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On the Quantization of Poisson Brackets

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In this paper we introduce two classes of Poisson brackets on algebras (or on sheaves of algebras). We call them locally free and nonsingular Poisson brackets. Using the Fedosov's method we prove that any locally free nonsingular Poisson

about the quantization of nondegenerate Poisson brackets on smooth manifolds follows from the main result of this paper as well.

1. INTRODUCTION

From an algebraic point of view to quantize a commutative associative algebra A over a field $\mathbf{k}$ is to deform it as an associative algebra in such a way that the deformed algebra is noncommutative. This concept of deformation quantization was proposed in [1]. In many interesting cases an algebra A has no nontrivial commutative deformations, for example, algebras of functions on smooth or analytic manifolds. However these algebras admit many nonequivalent noncommutative deformations.

A general theory of deformations of associative algebras has been developed in the fundamental works of Gerstenhaber [2–5] (see also [6]). The Hochschild 2-cocycles play the role of infinitesimal objects of such deformations, hence, the tangent space (or deformations of order one) to a "versal family" of deformations of an associative algebra A is the second Hochschild cohomology space $H^2(A, A)$. In the case of commutative algebra, it is natural to begin a "pure" noncommutative deformation with a skew-symmetric Hochschild cocycle, which gives a deformation of order one. Then, a skew-symmetric bilinear form $A \times A \to A$ is a Hochschild cocycle if and only if it defines a biderivation with respect to the original multiplication, i.e. satisfies the Leibniz rule. This form must also satisfy the Jacobi identity if there exists an extension of the deformation up to order

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two (see Section 2), so, the natural initial object for deformation (or quantization) of a commutative algebra A is a Poisson bracket f on this algebra. Given a Poisson bracket f on A, a natural question arises: whether there exists a deformation of A over $\mathbf{k}[[\hbar]]$ with f as tangent vector? The author does not know any example of nonquantizable Poisson bracket on "good" algebras. However, there are lots of examples of Poisson brackets admitting quantization. First of all, any Poisson bracket on a two-dimensional smooth manifold can be quantized. Any nondegenerate Poisson bracket on a smooth manifold of arbitrary dimension can be quantized as well (see [14, 16–18]). Universal enveloping algebras are examples of quantizations of linear Poisson brackets (degenerate, of course) on polynomial algebras. It is shown in [8] that any quadratic Poisson bracket on the polynomial algebra of three variables can be quantized. Note that quadratic Poisson brackets correspond to deformations of polynomial algebras as quadratic algebras. Quantum groups are quantizations of so-called R-matrix Poisson brackets on Lie groups (see [7]). On the other hand, the R-matrix Poisson brackets on Lie groups induce, in several ways, Poisson brackets on some homogeneous spaces of these groups. In certain cases, these Poisson brackets can be quantized (see [9–11]). Other examples of quantizations can be found in [6].

In this paper we introduce two classes of Poisson brackets on algebras (or on sheaves of algebras)—the class of locally free and its subclass of locally free nonsingular Poisson brackets (see Section 5). We prove that any locally free nonsingular Poisson bracket can be quantized. It should be noted that the class of locally free nonsingular Poisson brackets includes nondegenerate brackets on algebras of functions on manifolds and all Poisson brackets on fields. The class of locally free Poisson brackets includes almost everywhere nondegenerate Poisson brackets on a manifold. In particular, the last class contains R-matrix brackets on symmetric spaces (see [11]). The quantization of these brackets is given in [12].

I would like to underline that methods of the work [16] have influenced this paper considerably.

2. POISSON BRACKETS AND DEFORMATIONS OF COMMUTATIVE ALGEBRAS

Let A be an associative algebra with unit over a field $\mathbf{k}$ of characteristic zero. We will consider deformations of A over the algebra of formal power series $\mathbf{k}[[\hbar]]$ in a variable h.

By a deformation of A we mean an algebra A_h over $\mathbf{k}[[\hbar]]$ that is isomorphic to $A[[\hbar]] = A \hat{\otimes}_{\mathbf{k}} \mathbf{k}[[\hbar]]$ as a $\mathbf{k}[[\hbar]]$-module and $A_h/\hbar A_h = A$ (the symbol $\hat{\otimes}$ denotes the tensor product completed in the $\hbar$-adic topology). We will also denote A as A_0.

If $A'_\hbar$ is another deformation of A, we call the deformations $A_\hbar$ and $A'_\hbar$ equivalent if there exists a $\mathbf{k}[[\hbar]]$-algebra isomorphism $A_\hbar \to A'_\hbar$ which induces the identity automorphism of A_0.

In other words, $A_\hbar$ consists of elements of the form

$$x = \sum_{i=0}^{\infty} x_i \hbar^i, \qquad x_i \in A.$$

The multiplication in $A_\hbar$ is given by a $\mathbf{k}$-bilinear map $F_\hbar \colon A \times A \to A[[\hbar]]$ written as

$$F_\hbar(x, y) = \sum_{i \geqslant 0} \hbar^i F_i(x, y), \qquad x, y \in A,$$

where $F_0(x, y) = xy$ is the multiplication in A. The terms F_i, $i > 0$, are $\mathbf{k}$-bilinear forms $A \times A \to A$. Associativity means that $F = F_\hbar$ satisfies the following equation

$$F(F(x, y), z) = F(x, F(x, y)).$$

Collecting the terms by the powers of $\hbar$, we obtain

$$\sum_{i+j=n} (F_i(F_j(x, y), z) - F_i(x, F_j(y, z))) = 0, \qquad n \geqslant 0. \tag{1}$$

If the multiplication in $A'_\hbar$ is given by a bilinear map $F'_\hbar \colon A \times A \to A[[\hbar]]$, then the equivalence of $A_\hbar$ and $A'_\hbar$ can be given as a power series

$$Q = Id + \sum_{i \geqslant 1} \hbar^i Q_i,$$

where Q_i are $\mathbf{k}$-linear maps $Q_i \colon A \to A$, such that

$$F'(x, y) = Q^{-1}(F(Q(x), Q(y))). \tag{2}$$

Let us consider the element F_1. From Eq. (1) for $n = 1$ we get the following relation

$$xF_1(y, z) - F_1(xy, z) + F_1(x, yz) - F_1(x, y)\, z = 0. \tag{3}$$

This means that F_1 is a Hochschild 2-cocycle. From (1) for $n = 2$ we have

$$\begin{aligned} &F_1(F_1(x, y), z) - F_1(x, F_1(y, z)) \\ &\qquad = xF_2(y, z) - F_2(xy, z) + F_2(x, yz) - F_2(x, y)\, z, \end{aligned}$$

i.e. the element $F_1(F_1(x, y)), z) - F_1(x, F_1(yz))$ is 3-coboundary. If the series F' gives an equivalent deformation of A, then we have from (2)

$$F'_1(x, y) - F_1(x, y) = xQ_1(y) - Q_1(xy) + Q_1(x)\, y,$$

i.e. $F'_1(x, y)$ and $F_1(x, y)$ are cohomologous.

Throughout the remainder of this paper we will be interested in deformations of commutative algebras. If such is the case, we have

PROPOSITION 2.1. *Let A be a commutative algebra, $F_1(x, y)$, $x, y \in A$ a Hochschild cocycle, then $F'_1(x, y) = F_1(y, x)$ is a Hochschild cocycle.*

Proof. Straightforward computation.

Thus, in the case of commutative algebra any Hochschild cocyle $F_1(x, y)$ can be decomposed into two cocycles: $F_1(x, y) = \alpha(x, y) + \beta(x, y)$, where $\alpha(x, y) = (F_1(x, y) + F_1(y, x))/2$ and $\beta(x, y) = (F_1(x, y) - F_1(y, x))/2$. The cocycle $\alpha(x, y)$ is commutative, i.e. $\alpha(x, y) = \alpha(y, x)$, therefore it determines an infinitesimal commutative deformation of A. We will consider noncommutative deformations of a commutative algebra (or quantizations), so we will put $\alpha(x, y) = 0$. Moreover, in many cases algebra A will have no commutative deformations, and any commutative Hochschild cocycle will be a coboundary, so in these cases any deformation is equivalent to a deformation with skew-symmetric F_1, i.e. we can suppose that $F_1(x, y) = \beta(x, y)$. Put $(x, y) = \beta(x, y)$. We have $(x, y) = -(y, x)$.

PROPOSITION 2.2. (a) *A skew-symmetric bilinear map $A \otimes A \to A$, $x \otimes y \mapsto (x, y)$ is a Hochschild cocyle if and only if it obeys the Leibniz rule*

$$(xy, z) = x(y, z) + y(x, z).$$

(b) *If $((x, y), z) - (x, (y, z))$ is a Hochschild* 3-*coboundary, then the Jacobi identity holds*:

$$((x, y), z) + ((y, z), x) + ((z, x), y) = 0.$$

Proof. (a) If (x, y) satisfies the Leibniz rule a straightforward computation shows that it is a Hochschild cocycle. Let (x, y) be a cocycle. Then

$$x(y, z) - (xy, z) + (x, yz) - (x, y)\, z = 0,$$

$$y(x, z) - (yx, z) + (y, xz) - (y, x)\, z = 0,$$

$$(-1)[x(z, y) - (xz, y) + (x, zy) - (x, z)\, y] = 0.$$

Adding these equations, we will get the Leibniz rule.

(b) Suppose, there exists a 2-cochain $\{x, y\}$ such that

$$((x, y), z) - (x, (y, z)) = x\{y, z\} - \{xy, z\} + \{x, yz\} - \{x, y\} z.$$

Let us also write five similar equations for permutations of x, y, z. Let (abc) be a permutation of x, y, z. Multiplying the equation corresponding to (abc) by $\operatorname{sign}(abc)$ and adding all these equations, we will get the Jacobi identity. The proposition is proved.

The latter proposition shows that Poisson brackets are natural initial infinitesimal objects for noncommutative deformations of commutative algebras. The Leibniz rule and the Jacobi identity are needed for existence of deformations of degrees one and two in $\hbar$, respectively.

3. KOSZUL COMPLEX AND WEYL ALGEBRA

Let A be a commutative associative algebra with unit over a field $\mathbf{k}$ of characteristic zero.

Let B be a bigraded A-algebra (noncommutative). We will regard B as a super-algebra, an element of which $x \in B$ of degree (p, q) is even (odd) if the number q is even (odd). Denote by $\tilde{x}$ the parity of x. Then the commutator of two elements $x, y \in B$ is defined as $[x, y] = xy - (-1)^{\tilde{x}\tilde{y}} yx$.

An A-linear operator in B is said to be of degree (r, s) if it sends elements of degree (p, q) into elements of degree $(p+r, q+s)$. So the set of A-linear operators in B may also be considered as bigraded super-algebra: an operator of degree (r, s) is even or odd depending on the parity of s.

An operator D is called a derivation of degree $\tilde{D}$ if the following equality holds

$$D(xy) = D(x)\, y + (-1)^{\tilde{x}\tilde{D}}\, xD(y).$$

Note that all derivations form a Lie super-algebra with respect to (super)-commutator. In particular, if D is an odd derivation, then D^2 is an even one, and the Bianchi identity holds:

$$[D, D^2] = D^3 - D^3 = 0.$$

Given an A-module E, we denote by $T(E)$, $S(E)$, and $\bigwedge E$ the tensor, symmetric, and exterior algebra over A, respectively.

Suppose $u: E \to F$ is a morphism of A-modules. Let us define an operator $d_u = d$ on the bigraded A-module $T(E) \otimes \bigwedge F$ in the following way. If $x = x_1 \otimes \cdots \otimes x_m \otimes y_1 \wedge \cdots \wedge y_n \in T^m(E) \otimes \bigwedge^n F$, we put

$$dx = \sum_i x_1 \otimes \cdots \otimes \hat{x}_i \otimes \cdots \otimes x_m \otimes u(x_i) \wedge y_1 \wedge \cdots \wedge y_n.$$

Similarly, we define an operator $\partial_u = \partial$ on the bigraded A-module $S(F) \otimes T(E)$ assigning to $x = y_1 \odot \cdots \odot y_m \otimes x_1 \otimes \cdots \otimes x_n \in S^m F \otimes T^n E$ the element

$$\partial x = \sum_i (-1)^{i-1} y_1 \odot \cdots \odot y_m \odot u(x_i) \otimes x_1 \otimes \cdots \otimes \hat{x}_i \otimes \cdots \otimes x_n.$$

Proposition 3.1. *The operators d and ∂ are A-linear derivations of degrees $(-1, 1)$ and $(1, -1)$, respectively, on corresponding algebras considered as super-algebras with respect to the second degree. Moreover, $d^2 = 0$, $\partial^2 = 0$.*

Proof. Straightforward computation.

Given a bilinear skew-symmetric form $\varphi : \bigwedge^2 E \to A$, denote by I the ideal in $T(E)$ generated by relations

$$x \otimes y - y \otimes x - \varphi(x, y) = 0. \tag{1}$$

We will call $W(E) = T(E)/I$ the Weyl algebra associated to φ. The operator d induces a derivation on $W(E) \otimes \bigwedge F$. Indeed, d applied to the left side of (1) gives zero.

Analogously, given a bilinear symmetric form $\psi : S^2(E) \to A$, denote by J the ideal in $T(E)$ generated by relations

$$x \otimes y + y \otimes x - \psi(x, y) = 0. \tag{2}$$

Since ∂ applied to (2) gives zero, ∂ induces a derivation on $S(F) \otimes C(E)$, where $C(E) = T(E)/J$ (the Clifford algebra associated to ψ).

In particular, if $\varphi = 0$ we get Koszul complexes $\mathrm{Kos}^\bullet(u) = (S(E) \otimes \bigwedge F, d)$ and $\mathrm{Kos}_\bullet(u) = (S(F) \otimes \bigwedge E, \partial)$ (see e.g. [13], pp. 107–113).

Let us introduce an independent variable $\hbar$ and consider the modules $E[\hbar] = E \otimes_{\mathbf{k}} \mathbf{k}[\hbar]$ and $F[\hbar] = F \otimes_{\mathbf{k}} \mathbf{k}[\hbar]$ over the algebra $A[\hbar] = A \otimes_{\mathbf{k}} \mathbf{k}[\hbar]$. In this case we consider the following relations

$$x \otimes y - y \otimes x - \hbar\varphi(x, y) = 0, \tag{3}$$

$$x \otimes y + y \otimes x - \hbar\psi(x, y) = 0 \tag{4}$$

rather than (1) and (2).

Let us form as above the algebras $W(E[\hbar])$ and $C(E[\hbar])$. If we now regard $\hbar$ as being of degree two, then the relations (3) and (4) are homogeneous and $W(E[\hbar])$ and $C(E[\hbar])$ become graded algebras. Moreover, d and ∂ become derivations of degrees $(-1, 1)$ and $(1, -1)$.

Throughout the remainder of this paper we will often drop the reference to $\hbar$ in our notations. Thus, we will write E, A rather than $E[\hbar]$, $A[\hbar]$, and so on.

Consider in more detail the case in which $u\colon E \to F$ is an isomorphism of free modules of finite rank. If such is the case, the Koszul complexes $\mathrm{Kos}^{\bullet}(u)$ and $\mathrm{Kos}_{\bullet}(u^{-1})$ are equal to $S(E) \otimes \bigwedge F$ as algebras and we get the derivations d of degree $(-1, 1)$ and ∂ of degree $(1, -1)$ on this algebra.

PROPOSITION 3.2. *Let $u\colon E \to F$ be an isomorphism of free modules of finite rank. Then $(\partial d + d\partial)\, x = (p+q)\, x$ for $x \in S^p(E) \otimes \bigwedge^q F$.*

Proof. Straightforward computation.

In particular, $(\partial d + d\partial)\, x = 0$ if and only if $x \in S^0(E) \otimes \bigwedge^0 F = A$.

Notice, that there is a natural embedding of A-modules

$$\sigma\colon S(E) \otimes \bigwedge F \to T(E) \otimes \bigwedge F,$$

generated by the embeddings $S^n(E) \to T^n(E)$ for any n.

Let φ be a skew-symmetric form on E, and $W(E)$ the corresponding quotient algebra of $T(E)$, as above. Denote by π the natural projection $T(E) \wedge F \to W(E) \wedge F$. By the Poincaré–Birkhoff–Witt theorem the composition $\pi\sigma$ gives an isomorphism of A-modules $S(E) \otimes \bigwedge F \to W(E) \otimes \bigwedge F$. Due to this isomorphism, the operator ∂ can be carried onto $W(E) \otimes \bigwedge F$. Of course, it will not be a derivation, but the relation $(\partial d + d\partial)\, x = (p+q)\, x$ for $x \in W(E) \otimes \bigwedge F$ remains true.

Given $a \in W(E) \otimes \bigwedge F$, we put $s(a) = (\pi\sigma)^{-1}(a) \in S(E) \otimes \bigwedge F$ and call $s(a)$ the symbol of a. We say that a has s-degree (n, m) if $s(a) \in (S^n(E) \otimes \bigwedge^m F)[\hbar]$. We will also say that a has s-degree n if $s(a) \in (S^n(E) \otimes \bigwedge F)[\hbar]$. When a has s-degree n, then da and ∂a have s-degrees $(n-1)$ and $(n+1)$, respectively. It follows from the fact that the operators d and ∂ commute with $\pi\sigma$ and from the explicit forms of d and ∂ in $S(E) \otimes \bigwedge F$.

So, we get

PROPOSITION 3.3. *Let $u\colon E \to F$ be an isomorphism of free A-modules of finite rank. Given a skew-symmetric form φ on E, suppose $W(E) = W(E[\hbar])$ is the corresponding graded algebra. Then there exist a derivation d on $W(E) \otimes \bigwedge F$ of degree $(-1, 1)$ and an A-linear operator ∂ of degree $(1, -1)$ such that the following equality holds*

$$(\partial d + d\partial)\, x = (p+q)\, x \qquad \text{if} \quad s(x) \in W^p(E) \otimes \bigwedge^q F. \tag{5}$$

Moreover, d and ∂ have s-degree $(-1, 1)$ and $(1, -1)$, respectively.

Let us consider the complex

$$W^{\bullet}\colon W(E) \xrightarrow{d} W(E) \otimes \bigwedge^1 F \xrightarrow{d} W(E) \otimes \bigwedge^2 F \xrightarrow{d} \cdots.$$

COROLLARY 3.4. *In the hypothesis of Proposition* 3.3 *we have*

$$H^0(W^{\bullet}) = A, \qquad H^i(W^{\bullet}) = 0 \qquad \textit{for} \quad i > 0.$$

Proof. It follows from equality (5).

Remark. Let us assume that A is a sheaf of algebras over a topological space M, $u\colon E \to F$ is an isomorphism of locally free sheaves of A-modules, and φ is a mapping of sheaves $\bigwedge^2 E \to A$. Then all constructions above make sense in this situation, including Proposition 3.3 and Corollary 3.4, where $W(E) \otimes \bigwedge F$ becomes a sheaf of A-algebras.

4. COMPLETED WEYL ALGEBRA AND DERIVATIONS

Suppose E is a free A-module of finite rank, φ is a skew-symmetric form on E. Let us construct the Weyl algebra $W(E) - W(E[\hbar])$ with respect to φ, as in the preceding section. This algebra is a graded algebra, in which the elements from E have degree one and $\hbar$ is of degree two.

The gradation in $W(E)$ induces a decreasing filtration in this algebra with submodules $W_p(E)$, generated by elements of degree $\geqslant p$. A non-homogeneous element $a \in W(E)$ will be called of degree p, if p is the maximal number such that $a \in W_p(E)$.

Let us form the completion $\hat{W}(E)$ of $W(E)$ with respect to this filtration. In this case $\hat{W}(E)$ can be regarded as a module over the algebra $\hat{A} = A[[\hbar]]$ of power series in $\hbar$.

Let $u\colon E \to F$ be an isomorphism of modules. It is clear that one can complete the complex $W^{\bullet} = (W(E) \otimes \bigwedge F, d)$ defined in the preceding section and get the complex $\hat{W}^{\bullet} = (\hat{W}(E) \otimes \bigwedge F, d)$. Moreover, the operator ∂ makes sense in this case and Propositions 3.2 and 3.3 remain true. Further we will use the notations $W(E)$ and $W^{\bullet}$ for the completed algebras $\hat{W}(E)$ and $\hat{W}^{\bullet}$.

As in Section 3 we will consider $W(E) \otimes \bigwedge F$ as a bigraded algebra and as a super-algebra: an element of this algebra of the degree (p, q) is even or odd depending on the parity of q. Thus, by derivations of $W(E) \otimes \bigwedge F$ we mean super-derivations, by commutator of two elements from $W(E) \otimes \bigwedge F$ we mean super-commutator, and so on.

Later we will need the following description of A-linear derivations of nondegenerate Weyl algebras. Assume φ is a nondegenerate skew-symmetric form on E. This means that the mapping $\phi\colon E \to E^*$, $x \mapsto \varphi(x, \cdot)$, from E into the dual module is an isomorphism. The Weyl algebra $W(E)$ associated to such a φ will be called nondegenerate.

PROPOSITION 4.1. *Any A-linear derivation D of a nondegenerate Weyl algebra $W(E)$ is an inner one, i.e. there exists an element $v \in W(E)$ such that $D(x) = (1/\hbar)[v, x]$ for any $x \in W(E)$. (Recall, that we use the form $\hbar\varphi$ in the definition of $W(E)$).*

Proof. Let us consider the complex $(W(E) \otimes \bigwedge E^*, d)$ associated to the mapping $\phi\colon E \to E^*$, $x \mapsto \varphi(\cdot, x)$. Let e_i be a basis in E over A and e^i be the dual basis in E^*, $e^i(e_j) = \delta^i_j$. It is easy to verify that the operator d has the form $1/\hbar[\bar{d}, \cdot]$, where

$$\bar{d} = \sum_i e_i \otimes e^i \in E \otimes \bigwedge^1 E^*.$$

One has the equalities

$$[De_i, e_j] + [e_i, De_j] = 0 \qquad \text{for any } i, j.$$

Form the element

$$\bar{D} = \frac{1}{\hbar} \sum_i De_i \otimes e^i \in W(E) \otimes \bigwedge^1 E^*.$$

The last equation implies that $d\bar{D} = 0$. It follows from the exactness of the complex $(W(E) \otimes \bigwedge E^*, d)$ that there exists an element $v \in W(E)$ such that

$$-dv = \frac{1}{\hbar} \sum_i [v, e_i] \otimes e^i = \frac{1}{\hbar} \sum_i De_i \otimes e^i,$$

i.e. $1/\hbar[v, e_i] = De_i$ for all i. Thus the operator $1/\hbar[v, \cdot]$ coincides with D. The proposition is proved.

Let us consider derivations in the superalgebra $W(E) \otimes \bigwedge F$ which (super)-commute with the multiplication by elements from $\bigwedge F$. Thus, the operator d is such a derivation. It follows from the last proposition that such derivations are inner.

Remark. As in the preceding section, all the constructions of this section make sense in the global case as well, when A is a sheaf of algebras over a topological space M, etc. In this case free A-modules are replaced with locally free sheaves of A-modules. Proposition 4.1 is true locally.

In order for any global derivation D in that proposition to be inner, one needs to suppose that $H^1(A, M) = 0$, where $H^i(\mathcal{F}, M)$ denotes the ith cohomology of a sheaf $\mathcal{F}$ over M.

5. POISSON BRACKETS AND HAMILTONIAN DERIVATIONS

We recall that a Poisson bracket on a **k**-algebra A is a skew-symmetric **k**-linear form $f\colon \bigwedge^2 A \to A$ which satisfies two conditions:
the Leibniz rule

$$f(ab, c) = af(b, c) + bf(a, c),$$

and the Jacobi identity

$$f(f(a, b), c) + f(f(b, c), a) + f(f(c, a), b) = 0.$$

It follows from the Leibniz rule that f defines a mapping $\bar{f}\colon A \to \mathrm{Der}(A)$, namely, $\bar{f}(a) = f(a, \cdot)$ where $\mathrm{Der}(A)$ denotes the A-module of derivations of the algebra A.

Note that the Jacobi identity implies that $H = \mathrm{Im}(\bar{f})$ forms a Lie algebra over **k**. Indeed, let $a, b, c \in A$ and $\bar{a} = \bar{f}(a)$, $\bar{b} = \bar{f}(b)$. We have by definition $\bar{a}c = f(a, c)$, $\bar{b}c = f(b, c)$, hence $\bar{a}\bar{b}c = f(a, f(b, c))$. Similarly, $\bar{b}\bar{a}c = f(b, f(a, c))$. Therefore,

$$(\bar{a}\bar{b} - \bar{b}\bar{a})\, c = f(a, f(b, c)) - f(b, f(a, c)) = f(f(a, b), c)$$

by the Jacobi identity. This means that

$$\bar{a}\bar{b} - \bar{b}\bar{a} = \overline{f(a, b)},$$

i.e. $\bar{a}\bar{b} - \bar{b}\bar{a}$ is the image of $f(a, b)$ by the mapping $\bar{f}$.

It should be noted that H is not an A-module. We will call elements from H strong Hamiltonian derivations. The bracket f forms a **k**-linear non-degenerate skew-symmetric form φ on H with values in A in the following way. Let $x = \bar{f}(a)$, $y = \bar{f}(b)$, then we put $\varphi(x, y) = f(a, b)$. It is clear that such a definition of φ makes sense. Indeed, suppose $x_i \in A$, and $\bar{x}_i$ are the corresponding strong Hamiltonian derivations. We must show that if $D = \sum_i a_i \bar{x}_i = 0$, $a_i \in A$, then $\sum_i a_i \varphi(\bar{x}_i, \bar{b}) = 0$ for any $b \in A$. But that follows from the following chain of equalities

$$\sum_i a_i \varphi(\bar{x}_i, \bar{b}) = \sum_i a_i f(x_i, b) = Db = 0.$$

Let us denote by E the A-submodule in $\mathrm{Der}(A)$ generated by H and call the elements of it weak Hamiltonian derivations. It is easy to see that E

forms a Lie subalgebra in $\mathrm{Der}(A)$ over A, and the form φ can be extended by linearity to E. So, we obtain on E an A-linear skew-symmetric form $\varphi\colon \bigwedge^2 E \to A$ associated to our Poisson bracket on A. We assume further that E is an A-module of finite type.

Note that the mapping $\phi\colon E \to E^*$, $x \mapsto \varphi(x, \cdot)$, is always a monomorphism. This follows from the fact that if $\varphi(x, y) = 0$ for a fixed x and any y, $x, y \in E$, then $x = 0$. It is sufficient to prove the fact for all y of the form $\bar{b}$, where $b \in A$. But, if $x = \sum_i a_i \bar{x}_i$, $a_i \in A$, is zero, where $\bar{x}_i$ are strong Hamiltonian derivations corresponding to $x_i \in A$, then

$$\varphi(x, \bar{b}) = \sum_i a_i f(x_i, b) = x(b),$$

for any $b \in A$; i.e. the derivation x applied to b is equal to zero.

The module E need not be free (or locally free in the global situation). We will call a Poisson bracket free (locally free in the global case) if the associated module (or sheaf) of weak Hamiltonian derivations E is (locally) free. We will call a Poisson bracket f nonsingular if the corresponding map $\phi\colon E \to E^*$ is an isomorphism.

Given a locally free Poisson bracket f on A, let us construct the Weyl algebra $W(E)$ and the algebra $W^\bullet(E) = (W(E) \otimes \bigwedge E^*, d)$ associated to the monomorphism $\phi\colon E \to E^*$, as in preceding sections.

The Lie algebra E acts on A as derivations, so one can associate to any $a \in A$ the differential form ∇a, $\nabla a(x) = xa$. Hence, ∇ can be considered as a mapping $A \to \bigwedge^1 E^*$ with the property $\nabla(ab) = a\nabla(b) + b\nabla(a)$.

Moreover, the operator ∇ can be extended to a derivation on the exterior algebra $\bigwedge E^*$ in the following way. Consider $\bigwedge^n E^*$ as the algebra of A-linear skew-symmetric functions on E of n variables with values in A. Then for $\rho \in \bigwedge^{n-1} E^*$ we define $\nabla(\rho) \in \bigwedge^n E^*$ by

$$\begin{aligned}\nabla(\rho)(x_1, ..., x_n) = &\sum_{1 \leqslant i < j \leqslant n} (-1)^{i+j} \rho([x_i, x_j], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_n)\\ &+ \sum_{1 \leqslant i \leqslant n} (-1)^{i-1} x_i \rho(x_1, ..., \hat{x}_i, ..., x_n).\end{aligned}$$

So the definition of ∇ is the same as for the de Rham complex and for the cohomology complex for Lie algebra. It is easy to verify that ∇ is an odd derivation of the algebra $\bigwedge E^*$ and $\nabla^2 = 0$ on $\bigwedge E^*$. It turns out that ∇ can be extended to the whole algebra $W^\bullet = W(E) \otimes \bigwedge E^*$.

PROPOSITION 5.1. *Let f be a locally free Poisson bracket on A. Then the derivation ∇ can be extended to a derivation on $W^\bullet(E)$ of degree* $(0, 1)$ *with the property*

$$d\psi = 0, \tag{1}$$

where $\psi \in E \otimes \bigwedge^2 E^*$ *is the tensor of torsion defined as* $\psi(x, y) = \nabla_x(y) - \nabla_y(x) - [x, y]$ *for any* $x, y \in E$ *(here* $[\cdot, \cdot]$ *denotes the Lie bracket in* E*).*

Proof. First of all, reduce the proposition to the case in which E admits a basis over A consisting of strong Hamiltonian derivations.

Let Spec(A) be the spectrum of A with Zariski topology and structure sheaf $\mathscr{A}$, i.e. the affine scheme associated to A. The sheaf $\mathscr{A}$ is a sheaf of local algebras such that $A = H^0(\mathscr{A})$. The Poisson bracket f induces in a natural way the Poisson bracket $\tilde{f}$ on $\mathscr{A}$. On the other hand, there is the sheaf $\mathscr{E}$ on Spec(A) corresponding to E, which is a locally free $\mathscr{A}$-module, and $E = H^0(\mathscr{E})$. It is easy to see that $\mathscr{E}$ will be the sheaf of germs of weak Hamiltonian derivations of $\tilde{f}$. Denote by $\mathscr{E}_p$ and $\mathscr{A}_p$ the stalks of the sheaves $\mathscr{E}$ and $\mathscr{A}$ over the point $p \in \mathrm{Spec}(A)$. By definition, $\mathscr{E}_p$ is generated by strong Hamiltonian derivations as $\mathscr{A}_p$-module. Since $\mathscr{E}_p$ is a free module over the local algebra $\mathscr{A}_p$, one can extract from these Hamiltonian derivations a finite basis of $\mathscr{E}_p$ over $\mathscr{A}_p$. This basis will be a basis of the A_U-module E_U, where A_U and E_U denote the spaces of sections of $\mathscr{A}$ and $\mathscr{E}$ over some neighborhood U of p. So, we see that any point of Spec(A) has a neighborhood in which the module of weak Hamiltonian derivations admits a basis consisting of strong Hamiltonian derivations. Consider a covering U_i of $M = \mathrm{Spec}(A)$ by such neighborhoods and suppose that the proposition is true for the algebras A_{U_i} with induced Poisson brackets f_{U_i}. Now, let us apply the arguments from the proof of Proposition 5.2 below to the Poisson bracket $\tilde{f}$ given on the sheaf $\mathscr{A}$ over the space $M = \mathrm{Spec}(A)$. Note that in this case $H^1(\mathscr{T}, M) = 0$, because $\mathscr{T}$ is a quasi-coherent sheaf on affine algebraic space M (see the definition of $\mathscr{T}$ just before Proposition 5.2). So we conclude that the proposition will be true if it is true for each pair A_{U_i}, f_{U_i}. Thus, it suffices to prove the existence of the required derivation ∇ in supposing that A-module E admits a strong Hamiltonian basis.

Let e_i be a basis in E over A consisting of strong Hamiltonian derivations. Let us define $\nabla_{e_i}(e_j) = [e_i, e_j]$ (here we mean by $[\cdot, \cdot]$ the bracket in the Lie algebra E). If $x = \sum_i a_i e_i$, $a_i \in A$, is an element from E we set $\nabla_{e_j}(x) = \nabla_{e_j}(a_i)\, e_i + a_i \nabla_{e_j}(e_i)$. If y is another element from E, it can also be presented uniquely as a linear combination $\sum_i c_i e_i$ with $c_i \in A$. Hence, defining $\nabla_y(x) = \sum_i c_i \nabla_{e_i}(x)$, we get a linear mapping $\nabla\colon E \to E \otimes \bigwedge^1 E^*$, $x \mapsto \nabla \cdot (x)$. The operator ∇ defined just above is a connection along weak Hamiltonian derivations.

Let us prove now that the form φ on E associated to the Poisson bracket f is an invariant one relative to the connection ∇, i.e.

$$\nabla_z(\varphi(x, y)) = \varphi(\nabla_z x, y) + \varphi(x, \nabla_z y) \qquad \text{for any} \quad x, y, z \in E.$$

It is sufficient to verify it for the elements e_i of our basis. Suppose $a_i \in A$ are such elements that $e_i = f(a_i, \cdot)$. Then, if we take $x = e_i$, $y = e_j$, $z = e_k$, the previous equation will be equivalent to the relation $f(a_k, f(a_i, a_j)) = f(f(a_k, a_i), a_j) + f(a_i, f(a_k, a_j))$, which is true by the Jacobi identity.

Now we can extend ∇ on the whole algebra $T(E) \otimes \bigwedge E^*$ as an odd operator by the Leibniz rule.

Let us prove that the ideal $I \subset T(E)$ generated by the relations

$$x \otimes y - y \otimes x = \hbar\varphi(x, y), \qquad x, y \in E, \tag{2}$$

is invariant under the action of ∇. Applying ∇ to the left side of (2) we obtain

$$\nabla[x, y] = [\nabla x, y] + [x, \nabla y], \tag{3}$$

where $[x, y] = x \otimes y - y \otimes x$, the commutator in $T(E)$. Applying ∇ to the right side of (2) we obtain

$$\hbar\nabla(\varphi(x, y)) = \hbar(\varphi(\nabla x, y) + \varphi(x, \nabla y)) \tag{4}$$

because of the invariance of φ. It is obvious that the difference between (3) and (4) belongs to I, which proves the invariance of I under the action of ∇. It follows from this that ∇ induces a well-defined derivation on $W^{\bullet}(E) = T(E)/I \otimes \bigwedge E^*$.

To verify (1), note that for elements of the basis we have

$$\psi(e_i, e_j) = \nabla_{e_i}(e_j) - \nabla_{e_j}(e_i) - [e_i, e_j] = [e_i, e_j]$$

by definition. On the other hand, the element ψ is obviously A-bilinear. Hence, ψ has the form $\psi = \sum_{i,j} [e_i, e_j]\, e^i \wedge e^j$. Recall that the operator d has the form $1/\hbar[\bar{d}, \cdot]$ (see the proof of Proposition 4.1). Using (2) we get $d\psi(e_i, e_j, e_k) = \mathrm{Alt}_{ijk}\, \varphi(e_i, [e_j, e_k]) = f(a_i, f(a_j, a_k)) + f(a_j, f(a_k, a_i)) + f(a_k, f(a_i, a_j)) = 0$, by the Jacobi identity. It implies that this equality holds for any elements $x, y \in E$. This completes the proof.

Now we want to extend the last proposition to the global situation. Thus, let A be a sheaf of algebras over a topological space M endowed with a Poisson bracket f, which is a global section of the sheaf $Hom(\bigwedge A^2, A)$. We construct as above the sheaf E of weak Hamiltonian derivations with the A-linear form φ. The **k**-linear differential operator ∇ is defined on A as well.

Let us denote by $sp(E)$ the sheaf of germs of symplectic A-linear endomorphisms of E. By definition, such an endomorphism Q preserves the form φ, i.e. $\varphi(Qx, y) + \varphi(x, Qy) = 0$, $x, y \in E$. Denote by $\mathscr{T}$ the subsheaf of sections s of $sp(E)\otimes E^*$ such that $ds=0$ for any $x, y\in E$. (Here we consider $sp(E)\otimes E^*$ as a subsheaf of $E\otimes E^*\wedge E^*$.) Since d is A-linear operator, $\mathscr{T}$ is a sheaf of A-modules.

PROPOSITION 5.2. *Given a locally free Poisson bracket on a sheaf of algebras over a topological space M, suppose that $H^1(\mathscr{T}, M)=0$. Then the operator ∇ can be extended to a derivation on $W^\bullet(E)$ of degree* $(0, 1)$ *with the property* (1).

Proof. Let $\{U_i\}$ be an open covering of M such that the sheaf E is free over each U_i. By Proposition 5.1, there exist extensions ∇_i of ∇ over each U_i. A direct check shows that the differences $\nabla_{ij}=\nabla_i-\nabla_j$ form A-linear derivations of degree $(0, 1)$ over $U_{ij}=U_i\cap U_j$, and are sections of the sheaf $\mathscr{T}$ over U_{ij}. Moreover, they obviously form a Čech cocycle on M. The condition $H^1(\mathscr{T}, M)=0$ means that there exist sections s_i of $\mathscr{T}$ over U_i such that $s_i-s_j=\nabla_{ij}=\nabla_i-\nabla_j$. Hence, $\nabla_i-s_i=\nabla_j-s_j$ is a global operator ∇' on A. It is easy to see that the operator ∇' is the required derivation on $W^\bullet(E)$. The proposition is proved.

The operator ∇ constructed above is a connection along weak Hamiltonian derivations. This connection also determines derivations on the algebras $T(E)\otimes\bigwedge E$ and $S(E)\otimes\bigwedge E$ in the same way. Moreover, this connection commutes with taking of symbol, i.e.

$$s(\nabla(a))=\nabla(s(a)) \qquad \text{for any} \quad a\in W(E)\otimes\bigwedge E^*.$$

It is easy to verify that the operators ∇^2 and $\nabla d+d\nabla$ are A-linear derivations on $W(E)\otimes\bigwedge E^*$ (super)commuting with elements from $\bigwedge E^*$. Let us assume that φ is a nondegenerate form. Then, as was shown in the preceding section, there exist such elements $\alpha, \beta\in W(E)\otimes\bigwedge^2 E^*$ that ∇^2 and $\nabla d+d\nabla$ have the forms $1/h\,\mathrm{ad}\,\alpha=1/h[\alpha, \cdot\,]$ and $1/h\,\mathrm{ad}\,\beta=1/h[\beta, \cdot\,]$, respectively.

We will need the following.

LEMMA 5.3. *Let ∇ satisfy the property* (1). *Then the elements α and β satisfy the following relations*:

$$\nabla\alpha=0,$$

$$d\beta=0,$$

$$(d+\nabla)(\alpha+\beta)=0.$$

Proof. There are no difficulties in seeing that the operator ∇ preserves s-degree and that the elements α and β have to be of s-degree $(2, 2)$ and

(1, 2). By the Jacobi identity, $\mathrm{ad}(\nabla\alpha) = [\nabla, \nabla^2] = 0$. It follows from this that $\nabla\alpha$ must commute with all elements from $W(E)$, i.e., be of s-degree zero. But on the other hand, $\nabla\alpha$ has s-degree 2. This implies that $\nabla\alpha = 0$, because the center of the algebra $W(E)$ consists of elements of s-degree zero, which proves the first relation.

Further, the operator d is realized as $1/\hbar[\bar{d}, \cdot\,]$ where $\bar{d} = \sum_i e_i \otimes e^i$, so $\mathrm{ad}(\beta) = \mathrm{ad}(\nabla\bar{d})$. This means that $\beta = \nabla\bar{d} + c$, where c is some central element. It implies that $d\beta = d(\nabla\bar{d})$. But it is easy to check that $\nabla\bar{d} = (\nabla_{e_i} e_j - \nabla_{e_j} e_i - [e_i, e_j]) \otimes e^i \wedge e^j = \psi$. So, the second relation follows from the property (1).

To prove the third relation, note that $a = (d + \nabla)(\alpha + \beta) = d\alpha + \nabla\beta$ due to the first two relations. On the other hand, $\mathrm{ad}\, a = [d + \nabla, (d + \nabla)^2] = 0$ by the Bianchi identity (see 3). But the s-degree of a has to be equal to 1, because two summands of a have this s-degree. So, a must be equal to zero, which proves the third relation of the lemma.

6. A TOPOLOGICAL LEMMA

In this section we will denote by E an Abelian group with filtration $E = \cdots \supset E_i \cdots$. We will assume that i runs over all the integers and $\bigcup E_i = E$, $\bigcap E_i = 0$. The degree of an element $x \in E$ is the maximal number i such that $x \in E_i$. We will suppose that any element from E has a finite degree, and denote by $deg(x)$ the degree of x.

The filtration defines on E a topology: the E_i form a fundamental system of neighborhoods of zero. Every group with such a filtration can be completed with respect to that topology. Henceforth we assume that E is complete.

Let $\Phi\colon E \to E$ be an arbitrary mapping (in the set-theoretic sense). The following simple lemma gives a criteria for the operator $Id + \Phi$ to be invertible.

LEMMA 6.1. *Let E be a complete Abelian group with filtration. Suppose that an operator $\Phi\colon E \to E$ satisfies the following condition*:

$$deg(\Phi(x) - \Phi(y)) > deg(x - y).$$

Then the operator $Id + \Phi$ is invertible.

Proof. The lemma will be proved if we establish the existence and uniqueness of a solution of the equation $b = x + \Phi(x)$, where $b \in E$ is given. Put

$$x_0 = b - \Phi(b), \qquad x_1 = b - \Phi(x_0), \ldots, x_k = b - \Phi(x_{k-1}), \ldots$$

The sequence (x_k) is convergent. Indeed, $x_{k+1}-x_k=\Phi(x_k)-\Phi(x_{k-1})$. This implies that

$$deg(x_{k+1}-x_k)=deg(\Phi(x_k)-\Phi(x_{k-1}))>deg(x_k-x_{k-1}),$$

which proves the convergence. Let x be the limit of this sequence. Then it is obvious that x is a solution of our equation. If x' is an other solution, then it should be $deg(x-x')=deg(\Phi(x')-\Phi(x))>deg(x'-x)$, which is impossible. This proves uniqueness and the lemma.

There are no difficulties in seeing that both operators $Id+\Phi$ and its inverse are continuous in the topology associated to the filtration.

7. COMPLEX ASSOCIATED TO POISSON BRACKET

In this section we will again suppose that A is an algebra with a Poisson bracket f, and E is the module of weak Hamiltonian derivations of A. We also suppose that f is locally free and nonsingular. This means that the induced morphism $\phi: E\to E^*$ is an isomorphism (see 5). So, we can construct the completed complex

$$W^\bullet(E)=\left(W(E)\otimes\bigwedge E^*, d\right),$$

as in 4. Recall that the operator ∂ is well defined on this complex. It is clear that the operator ∇ constructed in 5 being of degree $(0, 1)$ is well defined in this complex as well. Denote by $W_2(E)$ the set of elements from $W(E)$ of degree $\geqslant 2$.

Given $r\in W^\bullet(E)$, let us denote by $\operatorname{ad} r$ the inner derivation $[r,\cdot]$ in $W^\bullet(E)$, where the bracket $[\cdot,\cdot]$ is regarded as a commutator in super-algebra.

We want to construct such an element $r\in W_2(E)\otimes\bigwedge^1 E^*$ that the derivation $D=d+\nabla+1/h\operatorname{ad} r$ will satisfy the property $D^2=0$. Let us find such r.

We have

$$D^2=\nabla^2+d\nabla+\nabla d+\frac{1}{h}[d,\operatorname{ad} r]+\frac{1}{h}[\nabla,\operatorname{ad} r]+\left(\frac{1}{h}\operatorname{ad} r\right)^2. \tag{1}$$

Using the fact that ∇, d, $\operatorname{ad} r$ are odd, and using the Jacobi identity in the super-case, we get that $[d,\operatorname{ad} r]=\operatorname{ad}(dr)$, $[\nabla,\operatorname{ad} r]=\operatorname{ad}(\nabla r)$, $(\operatorname{ad} r)^2=\frac{1}{2}\operatorname{ad}[r,r]$. Moreover, a direct computation shows that the derivation $B=\nabla^2+d\nabla+\nabla d$ is A-linear. So, by Proposition 4.1 (see also Lemma 5.3) there exists an element $b=\alpha+\beta\in W_1(E)\otimes\bigwedge^2 E^*$ such that $B=1/\hbar[b,\cdot\,]=1/\hbar\ \mathrm{ad}\ b$. So we can rewrite (1) in the form

$$D^2=\mathrm{ad}\left(\frac{1}{\hbar}b+\frac{1}{\hbar}dr+\frac{1}{\hbar}\nabla r+\frac{1}{2}\left[\frac{1}{\hbar}r,\frac{1}{\hbar}r\right]\right). \tag{2}$$

Therefore, if we will be able to find an element r such that the equations

$$\frac{1}{\hbar}b+\frac{1}{\hbar}dr+\frac{1}{\hbar}\nabla r+\frac{1}{2}\left[\frac{1}{\hbar}r,\frac{1}{\hbar}r\right]=0 \tag{3}$$

holds, then the condition $D^2=0$ will be satisfied with this r.

PROPOSITION 7.1. *There exists $r\in W_2(E)\otimes\bigwedge^1 E^*$ such that Eq. (3) holds.*

Proof. At first, let us slightly modify the operator ∂. We set $\delta(a)=(1/p+q)\,\partial(a)$ for elements $a\in W^\bullet(E)$ of s-degree (p,q). Therefore, when a is of s-degree (p,q) with $q>0$ we have

$$(d\delta+\delta d)\,a=a. \tag{4}$$

Of course, $\delta^2=0$ as well.

The element $-\delta(b)$ obviously belongs to $W_2(E)\otimes\bigwedge^1 E^*$. Let us consider the following equation in $W_2(E)\otimes\bigwedge^1 E^*$

$$-\delta(b)=r+\delta\left(\nabla r+\frac{1}{2\hbar}[r,r]\right). \tag{5}$$

It is easy to see that the operator $\Phi(r)=\delta(\nabla r+1/2\hbar[r,r])$ satisfies the hypothesis of Lemma 6.1, because δ increases degree by one with respect to the filtration defined on $W(E)$. So, using this lemma we can find $r\in W_2(E)\otimes\bigwedge^1 E^*$ satisfying (5).

Let us show that this r satisfies (3) as well. Denote by a the left side of (3). Note that $\delta r=0$ by (5), so $\delta\,dr=r$ by (4). It implies that $\delta a=0$.

A direct computation shows that

$$\left(d+\nabla+\frac{1}{\hbar}\mathrm{ad}\ r\right)a=(d+\nabla)\left(\frac{1}{\hbar}b\right)=\frac{1}{\hbar}(d+\nabla)(\alpha+\beta).$$

But the right-hand side expression is equal to zero due to Lemma 5.3. So,

$$Da=da+\nabla a+\frac{1}{\hbar}[r,a]=0. \tag{6}$$

Since $\delta a = 0$, $\delta\, da = a$ by (4), and we get from (6) $a = -\delta(\nabla a + 1/\hbar[r, a])$. But this is possible only if $a = 0$, because the degree of the right side of this equation is greater at least by one than the degree of the left side. It proves that r satisfies the equation (3). The proposition is proved.

So, we have constructed a derivation on the algebra $W^\bullet(E)$ of the form $D = d + \nabla + 1/\hbar\, \mathrm{ad}\, r$ and such that $D^2 = 0$. Therefore, one can write the following complex

$$DW^\bullet\colon W(E) \xrightarrow{D} W(E) \otimes \bigwedge^1 E^* \xrightarrow{D} W(E) \otimes \bigwedge^2 E^* \xrightarrow{D} \cdots.$$

The cohomology $H^0(DW^\bullet)$ of this complex is a subalgebra $A_\hbar$ of the algebra $W(E)$. In the next section we will show that the complexes $DW^\bullet = (W^\bullet, D)$ and $(W^\bullet, d)$ are isomorphic as $\mathbf{k}[[\hbar]]$-modules, so $A_\hbar$ coincides with $A[[\hbar]]$ as $\mathbf{k}[[\hbar]]$-module. Moreover, we will see that $A_\hbar$ is a quantization of A by our Poisson bracket f.

Remark. As in the preceding sections, note that in the case when A is a sheaf of algebras over a topological space M construction of the derivation D can be realized globally.

8. QUANTIZATION

PROPOSITION 8.1. *There exists a* $\mathbf{k}[[\hbar]]$*-linear operator* Q *on* $W(E)$ *such that* $d = QDQ^{-1}$*, therefore, this operator gives an isomorphism of the complexes* $DW^\bullet = (W^\bullet(E), D)$ *and* $(W^\bullet(E), d)$*. (We assume here that* Q *acts on* $W(E) \otimes \bigwedge E^*$ *as* $Q \otimes 1$*.)*

Proof. Let us put $Q = Id + \delta(\nabla + 1/\hbar\, \mathrm{ad}\, r)$ and prove that it is an operator as required in the proposition. First of all, Q is invertible by Lemma 6.1, because δ increases degree. We have to show that $dQ - QD = 0$, i.e.

$$d(Id + \delta(D - d)) - (Id + \delta(D - d))\, D = 0. \tag{1}$$

But $\delta(D - d)\, D = -\delta\, dD = -\delta d(D - d)$, because $D^2 = d^2 = 0$. Using this in (1) we get $d - D + (d\delta + \delta d)(D - d) = 0$, which is true because $d - D$ is a derivation of degree 1 with respect to the second degree, so $(d\delta + \delta d)(D - d) = D - d$. Proposition is proved.

Thus, the subalgebras $A[[\hbar]]$ and $A_\hbar$ of $W(E)$ are isomorphic as $k[[\hbar]]$-modules and $Q^{-1}\colon A[[\hbar]] \to A_\hbar$ realizes this isomorphism.

The operator Q^{-1} has the form

$$Q^{-1} = Id - \delta\left(\nabla + \frac{1}{\hbar}\,\mathrm{ad}\,r\right) + \left(\delta\left(\nabla + \frac{1}{\hbar}\,\mathrm{ad}\,r\right)\right)^2 + \cdots.$$

Let us apply it to elements $a, b \in A$. We obtain

$$Q^{-1}a = a - \delta\nabla(a) + \cdots, \qquad Q^{-1}b = b - \delta\nabla(b) + \cdots.$$

Taking into account that A lies in the center of $W(E)$, we get

$$[Q^{-1}a, Q^{-1}b] = [\delta\nabla(a), \delta\nabla(b)] + \cdots. \tag{2}$$

But the first bracket in the right-hand side expression is equal to $\hbar f(a, b)$, which follows from the definitions of action of ∇ on elements from A and commutation in $W(E)$. Using the operator Q, the algebra $A_\hbar$ can be identified with $A[[\hbar]]$ as $\mathbf{k}[[\hbar]]$-module and the new multiplication in A has the form $a *_\hbar b = Q(Q^{-1}(a)\, Q^{-1}(b))$. Taking into account that the operator $\delta(\nabla + 1/\hbar\,\mathrm{ad}\,r)$ increases s-degree by one and the fact that the element $a *_\hbar b$ has s-degree zero, one can deduce, using (2), that the coefficient of $\hbar$ in $Q(Q^{-1}a, Q^{-1}b)$ is equal to $f(a, b)$ and the other terms have order in $\hbar$ greater than one. So, we have proved

PROPOSITION 8.2. *Let A be a sheaf of algebras on a topological space M, f a locally free nonsingular Poisson bracket on A. Suppose that $H^1(\mathscr{F}, M) = 0$ for the sheaves of A-modules over M (mentioned in preceding sections). Then there exists a quantization $A_\hbar$ of A by the bracket f.*

Remarks. (a) The construction of quantization shows that if $a \in A$ is an element such that $f(a, b) = 0$ for any $b \in A$, then a lies in the center of $A_\hbar$ with respect to the new multiplication.

(b) If a family f_t of Poisson brackets is given, the construction of quantization shows that this family can be quantized simultaneously.

COROLLARY 8.3. *Let K be a field of finite transcendence degree over* **k**. *Then any Poisson bracket on K can be quantized.*

Proof. Indeed, consider K as a sheaf over a point. Since K is of finite transcendence degree over **k**, the weak Hamiltonian derivations E form a finite-dimensional vector space over K, therefore E is a free K-module. The mapping $\phi: E \to E^*$ induced with the Poisson bracket, being a monomorphism (see 5) of vector spaces of the same dimensions, is an isomorphism. Hence, any Poisson bracket on the field is locally free and nonsingular, by definition, and the corollary follows from Proposition 8.2.

COROLLARY 8.4 ([14, 15, 16]). *Let M be a smooth manifold. Then any nondegenerate Poisson bracket on M can be quantized.*

Proof. It is clear that any nondegenerate Poisson bracket on M will be locally free and nonsingular. Moreover, $H^1(\mathscr{F}, M) = 0$ for any sheaf of modules over the sheaf of algebras of smooth functions on M.

The same argument shows that Corollary 8.4 remains true if one replaces the smooth manifold M by a complex analytic Stein manifold or an affine algebraic smooth variety. Indeed, all the sheaves $\mathscr{F}$ from Proposition 8.2 are coherent sheaves of modules over the structure sheaf. But, for these classes of spaces $H^1(\mathscr{F}, M) = 0$ for any coherent sheaf of modules.

We can also formulate an assertion on the quantization of an arbitrary Poisson bracket on Stein analytic spaces or affine algebraic varieties.

For example, let X be a reduced (i.e. without nilpotent elements in the structure sheaf) complex analytic space, and A the structure sheaf of X. Assume that $\mathscr{F}$ is a coherent analytic sheaf on X. It follows from the existence of a free resolution of $\mathscr{F}$ that one can find an analytic subset $Y \subset X$ of codimension one such that $\mathscr{F}$ will be locally free over A_Y. Here A_Y denotes the sheaf of meromorphic functions on X with poles in Y (see [19]).

It follows from this that if f is an arbitrary Poisson bracket on X, then there exists an analytic subset $Y \subset X$ of codimension one such that the corresponding sheaf E of weak Hamiltonian derivations is locally free, and the monomorphism $\phi: E \to E^*$ induces an isomorphism $A_Y \otimes_A E \to A_Y \otimes_A E^*$ of A_Y-modules. Therefore f determines a locally free nonsingular Poisson bracket on A_Y. So, we obtain from Proposition 8.2.

COROLLARY 8.5. *Let X be an reduced complex analytic Stein space. Suppose f is a Poisson bracket on X. Then there exists an analytic subset $Y \subset X$ of codimension one such that the algebra A_Y can be quantized by this bracket.*

A similar statement is valid for affine algebraic varieties.

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