



Phantom space–times in fake supergravity



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ABSTRACT

We discuss phantom metrics admitting Killing spinors in fake $N = 2$, $D = 4$ supergravity coupled to vector multiplets. The Abelian $U(1)$ gauge fields in the fake theory have kinetic terms with the wrong sign. We solve the Killing spinor equations for the standard and fake theories in a unified fashion by introducing a parameter which distinguishes between the two theories. The solutions found are fully determined in terms of algebraic conditions, the so-called stabilisation equations, in which the symplectic sections are related to a set of functions. These functions are harmonic in the case of the standard supergravity theory and satisfy the wave-equation in flat $(2 + 1)$ -space–time in the fake theory. Explicit examples are given for the minimal models with quadratic prepotentials.

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1. Introduction

In recent years, a good amount of research activity has been focused on the classification of solutions preserving fractions of supersymmetry in supergravity theories in various space–time dimensions. Finding new gravitational solutions by solving first order Killing spinors differential equations is certainly an easier task than solving for the coupled second order Einstein equations of motion. Building on the work of Gibbons and Hull [1], Tod in [2] performed the first systematic classification for all metrics admitting Killing spinors in four-dimensional Einstein–Maxwell theory. The solutions with time-like Killing spinors turn out to be the known IWP solutions [3] which in the static limit reduce to the MP solutions [4]. More recently, techniques, partly based on [5], were implemented in the classifications of supersymmetric solutions. This was first done in [6] and later has been a very powerful tool in the classification of solutions in supergravity theories in four and five space–time dimensions (see for example [7]). This classification included, in addition to the standard ungauged and gauged supergravities, fake de Sitter supergravity theories which can be obtained by analytic continuation of anti de Sitter supergravity. It must be noted that de Sitter supergravities can also be obtained by a non-linear Kaluza–Klein reduction of the so called * theories of Hull [8]. The reduction of IIB* string theory and M* theory produced de Sitter supergravities with vector multiplets in four and five space–time

dimensions [9]. A new feature about these theories is that they come with gauge fields with the non-conventional sign of kinetic terms in the action. We shall refer to such gauge fields as anti or phantom fields and gravitational solutions to such theories as phantom solutions.

Phantom black hole solutions have been considered and analysed in [10]. Also, phantom solutions have been used by many authors in astrophysics and in the field of dark matter (see for instance [11] and references therein). In a recent work [12], metrics with space-like Killing vectors admitting Killing spinors in four-dimensional Einstein gravity coupled to a phantom Maxwell field were found. These solutions can be considered as the time-dependent analogues of the IWP metrics of the canonical Einstein–Maxwell theory. While the IWP metrics are expressed in terms of a harmonic complex function, the phantom analogue is expressed in terms of a complex function satisfying the wave-equation in a flat $(2 + 1)$ -space–time.

Generalisations of the IWP solutions in the context of $N = 2$ supergravity action coupled to matter multiplets were found some-time ago in [13]. These stationary solutions are generalisations of the double-extreme and static black hole solutions found in [14]. In our present work, we will generalise the results of [12] to four-dimensional $N = 2$ supergravity theory coupled to vector multiplets. We shall consider the action

$$e^{-1}\mathcal{L} = \frac{1}{2}R - g_{A\bar{B}}\partial_{\mu}z^A\partial^{\mu}\bar{z}^B - \frac{\kappa^2}{4}\left(\text{Im}\mathcal{N}_{IJ}F^I \cdot F^J + \text{Re}\mathcal{N}_{IJ}F^I \cdot \tilde{F}^J\right), \quad (1.1)$$

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where we used the notation $F^I \cdot F^J = F^I_{\mu\nu} F^{J\mu\nu}$ and $\tilde{F}^J = * F^J$, $I = 0, \dots, n$. For $\kappa = i$, this is the action of the standard $N = 2$, $D = 4$ supergravity theory coupled to vector multiplet. For $\kappa = 1$, this represents the action of a fake theory where the gauge field terms in the action come with the opposite sign. The n complex scalar fields, z^A , of $N = 2$ vector multiplets are coordinates of a special Kähler manifold and $g_{A\bar{B}} = \partial_A \partial_{\bar{B}} K$ is the Kähler metric with K being the Kähler potential. The structure of the scalar fields and relations of special geometry remain unaltered in the fake case. For details of special geometry, we refer the reader to [15] and references therein.

In what follows, we give some details and relations of special geometry which will be relevant to our discussions. A useful definition of a special Kähler manifold can be given by introducing a $(2n + 2)$ -dimensional symplectic bundle over the Kähler–Hodge manifold with the covariantly holomorphic sections \mathcal{V} ,

$$\mathcal{V} = \begin{pmatrix} L^I \\ M_I \end{pmatrix} = e^{K/2} \begin{pmatrix} X^I \\ F_I \end{pmatrix}, \quad I = 0, \dots, n, \quad \mathcal{D}_{\bar{A}} \mathcal{V} = 0, \quad (1.2)$$

where $\mathcal{D}_{\bar{A}} \mathcal{V} = (\partial_{\bar{A}} - \frac{1}{2} \partial_{\bar{A}} K) \mathcal{V}$ and $\mathcal{D}_A \mathcal{V} = (\partial_A + \frac{1}{2} \partial_A K) \mathcal{V}$. These sections obey the symplectic constraint

$$i (\bar{L}^I M_I - L^I \bar{M}_I) = 1. \quad (1.3)$$

The Kähler potential can be obtained from the holomorphic sections by

$$e^{-K} = i (\bar{X}^I F_I - X^I \bar{F}_I). \quad (1.4)$$

The coupling matrix, \mathcal{N}_{IJ} , can be defined by

$$F_I(z) = \mathcal{N}_{IJ} X^J(z), \quad \mathcal{D}_A F_I(z) = \bar{\mathcal{N}}_{IJ} \mathcal{D}_A X^J(z). \quad (1.5)$$

We also note the very useful relations

$$g^{A\bar{B}} \mathcal{D}_A L^M \mathcal{D}_{\bar{B}} \bar{L}^I = -\frac{1}{2} (\text{Im } \mathcal{N})^{MI} - \bar{L}^M L^I, \quad (1.6)$$

$$F_I \partial_\mu X^I - X^I \partial_\mu F_I = 0. \quad (1.7)$$

Also, one can derive the relations [16]

$$\mathcal{D}_A L^I dz^A = (d + iA) L^I, \quad (1.8)$$

$$dM_I - 2 \text{Im } \mathcal{N}_{IJ} L^J A = \bar{\mathcal{N}}_{IJ} dL^J, \quad (1.9)$$

$$A = M_I d\bar{L}^I - L^I d\bar{M}_I, \quad (1.10)$$

where the $U(1)$ Kähler connection A is defined by

$$A = -\frac{i}{2} (\partial_A K dz^A - \partial_{\bar{A}} K d\bar{z}^{\bar{A}}). \quad (1.11)$$

The Killing spinor equations we shall analyse are given by

$$\begin{aligned} (\nabla_\mu + \frac{i}{2} \mathcal{A}_\mu \gamma_5 \\ + \frac{\kappa}{4} \text{Im } \mathcal{N}_{IJ} \gamma \cdot F^I (\text{Im } L^J - i\gamma_5 \text{Re } L^J) \gamma_\mu) \varepsilon = 0, \end{aligned} \quad (1.12)$$

and

$$\begin{aligned} \frac{\kappa}{2} (\text{Im } \mathcal{N})_{IJ} \gamma \cdot F^J \left[\text{Im} (g^{A\bar{B}} \mathcal{D}_{\bar{B}} \bar{L}^I) - i\gamma_5 \text{Re} (g^{A\bar{B}} \mathcal{D}_{\bar{B}} \bar{L}^I) \right] \varepsilon \\ + \gamma^\mu \partial_\mu (\text{Re } z^A - i\gamma_5 \text{Im } z^A) \varepsilon = 0. \end{aligned} \quad (1.13)$$

Here $\nabla_\mu = (\partial_\mu + \frac{1}{4} \gamma \cdot \omega_\mu)$ and ε are Dirac spinors. For $\kappa = i$, those represent the vanishing of the supersymmetry variations, in a bosonic background, of the gravitini and gaugini in the standard $N = 2$, $D = 4$ supergravity theory coupled to vector multiplet. For

$\kappa = 1$, those represent the vanishing of fake supersymmetry transformations for a theory where all the gauge fields terms in the action come with the opposite sign.

In our analysis of the Killing spinor equations, we follow the method of spinorial geometry. We write the spinors as complexified forms on \mathbb{R}^2 . A generic spinor, ε , can therefore be written as

$$\varepsilon = \lambda 1 + \mu_i e^i + \sigma e^{12}, \quad (1.14)$$

where e^1, e^2 are 1-forms on \mathbb{R}^2 , and $i = 1, 2$; $e^{12} = e^1 \wedge e^2$. λ, μ_i and σ are complex functions.

The action of γ -matrices on these forms is given by

$$\begin{aligned} \gamma_0 = -e^2 \wedge + i e^2, \quad \gamma_1 = e^1 \wedge + i e^1, \\ \gamma_2 = e^2 \wedge + i e^2, \quad \gamma_3 = i(e^1 \wedge - i e^1). \end{aligned} \quad (1.15)$$

and γ_5 is defined by $\gamma_5 = i\gamma_{0123}$ where

$$\gamma_5 1 = 1, \quad \gamma_5 e^{12} = e^{12}, \quad \gamma_5 e^i = -e^i, \quad i = 1, 2. \quad (1.16)$$

Using the results of [17], we define

$$\begin{aligned} \gamma_+ &= \frac{1}{\sqrt{2}} (\gamma_2 + \gamma_0) = \sqrt{2} i e^2, \\ \gamma_- &= \frac{1}{\sqrt{2}} (\gamma_2 - \gamma_0) = \sqrt{2} e^2 \wedge, \\ \gamma_1 &= \frac{1}{\sqrt{2}} (\gamma_1 + i\gamma_3) = \sqrt{2} i e^1, \\ \gamma_{\bar{1}} &= \frac{1}{\sqrt{2}} (\gamma_1 - i\gamma_3) = \sqrt{2} e^1 \wedge, \end{aligned} \quad (1.17)$$

where the non-vanishing metric components in this null basis are given by $g_{+-} = 1, g_{\bar{1}\bar{1}} = 1$. The canonical forms of the spinor are basically representatives up to gauge transformations which preserve the Killing spinor equation. Using $Spin(3, 1)$ gauge transformations, it was shown in [17], that one finds the three canonical forms:

$$\varepsilon = 1 + \mu_2 e^2, \quad \varepsilon = 1 + \mu_1 e^1, \quad \varepsilon = e^2. \quad (1.18)$$

As in [12], we shall focus on the first canonical form. Plugging $\varepsilon = 1 + \mu e^2$ in (1.12) and (1.13) and using (1.17), the Killing spinor equations amount to two sets of equations:

$$\begin{aligned} \omega_{+,-1} &= 0, \\ \omega_{1,-1} &= 0, \\ \omega_{-,+1} &= 0, \\ \omega_{1,+1} &= 0, \\ \mu \omega_{-,-1} + i\kappa \sqrt{2} \text{Im } \mathcal{N}_{IJ} F_{-1}^I \bar{L}^J &= 0, \\ \mu \omega_{\bar{1},-1} - \frac{i\kappa}{\sqrt{2}} \text{Im } \mathcal{N}_{IJ} (F_{\bar{1}\bar{1}}^I + F_{-+}^I) \bar{L}^J &= 0, \\ \partial_- \log \mu - \frac{1}{2} (\omega_{-,\bar{1}\bar{1}} + \omega_{-,-+}) \\ - \frac{i}{2} \mathcal{A}_- - i \frac{\kappa}{\mu \sqrt{2}} \text{Im } \mathcal{N}_{IJ} (F_{\bar{1}\bar{1}}^I + F_{-+}^I) \bar{L}^J &= 0, \\ \partial_1 \log \mu - \frac{1}{2} (\omega_{1,\bar{1}\bar{1}} + \omega_{1,-+}) - \frac{i}{2} \mathcal{A}_1 &= 0, \\ \partial_+ \log \mu - \frac{1}{2} (\omega_{+,\bar{1}\bar{1}} + \omega_{+,-+}) - \frac{i}{2} \mathcal{A}_+ &= 0, \\ \omega_{1,-+} - \omega_{\bar{1}\bar{1},1} + i\mathcal{A}_1 &= 0, \end{aligned}$$

$$\begin{aligned}
 \omega_{-, -+} - \omega_{-, 1\bar{1}} + iA_- &= 0, \\
 \partial_{\bar{1}} \log \mu - \frac{1}{2}(\omega_{\bar{1}, 1\bar{1}} + \omega_{\bar{1}, -+}) - \frac{i}{2}A_{\bar{1}} + \frac{i\kappa}{\mu} \text{Im} \mathcal{N}_{\bar{1}J} F_{\bar{1}}^J L^J \sqrt{2} &= 0, \\
 \frac{1}{2}(\omega_{\bar{1}, -+} - \omega_{\bar{1}, 1\bar{1}} + iA_{\bar{1}}) - i\kappa \mu \text{Im} \mathcal{N}_{\bar{1}J} F_{\bar{1}}^J L^J \sqrt{2} &= 0, \\
 \frac{1}{2}(\omega_{+, -+} - \omega_{+, 1\bar{1}} + iA_+) - i \frac{\kappa \mu}{\sqrt{2}} \text{Im} \mathcal{N}_{\bar{1}J} (F_{-+}^J - F_{1\bar{1}}^J) L^J &= 0, \\
 \omega_{+, +1} - i\kappa \mu \text{Im} \mathcal{N}_{\bar{1}J} F_{+1}^J L^J \sqrt{2} &= 0, \\
 \omega_{\bar{1}, +1} + i \frac{\kappa \mu}{\sqrt{2}} \text{Im} \mathcal{N}_{\bar{1}J} (F_{1\bar{1}}^J - F_{-+}^J) L^J &= 0,
 \end{aligned} \tag{1.19}$$

and

$$\begin{aligned}
 -i\kappa g^{A\bar{B}} \mathcal{D}_{\bar{B}} \bar{L}^J (\text{Im} \mathcal{N})_{\bar{1}J} (F_{-+}^J - F_{1\bar{1}}^J) + \partial_- z^A \mu \sqrt{2} &= 0, \\
 -i\bar{\kappa} \bar{\mu} g^{A\bar{B}} \mathcal{D}_{\bar{B}} \bar{L}^J (\text{Im} \mathcal{N})_{\bar{1}J} (F_{1\bar{1}}^J - F_{-+}^J) + \partial_+ z^A \sqrt{2} &= 0, \\
 2i\bar{\kappa} \bar{\mu} g^{A\bar{B}} \mathcal{D}_{\bar{B}} \bar{L}^J (\text{Im} \mathcal{N})_{\bar{1}J} F_{\bar{1}}^J + \partial_{\bar{1}} z^A \sqrt{2} &= 0, \\
 2i\kappa g^{A\bar{B}} \mathcal{D}_{\bar{B}} \bar{L}^J (\text{Im} \mathcal{N})_{\bar{1}J} F_{+1}^J + \partial_1 z^A \mu \sqrt{2} &= 0.
 \end{aligned} \tag{1.20}$$

The analysis of the equations of (1.19) gives:

$$\begin{aligned}
 \text{Im} \mathcal{N}_{\bar{1}J} F_{\bar{1}}^J L^J &= -\frac{i\bar{\kappa}}{\sqrt{2}|\mu|^2} (\partial_{\bar{1}} + iA_{\bar{1}}) \bar{\mu}, \\
 \text{Im} \mathcal{N}_{\bar{1}J} (F_{-+}^J - F_{1\bar{1}}^J) L^J &= i\kappa \sqrt{2} (\partial_- + iA_-) \bar{\mu}, \\
 \text{Im} \mathcal{N}_{\bar{1}J} F_{+1}^J L^J &= -\frac{i\kappa}{\sqrt{2}} (\partial_1 + iA_1) \bar{\mu},
 \end{aligned} \tag{1.21}$$

with the condition

$$\mu \partial_- \bar{\mu} + \kappa^2 \partial_+ \log \bar{\mu} = -i(A_- |\mu|^2 + \kappa^2 A_+). \tag{1.22}$$

We also obtain the following relations for the spin connection

$$\begin{aligned}
 \omega_{1\bar{1}} &= \left(\partial_+ \log \frac{\mu}{\bar{\mu}} - iA_+ \right) \mathbf{e}^+ \\
 &\quad + iA_- \mathbf{e}^- + \partial_1 \log \mu \mathbf{e}^1 - \partial_{\bar{1}} \log \bar{\mu} \mathbf{e}^{\bar{1}}, \\
 \omega_{-1} &= \frac{\kappa^2}{|\mu|^2} (\partial_1 \log \mu - iA_1) \mathbf{e}^- + (\partial_- \log \mu - iA_-) \mathbf{e}^{\bar{1}}, \\
 \omega_{-+} &= (\partial_1 \log \mu - iA_1) \mathbf{e}^1 \\
 &\quad + (\partial_{\bar{1}} \log \bar{\mu} + iA_{\bar{1}}) \mathbf{e}^{\bar{1}} + \partial_+ \log \bar{\mu} \mu \mathbf{e}^+, \\
 \omega_{+1} &= (\partial_+ \log \bar{\mu} + iA_+) \mathbf{e}^{\bar{1}} + \kappa^2 \mu (\partial_1 \bar{\mu} + iA_1 \bar{\mu}) \mathbf{e}^+.
 \end{aligned} \tag{1.23}$$

The vanishing of torsion implies the conditions

$$\begin{aligned}
 d\mathbf{e}^1 + d \log \bar{\mu} \wedge \mathbf{e}^1 &= 0, \\
 d\mathbf{e}^+ &= -\left(\partial_- \log \frac{\mu}{\bar{\mu}} - 2iA_- \right) \mathbf{e}^{\bar{1}} \wedge \mathbf{e}^1 \\
 &\quad - \left(\frac{\kappa^2}{|\mu|^2} \mathbf{e}^- - \mathbf{e}^+ \right) \\
 &\quad \wedge \left((\partial_{\bar{1}} \log \bar{\mu} + iA_{\bar{1}}) \mathbf{e}^{\bar{1}} + (\partial_1 \log \mu - iA_1) \mathbf{e}^1 \right),
 \end{aligned} \tag{1.24}$$

and

$$\begin{aligned}
 d\mathbf{e}^- &= -\left(\partial_+ \log \frac{\bar{\mu}}{\mu} + 2iA_+ \right) \mathbf{e}^{\bar{1}} \wedge \mathbf{e}^1 + \partial_+ \log |\mu|^2 \mathbf{e}^+ \wedge \mathbf{e}^- \\
 &\quad - \kappa^2 \mathbf{e}^+ \wedge \left((\mu \partial_1 \bar{\mu} + iA_1 \mu \bar{\mu}) \mathbf{e}^1 \right. \\
 &\quad \left. + (\bar{\mu} \partial_{\bar{1}} \mu - i|\mu|^2 A_{\bar{1}}) \mathbf{e}^{\bar{1}} \right) \\
 &\quad - \frac{1}{|\mu|^2} \mathbf{e}^- \left((\bar{\mu} \partial_1 \mu - i\mu \bar{\mu} A_1) \mathbf{e}^1 \right. \\
 &\quad \left. + (\mu \partial_{\bar{1}} \bar{\mu} + i|\mu|^2 A_{\bar{1}}) \mathbf{e}^{\bar{1}} \right).
 \end{aligned} \tag{1.26}$$

An immediate result of the torsion free conditions and (1.22) is that $(\mu \bar{\mu} \mathbf{e}^+ - \kappa^2 \mathbf{e}^-)$ is a total differential

$$d(\mu \bar{\mu} \mathbf{e}^+ - \kappa^2 \mathbf{e}^-) = 0, \tag{1.27}$$

and that the vector V ,

$$V = |\mu|^2 \mathbf{e}^+ + \kappa^2 \mathbf{e}^- = |\mu|^2 \partial_- + \kappa^2 \partial_+, \tag{1.28}$$

is a Killing vector which is space-like for $\kappa^2 = 1$ and time-like for $\kappa^2 = -1$. Note that these two special vectors are related to the inner Hermitian products $\langle \gamma_0 \varepsilon, \gamma_a \varepsilon \rangle$ and $\langle \gamma_0 \varepsilon, \gamma_5 \gamma_a \varepsilon \rangle$.

The above conditions enable us to introduce the coordinates (t, x, y, z) , such that

$$\begin{aligned}
 \mathbf{e}^1 &= \frac{1}{\bar{\mu} \sqrt{2}} (dx + idy), \\
 \mathbf{e}^+ &= \frac{1}{|\mu|^2 \sqrt{2}} (dz + \kappa^2 |\mu|^2 (dt + \sigma)), \\
 \mathbf{e}^- &= -\frac{\kappa^2}{\sqrt{2}} (dz - \kappa^2 |\mu|^2 (dt + \sigma)),
 \end{aligned} \tag{1.29}$$

and the metric is independent of the coordinate t and is given by

$$\begin{aligned}
 ds^2 &= 2\mathbf{e}^1 \mathbf{e}^{\bar{1}} + 2\mathbf{e}^+ \mathbf{e}^- \\
 &= \kappa^2 |\mu|^2 (dt + \sigma)^2 + \frac{1}{|\mu|^2} (-\kappa^2 dz^2 + dx^2 + dy^2).
 \end{aligned} \tag{1.30}$$

Here σ is a one form, $\sigma = \sigma_x dx + \sigma_y dy + \sigma_z dz$, independent of the coordinate t and satisfies

$$d\sigma = -\frac{\kappa^2}{|\mu|^2} *_3 \left(id \log \frac{\mu}{\bar{\mu}} + 2A \right), \tag{1.31}$$

where $*_3$ is the Hodge dual with metric $(-\kappa^2 dz^2 + dx^2 + dy^2)$.

The first two equations in second set of conditions (1.20) imply that

$$(\mu \bar{\mu} \partial_- + \kappa^2 \partial_+) z^A = 0. \tag{1.32}$$

Thus the scalar fields are also independent of the coordinate t . Equations (1.32) and (1.11) imply that

$$\kappa^2 A_+ + \mu \bar{\mu} A_- = 0. \tag{1.33}$$

Going back to (1.22), we then deduce that

$$\partial_t \mu = 0. \tag{1.34}$$

Multiplying the relations (1.20) by $\mathcal{D}_A L^M$ and using the relations (1.6) and (1.8), we obtain the relations

$$\begin{aligned}
& \frac{i\kappa}{2} \left(F_{-+}^M - F_{1\bar{1}}^M \right) + i\kappa (\text{Im} \mathcal{N})_{IJ} \left(F_{-+}^J - F_{1\bar{1}}^J \right) \bar{L}^M L^I \\
& + (\partial_- + iA_-) L^M \mu \sqrt{2} = 0, \\
& -2i\bar{\kappa} \bar{\mu} (\text{Im} \mathcal{N})_{IJ} F_{-1}^J \bar{L}^M L^I - i\bar{\kappa} F_{-1}^M \bar{\mu} \\
& + (\partial_{\bar{1}} + iA_{\bar{1}}) L^M \mu \sqrt{2} = 0, \\
& -2i\kappa (\text{Im} \mathcal{N})_{IJ} F_{+1}^J \bar{L}^M L^I - i\kappa F_{+1}^M \mu \\
& + (\partial_1 + iA_1) L^M \mu \sqrt{2} = 0. \tag{1.35}
\end{aligned}$$

Using (1.21) and converting to space-time indices, we obtain for the gauge field strength two-form

$$\begin{aligned}
F^I &= d \left(i\kappa \mu L^I - i\bar{\kappa} \bar{L}^I \bar{\mu} \right) \wedge (dt + \sigma) \\
& - \frac{1}{|\mu|^2} *_{3} \left[\kappa \bar{\mu} d\bar{L}^I - \kappa \bar{L}^I d\bar{\mu} + \bar{\kappa} \mu dL^I - \bar{\kappa} L^I d\mu \right] \\
& - \frac{2i}{|\mu|^2} *_{3} \left(\bar{\kappa} \mu L^I - \kappa \bar{L}^I \bar{\mu} \right) \mathcal{A}. \tag{1.36}
\end{aligned}$$

Using (1.31), (1.36) can be rewritten in the form

$$\begin{aligned}
F^I &= d \left[\left(i\kappa \mu L^I - i\bar{\kappa} \bar{L}^I \bar{\mu} \right) (dt + \sigma) \right] \\
& - *_{3} d \left[\kappa \left(\frac{\bar{L}^I}{\mu} \right) + \bar{\kappa} \left(\frac{L^I}{\bar{\mu}} \right) \right]. \tag{1.37}
\end{aligned}$$

Calculating the dual \tilde{F}^I , we obtain

$$\begin{aligned}
\tilde{F}^I &= \frac{i}{|\mu|^2} *_{3} d \left[\kappa \bar{L}^I \bar{\mu} - \bar{\kappa} \mu L^I \right] \\
& + \left((\bar{\kappa} \bar{L}^I d\bar{\mu} - \kappa \mu dL^I) + (\kappa L^I d\mu - \bar{\kappa} \bar{\mu} d\bar{L}^I) \right) \wedge (dt + \sigma) \\
& - (2i\mathcal{A} (\kappa \mu L^I - \bar{\kappa} \bar{\mu} \bar{L}^I)) \wedge (dt + \sigma). \tag{1.38}
\end{aligned}$$

Again using (1.31) as well as (1.9), we obtain

$$\begin{aligned}
\text{Re} \mathcal{N}_{IJ} F^J - \text{Im} \mathcal{N}_{IJ} \tilde{F}^J &= d \left[(i\kappa \mu M_I - i\bar{\kappa} \bar{\mu} \bar{M}_I) (dt + \sigma) \right] \\
& - *_{3} d \left[\kappa \left(\frac{\bar{M}_I}{\mu} \right) + \bar{\kappa} \left(\frac{M_I}{\bar{\mu}} \right) \right]. \tag{1.39}
\end{aligned}$$

Then Bianchi identities and Maxwell equations

$$dF^I = 0, \quad d \left(\text{Re} \mathcal{N}_{IJ} F^J - \text{Im} \mathcal{N}_{IJ} \tilde{F}^J \right) = 0, \tag{1.40}$$

imply, respectively, the conditions

$$\left(\frac{\kappa \bar{L}^I}{\mu} + \frac{\bar{\kappa} L^I}{\bar{\mu}} \right) = \psi^I, \quad \left(\frac{\kappa \bar{M}_I}{\mu} + \frac{\bar{\kappa} M_I}{\bar{\mu}} \right) = \psi_I, \tag{1.41}$$

where

$$\begin{aligned}
\nabla^2 \psi^I &= \nabla^2 \psi_I = 0, \\
\nabla^2 &= \partial_x^2 + \partial_y^2 - \kappa^2 \partial_z^2. \tag{1.42}
\end{aligned}$$

Using (1.41), (1.3), (1.7) and (1.10), we obtain

$$\mathcal{A} = \frac{|\mu|^2}{2} (\psi_I d\psi^I - \psi^I d\psi_I) - \frac{i}{2} d \log \frac{\mu}{\bar{\mu}}. \tag{1.43}$$

Substituting (1.43) back in the expression of $d\sigma$, we obtain

$$d\sigma = -\kappa^2 *_{3} (\psi_I d\psi^I - \psi^I d\psi_I). \tag{1.44}$$

For $\kappa = i$, we obtain the known solutions of [13,18] which are generalisations of the solutions first obtained in [14]. The new derivation here, based on spinorial geometry, reveals that these

are the unique solutions with time-like Killing vector as has also been demonstrated in [19]. For $\kappa = 1$, we obtain new phantom solutions for theories with the wrong signs for the gauge kinetic terms. In this case, the functions ψ^I and ψ_I in (1.41) satisfy the wave-equation

$$\left(\partial_x^2 + \partial_y^2 \right) \psi^I = \partial_z^2 \psi^I, \quad \left(\partial_x^2 + \partial_y^2 \right) \psi_I = \partial_z^2 \psi_I. \tag{1.45}$$

These solutions are the unique solutions with space-like Killing vectors admitting Killing spinors.

2. Examples: Quadratic prepotentials

Supergravity minimal models are characterised by quadratic prepotentials F [20]. For these models we have

$$M_I = \partial_I F = Q_{IJ} L^J, \tag{2.1}$$

where Q_{IJ} is symmetric. Static black holes for the minimal models were considered in [20]. Without lack of generality, and as was explained in [20], Q_{IJ} can be taken to be purely imaginary. The stabilisation conditions (1.41) for these models then give

$$\left[\frac{\kappa \bar{L}^I}{\mu} + \frac{\bar{\kappa} L^I}{\bar{\mu}} \right] = \psi^I, \quad \left[\frac{\bar{\kappa} L^I}{\bar{\mu}} - \frac{\kappa \bar{L}^I}{\mu} \right] = Q^{IJ} \psi_J. \tag{2.2}$$

This can be solved by

$$L^I = \frac{\bar{\mu}}{2} \kappa (\psi^I + Q^{IJ} \psi_J). \tag{2.3}$$

The symplectic constraint (1.3), then implies that

$$\frac{1}{|\mu|^2} = \frac{i}{2} (Q_{IJ} \psi^J \psi^I - Q^{IJ} \psi_I \psi_J). \tag{2.4}$$

Using (1.37), the gauge fields are given by

$$F^I = d \left[i|\mu|^2 \kappa^2 Q^{IJ} \psi_J (dt + \sigma) \right] - *_{3} d\psi^I. \tag{2.5}$$

For $\kappa = 1$, as in [12], explicit solutions can be obtained if one assumes that the solution depends on the coordinate z only. In this case we have

$$\partial_z^2 \psi^I = \partial_z^2 \psi_I = 0, \tag{2.6}$$

and the solution can be given by

$$\psi^I = A^I + p^I z, \quad \psi_I = B_I + q_I z. \tag{2.7}$$

For $A^I = B_I = 0$, the solution is then given by

$$ds^2 = \frac{\gamma^2}{z^2} (dt)^2 + \frac{z^2}{\gamma^2} (-dz^2 + dx^2 + dy^2), \tag{2.8}$$

where we have defined $\gamma^{-2} = \frac{i}{2} (Q_{IK} p^K p^I - Q^{IM} q_M q_I)$. Setting

$$\tau = \frac{z^2}{2\gamma}, \quad x_3 = \sqrt{\frac{\gamma}{2}} t, \quad x_2 = \sqrt{\frac{2}{\gamma}} x, \quad x_1 = \sqrt{\frac{2}{\gamma}} y, \tag{2.9}$$

we get the Kasner metric [21]

$$ds^2 = -d\tau^2 + \tau (dx_2)^2 + \tau (dx_1)^2 + \frac{1}{\tau} (dx_3)^2, \tag{2.10}$$

where the gauge fields are given by

$$F^I = \frac{1}{2} \gamma \left(-i \frac{Q^{IJ} q_J}{\tau^{3/2}} d\tau \wedge dx_3 + p^I dx_2 \wedge dx_1 \right). \tag{2.11}$$

In summary, we have obtained new phantom metrics admitting Killing spinors in fake $N = 2$, $D = 4$ supergravity where the

Abelian $U(1)$ gauge fields have kinetic terms with the wrong sign. The solutions found are expressed in terms of algebraic constraints satisfied by the symplectic sections. The solutions are characterised in terms of a set of functions satisfying the wave-equation in flat $(2+1)$ -space-time. Explicit solutions are constructed for the supergravity models where the prepotential is quadratic. We note that our solutions for the fake supergravity cases are the only known time-dependent solutions to be found which admit Killing spinors and possess a space-like Killing vector. Moreover, our phantom solutions contain a subclass which constitutes novel generalisations of Kasner solutions of pure gravity to the models of $N = 2$ supergravity with Maxwell and scalar fields. Our time-dependent solutions should provide good models for the study of cosmological scenarios and could be of importance to the study of dark matter and astrophysics. Our analysis can be generalised to fake gauged supergravity theories as well as to the de Sitter supergravities constructed in [9]. Non-supersymmetric phantom solutions can also be analysed using the general framework presented in [22]. We hope to report on this in a future publication.

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