hold then there exists a solution u, h for which there exist at most $m+1$ disjoint closed intervals $\hat{J}_l$ in $[0,T)$ such that*

$$\begin{aligned} h'(t)&\equiv 0 \qquad \textit{in each } \hat{J}_l, \\ h'(t)&>0 \qquad \textit{in } [0,T)\setminus\hat{J}_l. \end{aligned}$$

We express the assertion by saying that the boundary has at most $m+2$ switchings.

Lemma 5.2. *Let w be a solution of*

$$\begin{aligned} -w_{xx}+w_t&=0 \qquad \textit{in } \Omega=\{(x,t);\, h(t)<x<L,\, 0<t<T\}, \\ w(h(t),t)&\geqslant 0 \qquad \textit{for} \quad 0<t<T, \\ w(L,t)&=0 \qquad \textit{for} \quad 0<t<T, \\ w(x,0)&=\phi(x) \qquad \textit{for} \quad 0<x<L,\ \phi \textit{ continuous.} \end{aligned}$$

Suppose that there exist m disjoint closed intervals J_l in $[h(0),L]$ such that

$$\begin{aligned} \phi(x)\leqslant 0, \qquad \phi(x)&\not\equiv 0 \qquad \textit{in each } J_l, \\ \phi(x)&>0 \qquad \textit{in } [h(0),L]\setminus\cup J_l. \end{aligned}$$

Then, for any $\sigma\in(0,T)$, there exist at most m disjoint open intervals in $(h(\sigma),L)$ where $w(x,\sigma)<0$.

Proof. Suppose there are at least p such intervals $\tilde{J}_l$ for $w(x,\sigma)$. Denote by S_l the component of $\{w<0\}$ in $\Omega\cap\{t\leqslant\sigma\}$ which contains $\tilde{J}_l$. Then

$$\bar{S}_k\cap\bar{S}_l=\varnothing \qquad \text{if} \quad k\neq l. \tag{5.3}$$

Indeed, otherwise, by the maximum principle,

$$w(x, \sigma) < 0 \qquad \text{on an } x\text{-interval containing both } \tilde{J}_l \text{ and } \tilde{J}_k. \tag{5.4}$$

Since $w = 0$ on $x = L$, $w \geqslant 0$ on $x = h(t)$, w cannot take its negative minimum in $\bar{S}_l$ on $x = L$ or on $x = h(t)$. Finally, if $\partial S_l \subset \{t > 0\}$ then $w \equiv 0$ in S_l, a contradiction. We conclude that $\bar{S}_l$ must intersect $\{t = 0\}$ at points where $w(x, 0) < 0$, i.e., $\bar{S}_l$ intersects some interval $J_{l'}$. If $\bar{S}_k$ with $k \neq l$ also intersects $J_{l'}$ then, again by the maximum principle, we deduce that (5.4) holds; a contradiction. We conclude that $p \leqslant m$.

Proof of Theorem 5.1. Let v be a solution of

$$\begin{aligned}
v_t - v_{xx} &= 0 && \text{if} \quad h(t) < x < L, 0 < t < T,\\
v(x, 0) &= g(0) && \text{if} \quad 0 < x < L,\\
v(h(t), t) &= g(t) && \text{if} \quad 0 < t < T,\\
v_x(L, t) &= 0 && \text{if} \quad 0 < t < T,
\end{aligned}$$

and set $w = -u_x + v$.

For any $\sigma \in (0, T)$ denote by $k(\sigma)$ the number of disjoint intervals I_j where $w(x, \sigma) < 0$ and let

$$s_1(\sigma) < s_2(\sigma) < \cdots < s_l(\sigma)$$

be their endpoints in $h(\sigma) < x < L$.

The proof of Lemma 5.2 shows that $k(\sigma)$ is monotone decreasing in σ.

Notice that any curve $x = s_j(t)$ cannot have discontinuity of the second kind. Indeed, if $\lim_{t \to \sigma} s_j(t)$ does not exist then $w(x, \sigma) = 0$ on some x-interval, so that $u_x(x, \sigma) - v(x, \sigma) \equiv 0$ (by analyticity of $u_x(x, \sigma)$, $v(x, \sigma)$ in x). Taking $x = L$ we get $v(L, \sigma) = 0$. But since $g > 0$, $v \geqslant 0$ and consequently v takes minimum at (L, σ). This implies (by the maximum principle) that $v_x(L, \sigma) < 0$, a contradiction.

If for some j, $w(x, t) < 0$ for $s_j(t) < x < s_{j+1}(t)$ if $t < \tau$ and $s_j(\tau - 0) = s_{j+1}(\tau - 0)$ then $k(\tau) < k(\tau - 0)$.

Consider now a maximal interval

$$\begin{gathered}
I_j \colon s_j(\sigma) < x < s_{j+1}(\sigma)\\
\text{where } w(x, \sigma) < 0, \text{ and } s_j(\sigma) > h(\sigma),\ s_{j+1}(\sigma) < L.
\end{gathered} \tag{5.5}$$

Since, as mentioned above, $w(x, \sigma)$ is analytic in x, there exist intervals

$$s_j(\sigma) - \delta \leqslant x < s_j(\sigma), \qquad s_{j+1}(\sigma) < x \leqslant s_{j+1}(\sigma) + \delta \qquad (\delta > 0)$$

where $w(x, \sigma) > 0$. By continuity we deduce that

$$w(s_j(\sigma) - \delta, t) > 0, \qquad w(s_{j+1}(\sigma) + \delta, t) > 0 \qquad \text{if } \sigma < t < \sigma + \varepsilon$$

for some $\varepsilon > 0$. If (5.5) holds for any maximal interval where $w(x, \sigma) < 0$, then we conclude that $k(t) = k(\sigma - 0)$ for $\sigma \leqslant t \leqslant \sigma + \varepsilon$.

If for a maximal interval I_j where $w(x, \sigma) < 0$ we have $s_j(\sigma) = h(\sigma)$, no new negative interval for w can start to the left of I_j for $t > \sigma$; this indeed follows from the proof of Lemma 5.2. A similar assertion holds in case $s_{j+1}(\sigma) = L$.

From the above analysis it follows that there exist at most $2m + 1$ domains Ω_i bounded by some of the curves $x = s_j(t)$, a segment on the x-axis and an arc Γ_i on the free boundary such that in each Ω_i either $w > 0$ or $w < 0$, and

$$\bigcup \Gamma_i \text{ coincides with the entire free boundary.}$$

Set

$$\Gamma_i = \{(h(t), t);\ \sigma_i \leqslant t \leqslant \tau_i\}.$$

Consider first a domain Ω_i where $w < 0$. Since $w(h(t), t) \geqslant 0$, we must have

$$w(h(t), t) = 0 \qquad \text{if} \quad \sigma_i \leqslant t \leqslant \tau_i,$$

and (1.5) gives $h'(t) \equiv 0$.

Consider next a domain Ω_i where $w > 0$. We claim: There exists a point $t_0 \in [\sigma_i, \tau_i]$ such that

$$\begin{aligned} h(t) &= 0 \qquad \text{if} \quad \sigma_i < t < t_0, \\ h'(t) &> 0 \qquad \text{if} \quad t_0 < t < \tau_i. \end{aligned} \tag{5.6}$$

To prove this suppose that

$$h'(t) = 0 \qquad \text{in some interval } t_1 \leqslant t \leqslant t_2$$

where $\sigma_i < t_1 < t_2 \leqslant \tau_i$, and

$$u(h(t_1), t_1) = 0.$$

Then

$$\begin{aligned} h(t) &= h(t_1) \qquad &\text{if} \quad t_1 \leqslant t \leqslant t_2. \\ w(h(t), t) &= h'(t) = 0 \qquad &\text{if} \quad t_1 \leqslant t \leqslant t_2. \end{aligned}$$

Since $w > 0$ in Ω_i, w attains minimum in $\bar{\Omega}_i$ at $(h(t), t)$ (for any $t \in (t_1, t_2)$) and the maximum principle gives

$$w_x(h(t), t) > 0.$$

Since $g' \geqslant 0$, v takes in $\{(x, t'); h(t') \leqslant x \leqslant L, 0 \leqslant t' \leqslant t\}$ maximum at the point $(h(t), t)$; hence

$$v_x(h(t), t) \leqslant 0.$$

Consequently

$$u_t = u_{xx} = v_x - w_x < 0 \qquad \text{at} \quad (h(t), t).$$

Since also $h'(t) = 0$, we get

$$\frac{d}{dt} u(h(t), t) < 0.$$

It follows that

$$u(h(t), t) < u(h(t_1), t_1) = 0 \qquad \text{if} \quad t_1 < t < t_2,$$

a contradiction.

We conclude that a maximal interval where $h'(t) = 0$ must be of the form $\sigma_i \leqslant t \leqslant t_0$ (it is therefore unique). Now define

$$t_0 = \inf\{t; \sigma_i \leqslant t \leqslant \tau_i, u(h(t), t) = 0\}.$$

Since $h'(t) = 0$ if $u(h(t), t) > 0$, the preceding result implies that

$$\begin{aligned} u(h(t), t) &> 0 \qquad \text{if} \quad \sigma_i < t < t_0, \\ u(h(t), t) &= 0 \qquad \text{if} \quad t_0 < t < \tau_i. \end{aligned} \tag{5.7}$$

Therefore $h'(t) = 0$ if $\sigma_i \leqslant t \leqslant t_0$. We next claim that $h'(t) > 0$ if $t_0 < t < \tau_i$. Indeed, otherwise there is a $t_1 \in (t_0, \tau_i)$ such that

$$h'(t_1) = 0. \tag{5.8}$$

Then, by (1.5),

$$w(h(t_1), t_1) = 0.$$

Since $w > 0$ in Ω_i, the maximum principle gives

$$w_x(h(t_1), t_1) > 0.$$

Since also $v_x(h(t_1), t_1) \leqslant 0$,

$$u_{xx}(h(t_1), t_1) = u_t(h(t_1), t_1) < 0. \tag{5.9}$$

On the other hand, by (5.7), (5.8),

$$0 = \left[\frac{d}{dt} u(h(t), t)\right]_{t=t_1} = [u_x h' + u_t](h(t_1), t_1) = u_t(h(t_1), t_1),$$

contradicting (5.9).

We have thus proved that

$$\begin{aligned} h'(t) &= 0 \qquad \text{if} \quad \sigma_i < t < t_0, \\ h'(t) &> 0 \qquad \text{if} \quad t_0 < t < \tau_i \end{aligned}$$

provided $w > 0$ in Ω_i.

Recalling also $h'(t) \equiv 0$ in $\sigma_i < t < \tau_i$ provided $w < 0$ in Ω_i, the proof of Theorem 5.1 is complete.

Remark. Suppose $g = \text{const}$ so that $v \equiv g(0)$. If $w < 0$ in Ω_i then

$$\begin{aligned} \frac{d}{dt} u(h(t), t) &= u_x(h(t), t)h'(t) + u_t(h(t), t) \\ &= u_t(h(t), t) = -w_x(h(t), t) > 0 \quad (\sigma_i < t < \tau_i) \end{aligned}$$

(since $w = 0$ on $x = h(t)$), so that u increases along the free boundary. On the other hand, if $w > 0$ in Ω_i then

$$\frac{d}{dt} u(h(t), t) = -w_x(h(t), t) < 0 \qquad \text{for} \quad \sigma_i < t < t_0$$

(by the maximum principle, since $w(h(t), t) = 0$), so that $u(h(t), t)$ is strictly decreasing.

6. Regularity of the Free Boundary

As mentioned in Remark 3.1, $h(t) \in C^{3/2}[0, T)$. In this section we show that this is the optimal regularity, i.e., in general,

$$h \notin C^{1+\alpha} \qquad \text{if} \quad \alpha > \tfrac{1}{2}. \tag{6.1}$$

We take for simplicity the case $g(t) \equiv \text{const}$ and $u_0(x)$ for which there is just one switching, and

$$u_0(0) > 0, \qquad -u_0'(0) + g(0) = 0.$$

Thus $h(t) = 0$ if $0 < t < t_0$, $h'(t) > 0$ if $t \geqslant t_0$. By the maximum principle, $w_x > 0$ at $(h(t_0), t_0)$, i.e.,

$$-u_{xx}(0, t_0) > 0. \tag{6.2}$$

We shall show that

$$h'(t) = -\frac{u_{xx}(0, t_0)}{\sqrt{\pi}}(t - t_0)^{1/2} + o(|t - t_0|^{1/2}); \tag{6.3}$$

this will establish (6.1).

For simplicity we replace $t - t_0$ by t. Then u is a solution of the Stefan problem for $t > 0$, with

$$u_x(L, t) = 0,$$
$$u(x, 0) = \phi(x) \qquad (\phi(x) = u(x, t_0)),$$
$$u(h(t), t) = 0, \qquad \frac{dh}{dt} = -u_x(h(t), t) + g.$$

We can represent the solution in the form

$$\begin{aligned} u(x, t) = & \int_0^t N(x, t; \xi, 0)\, \phi(\xi)\, d\xi \\ & + \int_0^t N(x, t; h(\tau), \tau)\, u_x(h(\tau), \tau)\, d\tau \end{aligned} \tag{6.4}$$

where

$$N(x, t; \xi, \tau) = K(L - x, t; L - \xi, \tau) + K(x - L, t; L - \xi, \tau),$$
$$K(x, t) = (4\pi t)^{-1/2} \exp(-x^2/4t).$$

Differentiating (6.4) with respect to x and letting $x \to h(t)$ we obtain, using standard jump relations (see [1, 3]),

$$\begin{aligned} \frac{1}{2}[g - H(t)] = & -N(h(t), t; \xi, 0)\phi(\xi)|_0^L + \int_0^L N(h(t), t; \xi, 0)\phi(\xi)\, d\xi \\ & + \int_0^t N_x(h(t), t; h(\tau), \tau)[g - H(\tau)]\, d\tau \end{aligned}$$

where $H(t) = h'(t)$. From this we easily obtain

$$\begin{aligned} H(t) = & \frac{1}{2} g - \int_0^L N(h(t), t; \xi, 0)[\phi'(0) + \phi''(0)\xi + o(\xi)]\, d\xi \\ & - g \int_0^t K_x(h(t), t; h(\tau), \tau)\, d\tau \\ & + \int_0^t K_x(h(t), t; h(\tau), \tau)\, H(\tau)\, d\tau + 0(t). \end{aligned}$$

Substituting

$$\eta = \frac{\xi - h(t)}{2\sqrt{t}}$$

in the first integral on the right with N replaced by K, we find that it is equal to

$$\frac{1}{2}\phi'(0) + \frac{\phi''(0)}{\sqrt{\pi}}\sqrt{t} + o(\sqrt{t}).$$

All the remaining integrals can be estimated by $o(t)$. Since $\phi'(0) = g$, (6.3) follows.

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