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On the zeros of a class of generalised Dirichlet series-XV

To Professor A. Schinzel on his fifty-sixth birthday

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§1. INTRODUCTION

We begin with a brief introduction. The most important of all Dirichlet series is

$$\zeta(s) = \sum_{n=1}^{\infty}\left(\frac{1}{n^s} - \int_n^{n+1}\frac{du}{u^s}\right) + \frac{1}{s-1}, \qquad \sigma > 0,$$

(we write $s = \sigma + it$ as usual) studied by L. Euler and B. Riemann and the most important problem about it is to prove Riemann Hypothesis which asserts (I prefer to state it this way) that $\zeta(s) \neq 0$ in $\sigma > \frac{1}{2}$. (Of course the well-known functional equation gives us the formulation that all the non-real zeros of $\zeta(s)$ lie on $\sigma = \frac{1}{2}$). Next the Riemann-von-Mangoldt formula tells us that there are $(T/2\pi)\log(T/2\pi) - (T/2\pi) + O(\log T)$ zeros of $\zeta(\frac{1}{2} + it)$ in $0 < t \leq T$. All this has only the status of a conjecture. (It may be of some interest to mention here references to some unconditional results on $\zeta(s)$. Let $N_*(\alpha, T)$ denote the number of zeros of $\zeta(s)$ in $(\sigma > \alpha, 0 < t \leq T)$. We have various unconditional upper bounds for $N_*(\alpha, T)$ valid for $\alpha \geq \frac{1}{2}$ and all $T > 0$, (see chapter IX of [9]). All these are very far from $N_*(\frac{1}{2}, T) = 0$ (or even $N_*(\frac{3}{4}, T) = 0$) for all $T > 0$. We mention two computational results. The first result due to J. van de Lune, H.J.J. te Riele and D.T. Winter is $N_*(\frac{1}{2}, T_0) = 0$ for $T_0 = 10^8(5.45439823215)$. Also they showed that the zeros of $\zeta(s)$ on the line segment $\sigma = \frac{1}{2}$, $0 < t \leq T_0$ are all simple. The second result due to A.M. Odlyzko is that if $n = 10^{20}$ then the nth zero of $\zeta(s)$ in the critical strip is

$$\tfrac{1}{2} + i\,10^{19}(1.52024401159207472686290299\cdots).$$

The second result is in a preprint with the title 'The (10^{20})th zero of the Riemann zeta-function and its 70 million of its neighbours'. A.M. Odlyzko acknowledges some help by A. Schonage. The reference to the first result is 'On the zeros of the Riemann zeta-function in the critical strip-IV', Math. Comp. 46 (1986), 667–681). But in the present series of papers of which this is 15th, we are interested in proving some unconditional results about the zeros of a class of generalised Dirichlet series. Let us start with a generalised Dirichlet series (for precise definitions see §2)

$$F_0(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}.$$

Let α_0 denote the abscissa of absolute convergence of $\sum_{n=1}^{\infty} (a_n \lambda_n^{-s})^2$. Suppose that $F_0(s)$ can be continued analytically in $(\sigma \geq \alpha_0 - \delta,\ t \geq t_0)$ and there $|F_0(s)| < t^A$, where $A > 0$, $t_0 > 0$, $\delta > 0$ are some constants of which δ can be arbitrarily small but fixed. The general question whether $F_0(s)$ has infinity of zeros on $\sigma = \alpha_0$ is not the right question since (although Hardy's theorem, see chapter X of [9], tells us that $\zeta(s)$ has infinity of zeros on $\sigma = \frac{1}{2}$) it is possible, using the functional equation, to prove that there exist uncountably many real numbers a for which $\zeta(s) - a$ has no zeros at all on $\sigma = \frac{1}{2}$ (these and related questions will form the subject matter of a forthcoming paper). Several questions can be asked about the zeros of $F_0(s)$. For example does $F_0(s)$ have infinity of zeros in $(\sigma \geq \alpha_0 - \delta,\ t_0)$? (This is answered by us in the affirmative in the earlier papers I and II of this series. For references see the paper III[1]). But the question whether $F_0(s)$ has infinity of zeros in $(\alpha_0 - \delta \leq \sigma \leq \alpha_0 + \delta,\ t \geq t_0)$ is very difficult and is very much open. Let $R_0(\alpha, T)$ denote the rectangle $(\sigma \geq \alpha,\ T \leq t \leq 2T)$ and $N_0(\alpha, T)$ the number of zeros of $F_0(s)$ in it (of course we must have $\alpha \geq \alpha_0 - \delta$, and $T \geq t_0$). Qualitatively speaking, the only method of proving that $N_0(\alpha_0 - \delta, T) - N_0(\alpha_0 + \delta, T)$ is unbounded as $T \to \infty$ seems to be to prove first that $N_0(\alpha_0 - \delta, T) \gg T \log T$ for a suitable sequence $T = T_\nu \to \infty$ and next to prove that $N_0(\alpha_0 + \delta, T) \ll T$ for all T. We cannot pretend to have solved this problem in this generality and we proceed to introduce some important special results (of some what reasonable generality) which we have considered in some previous papers and those to be considered in the present paper. The main difference between the previous papers and the present one is that in the present one (and also in the next paper XVI in this series to appear; see §6 for a reference to this forthcoming paper) functional equations play an important role.

In a previous paper XIV[3] with the same title we considered 'approximations' to $\zeta(s)$ by a class of generalised Dirichlet series (or briefly GDS; precise definitions will be given in §2) and proved that the number of zeros of any member of the class in $(\sigma \geq \frac{1}{2} - \delta,\ T \leq t \leq 2T)$ is $\gg T \log T$ and in $(\sigma \geq \frac{1}{2} + \delta,\ T \leq t \leq 2T)$ is $\ll T$. (In fact we considered approximations to more general Dirichlet series than $\zeta(s)$, for example $\sum_{n=1}^{\infty} ((-1)^{n-1} n^{-s} \exp(\sqrt{\log n}))$ and proved the same results for any member of this class of approximations). To be very precise we proved the lower and upper bounds mentioned above, for a large class of GDS of which a member is

$$(1) \qquad \zeta(s) + \sum_{n=1}^{\infty} ((n+\alpha_n)^{-s} - n^{-s}), \quad (\sigma > 0),$$

where $\{\alpha_n\}$ is any sequence of real numbers with $|\alpha_n|$ bounded above by a small positive absolute constant. (In §5 of the present paper we sketch a proof, following the method of the present paper, that $|\alpha_n| \leq 10^{-5}$ will be enough, and complete a nice theorem about algebraic number fields). In §6 we present a proliferation of the results of the earlier sections. Lastly in §7 we prove a result which figures in the note below the definition of the functional equation, namely that $0 < k < 1$ is not possible.

The main content of the present paper is that 'If $Z(s)$ is a GDS which has a functional equation of a certain type and $F(s)$ is any member of a class of GDS which approximates to $Z(s)$ then $F(s)$ has $\gg T \log T$ zeros in $(\sigma \geq \frac{1}{2} - \delta,$ $T \leq t \leq 2T)$ and sometimes $\ll T$ zeros in $(\sigma \geq \frac{1}{2} + \delta,\ T \leq t \leq 2T)$.' As remarked already precise statements will be made in §2. For example we will prove both the lower and upper bounds for the series

$$(2) \qquad \zeta^2(s) + \sum_{n=1}^{\infty} d(n)((n+\alpha_n)^{-s} - n^{-s}), \quad (\sigma > 0),$$

and only the lower bound for the series

$$(3) \qquad \zeta^k(s) + \sum_{n=1}^{\infty} d_k(n)((n+\alpha_n)^{-s} - n^{-s}), \quad (\sigma > 0),$$

where $k \geq 3$ is any integer. Here $|\alpha_n| \leq \frac{1}{3}$, but as will be clear from our proof we can relax this bound quite a bit. (Also we have defined $d_k(n)$ for $k \geq 2$ by $\zeta^k(s) = \sum_{n=1}^{\infty} d_k(n) n^{-s}$ in $\sigma > 1$ and $d(n) = d_2(n)$).

Notation. The notations are all standard. If $g > 0$ and $|f| g^{-1}$ is bounded above we write $f = O(g)$. Also we use $|f| \ll g$, whenever convenient to mean the same thing. When $f > 0$ and $g > 0$ we write $f \gg g$ to mean $g \ll f$. Some times we write $\ll_\varepsilon$ or $\gg_\varepsilon$ or $O_\varepsilon(\cdots)$ to indicate that the constant depends on ε (or the parameter or parameters in place of ε). Another notation is $f \asymp g$ which means $f \gg g \gg f$ and this is used when $f > 0$ and $g > 0$.

§2. DEFINITIONS AND MAIN THEOREMS

We will fix some absolute positive constants a, b with $a < b$ throughout.

Generalised Dirichlet series (GDS). Let $\{\lambda_n\}$ be any sequence of real numbers with $a < \lambda_1 < \lambda_2 < \cdots$, $\lambda_1 \leq b$, and $a \leq \lambda_{n+1} - \lambda_n \leq b$ for $n \geq 1$. Let $\{a_n\}$ be any sequence of complex numbers such that $a_1 \neq 0$ and

$$(4) \qquad Z(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}$$

is convergent for some complex s. $Z(s)$ is called a *generalised Dirichlet series*

(GDS). We remark that if $Z(s)$ is convergent at s_0 it is absolutely convergent at s_0+2. If $\sum_{n\le x}|a_n|^2 = O_\varepsilon(x^{1+\varepsilon})$ for every $\varepsilon>0$ then we say that $Z(s)$ is a *normalised generalised Dirichlet series* (NGDS). If $\sum_{n\le x}|a_n|^2 = O(x)$ then we call $Z(s)$ a *special normalised generalised Dirichlet series* (SNGDS).

Note. For different functions $Z(s)$, $\{\lambda_n\}$, $\{a_n\}$ may be different. The parameters $\{\lambda_n\}$ and $\{a_n\}$ may depend on T; but a and b are absolute positive constants.

Functional equation (FE). Let $Z(s)$ be an NGDS. It is said to have a *functional equation* if there exists $Z_1(s)$ which is a GDS such that $Z(s)$ and $Z_1(s)$ can be continued analytically in $|t|\ge t_0$ and

$$(5)\qquad Z(s) = \chi(s) Z_1(1-s)$$

where $\chi(s)$ is holomorphic in $|t|\ge t_0$ and there $|\chi(s)|t^{-k((1/2)-\sigma)}$ is bounded both above and below uniformly in every closed bounded σ-interval. Here k is a real constant and t_0 is a large positive constant. We assume further that $|Z(s)|<|t|^A$ and $|Z_1(s)|<|t|^A$ in every closed bounded σ-interval where $A(>0)$ depends on the interval and $|t|\ge t_0$.

Note. Trivially $k<0$ is not possible by simple convexity considerations. *In* §7 *of the present paper we prove that* $0<k<1$ *is not possible.* (Incidentally this gives the corollary that if $n\ge 1$ is any integer, the product of n Dirichlet L-functions is never the $(n+1)$th power of a holomorphic function in $t\ge t_0$, whatever t_0 be). However $k=0$ is possible since we can take $Z(s) = (\frac{1}{2})^{1-s} + 2^{-s}$ for example. It is quite likely that k has always to be an integer and can never be non-integral.

Perfect functional equation (PFE). Let $Z(s)$ have a functional equation. It is said to be a *perfect functional equation* if $Z_1(s)$ is also an NGDS and further $1<k\le 2$.

Special functional equation (SFE). Let $Z(s)$ have a functional equation. It is said to be a *special functional equation* if $Z(s)$ is an SNGDS, and further $k=1$.

If $Z(s)$ has a functional equation with $k\ge 1$, we will be interested in the zeros of $F(s)$ which is a GDS of the form

$$(6)\qquad F(s) = Z(s) + \sum_{n=1}^{\infty} a_n((\lambda_n+\alpha_n)^{-s} - \lambda_n^{-s}), \quad (\sigma>0),$$

where $|\alpha_n| \ll_\varepsilon n^\varepsilon$ for every fixed $\varepsilon>0$. Our main theorems are as follows.

Theorem 1. *Let $N(\alpha,T)$ denote the number of zeros of $F(s)$ in $(\sigma\ge\alpha$, $T\le t\le 2T)$ where $\alpha>0$. Then for every fixed $\delta>0$, we have $N(\frac{1}{2}+\delta,T)\ll T$ if at least one of the following two conditions is satisfied.*

(i) $k=1$.

(ii) $Z(s)$ *has a* PFE.

Remark. The same upper bound holds good for the zeros of $F(s) - \beta$ (in place of $F(s)$) where β is any fixed complex number, provided $F(s) - \beta$ is a GDS.

Theorem 2. *With the same notation for* $N(\alpha, T)$ *and with* $0 < \delta < \frac{1}{2}$ *we have* $N(\frac{1}{2} - \delta, T) \gg T \log T$ *if at least one of the following two conditions is satisfied.*

(i) $k > 1$.

(ii) $Z(s)$ *has an* SFE *and further* $|\alpha_n|$ *does not exceed a certain small positive constant independent of* δ.

Remark 1. The same lower bound holds good for the zeros of $F(s) - \beta$ (in place of $F(s)$) where β is any fixed complex number, provided $F(s) - \beta$ is a GDS.

Remark 2. As will be clear from our proof Theorems 1 and 2 hold good for more general Dirichlet series than $F(s)$ or $F(s) - \beta$.

§3. PROOF OF THEOREM 1

The following fundamental Theorem (see [5], see also equation 4.23 on page 60 of [8]) will be used at several places in the rest of the paper.

Theorem 3. (H.L. Montgomery and R.C. Vaughan). *Let* $\{x_n\}$, $n = 1, 2, 3, \ldots$ *be any sequence ofcomplex numbers and* $\{\lambda_n\}$, $n = 1, 2, 3, \ldots$ *be any increasing sequence of real numbers such that* $\lambda_1 > 0$ *and* $\lambda_{n+1} - \lambda_n$ *is bounded above and below. Then for* $H > 0$, *we have,*

$$(7)\qquad \int_0^H \left| \sum_{n=1}^{\infty} x_n \lambda_n^{-it} \right|^2 \mathrm{d}t = \sum_{n=1}^{\infty} |x_n|^2 (H + \theta_n),$$

where $|\theta_n| \leq 3\pi\Delta_n^{-1}$ *and* $\Delta_n = \min_{m \neq n} |\log(\lambda_m/\lambda_n)|$. *Also* LHS *is finite if* RHS *is.*

We begin our proof of Theorem 1 by proving

Lemma 1. *Under the conditions of Theorem* 1, *we have,*

$$(8)\qquad \frac{1}{T} \int_T^{2T} |Z(\tfrac{1}{2} + it)|^2 \,\mathrm{d}t = O_\varepsilon(T^\varepsilon).$$

Proof. Put $s = \frac{1}{2} + it$ and let h (> 0) be a large constant. Then by standard arguments

$$(9)\qquad \sum_{n=1}^{\infty} a_n \lambda_n^{-s} \exp\left(-\left(\frac{\lambda_n}{X}\right)^h\right) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+\infty} Z(s+w) X^w \Gamma\left(\frac{w}{h} + 1\right) \frac{dw}{w},$$

where $X = T^{1+\varepsilon}$ (with a small arbitrary positive constant ε) and $w = u + iv$ is a complex variable. If $k = 1$ we ignore the small contribution to the integral from $|v| \geq (\log T)^2$ and in the rest we move the line of integration to $u = -\frac{1}{2}h$. The pole $w = 0$ contributes $Z(s)$. Thus ignoring a small error term we have

(9)′ $$Z(s)=\sum_{n=1}^{\infty} a_n\lambda_n^{-s}\exp\left(-\left(\frac{\lambda_n}{X}\right)^h\right).$$

Now a simple application of Theorem 3 gives Lemma 1.

If $1<k\leq 2$, we write $Z_1(s)=\sum_{n=1}^{\infty} b_n\mu_n^{-s}$ and as before $s=\frac{1}{2}+it$, h (>0) a large constant, $X=T^{1+\varepsilon}$ and $Y=T^{k-1+\varepsilon}$. We proceed as before and obtain

(10) $$\begin{cases} Z(s)=\sum_{n=1}^{\infty} a_n\lambda_n^{-s}\exp\left(-\left(\frac{\lambda_n}{X}\right)^h\right) \\ \quad +\frac{1}{2\pi i}\int\limits_{u=-(h/2),\ |v|\leq(\log T)^2} Z(s+w)X^w\Gamma\left(\frac{w}{h}+1\right)\frac{dw}{w}, \end{cases}$$

where we have ignored a small error term. Here we use $Z(s+w)=\chi(s+w)Z_1(1-s-w)$ and then ignore the portion $\sum_{n>Y} b_n\mu_n^{s+w-1}$ which is a small term and then move back the line of integration to $u=-\varepsilon$. Thus we obtain

(11) $$\begin{cases} Z(s)=\sum_{n=1}^{\infty} a_n\lambda_n^{-s}\exp\left(-\left(\frac{\lambda_n}{X}\right)^h\right) \\ \quad +\frac{1}{2\pi i}\int\limits_{u=-\varepsilon,\ |v|\leq(\log T)^2}\left(\sum_{n\leq Y} b_n\mu_n^{s+w-1}\right)X^w\Gamma\left(\frac{w}{h}+1\right)\frac{dw}{w}. \end{cases}$$

Now a simple application of Theorem 3 gives Lemma 1. □

Remark. The method adopted above is due to the second of us (see [6] and [7]).

Lemma 2. *Under the conditions of Theorem 1, we have,*

(12) $$\frac{1}{T}\int_T^{2T}|Z(\tfrac{1}{2}+\delta+it)|^2\,dt\ll_\varepsilon 1$$

uniformly for $\delta\geq\varepsilon$, for every fixed $\varepsilon>0$.

Proof. It suffices to prove that (8) implies (12), under the assumption $\varepsilon\leq\delta\leq 2$. For this purpose we write $s=\frac{1}{2}+\delta+it$. We have, with $X=T^{1/2}$,

(13) $$\sum_{n=1}^{\infty} a_n\lambda_n^{-s}\exp\left(-\left(\frac{\lambda_n}{X}\right)^5\right)=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} Z(s+w)X^w\Gamma\left(\frac{w}{5}+1\right)\frac{dw}{w}$$

where $w=u+iv$ is a complex variable. In the integral in (13) we ignore the portion $|v|\geq(\log T)^2$ and in the rest we move the line of integration to $u=-\delta$. Since the pole $w=0$ contributes $Z(s)$, we now obtain Lemma 2 by a simple application of Theorem 3. □

Lemma 3. *Under the conditions of Theorem 1, we have,*

(14) $$\frac{1}{T}\int_T^{2T}|F(\tfrac{1}{2}+\delta+it)|^2\,dt\ll_\varepsilon 1$$

uniformly for $\delta\geq\varepsilon$ for every fixed $\varepsilon>0$.

Proof. Put $s = \frac{1}{2} + \delta + it$. It suffices to prove that

$$(15) \qquad \frac{1}{T} \int_{T}^{2T} |Z(s) - F(s))|^2 \, dt \ll_{\varepsilon} 1,$$

where $\varepsilon \leq \delta \leq 2$. For this purpose we write

$$(16) \qquad Z(s) - F(s) = \Sigma_1 + \Sigma_2 + \Sigma_3,$$

where

$$(17) \qquad \Sigma_1 = \sum_{n \leq T} a_n \lambda_n^{-s}, \Sigma_2 = - \sum_{n \leq T} a_n (\lambda_n + \alpha_n)^{-s}$$

and

$$(18) \qquad \Sigma_3 = Z(s) - F(s) - \Sigma_1 - \Sigma_2.$$

By a simple application of Theorem 3, it follows that

$$(19) \qquad \frac{1}{T} \int_{T}^{2T} (|\Sigma_1|^2 + |\Sigma_2|^2) \, dt \ll_{\varepsilon} 1,$$

Now

$$(20) \qquad \Sigma_3 = \sum_{n \geq T} a_n \int_{0}^{\alpha_n} \frac{s}{(\lambda_n + u)^{s+1}} \, du$$

and since the contribution from $n \geq T^{10}$ is very small,

$$(21) \qquad |\Sigma_3| \leq |s| \int_{-T^{\eta}}^{T^{\eta}} \left(\sum_{T^{10} \geq n \geq T} a_n \frac{ch(u, \alpha_n)}{(\lambda_n + u)^{s+1}} \right) du + 1$$

where $ch(u, \alpha_n) = 1$ if $u \in (0, \alpha_n)$ in case $\alpha_n \geq 0$ and $ch(u, \alpha_n) = 1$ if $u \in (\alpha_n, 0)$ in case $\alpha_n \leq 0$ and zero otherwise. Here $\eta > 0$ is an arbitrary small constant and $T \geq T_0(\eta)$. By a simple application of Theorem 3, it follows that

$$(22) \qquad \left\{ \begin{aligned} &\frac{1}{T} \int_{T}^{2T} |\Sigma_3|^2 \, dt \leq 2|3 + 2iT|^2 (2T^{\eta}) \\ &\qquad \times \int_{-T^{\eta}}^{T^{\eta}} \frac{1}{T} \int_{T}^{2T} \left| \sum_{T^{10} \geq n \geq T} \frac{a_n ch(u, \alpha_n)}{(\lambda_n + u)^{s+1}} \right|^2 du \, dt + 2 \end{aligned} \right.$$

$$(23) \qquad \ll T^{1+2\eta} \sum_{n \geq T} \frac{|a_n|^2}{n^{2+2\delta}} + 2$$

$$\ll_{\varepsilon} 1, \quad \text{by a suitable choice of } \eta.$$

This proves Lemma 3.

We are now in a position to prove Theorem 1. Actually from the inequality (14), the result $N(\frac{1}{2} + \delta, T) \ll_{\delta} T$ for every $\delta > 0$ follows just as Theorem 9.15 (A) on page 230 of [9] follows from the inequality

$$(24) \qquad \int_{T}^{2T} |\zeta(\sigma + it)|^2 \, dt = O_{\delta}(T)$$

uniformly in σ ($\sigma \geq \frac{1}{2} + \delta$, $\delta > 0$). The method of proof of Theorem 9.15 (A) mentioned above is due to J.E. Littlewood.

As a passing remark we mention that Theorem 1 is valid for all derivatives $F^{(l)}(s)$ $(l \geq 0)$ of $F(s)$ and also for $F^{(l)}(s) - \beta$ provided these are GDS. This follows from a corollary to Lemma 3 obtained by a simple application of Cauchy's theorem. □

§4. PROOF OF THEOREM 2

The method of proof is best explained by recalling a theorem due to E.C. Titchmarsh (see Theorem 9.14 on page 227 of [9]). Let ε (> 0) and δ (> 0) be small arbitrary constants. Then the theorem states that the rectangle $(\sigma \geq -1-\delta, |t-\tau| \leq \varepsilon)$ contains $\gg \log \tau$ zeros of $\zeta(s)$ for all $\tau \geq \tau_0(\varepsilon, \delta)$. To prove this Titchmarsh used $|\zeta(-1+it)| \gg t^{3/2}$ for all $t \geq 2$ besides some complicated arguments. Motivated by this theorem of Titchmarsh we proved Theorem 4 (stated below) and to state this theorem we find it convenient to begin with two definitions.

Definition. Let $F(s)$ be a GDS and $\alpha = \alpha(\tau) \geq 2$ $(\tau \geq 2)$ be real numbers with $|F(\alpha + i\tau)| \gg \tau^{\lambda}$ (where λ is a positive constant independent of τ) for a set of points $\alpha + i\tau$. Then we call $\alpha + i\tau$ a Titchmarsh point and the rectangle $(\sigma \geq \alpha - \delta, |t-\tau| \leq \varepsilon)$ (where ε (> 0) and δ (> 0) are small arbitrary constants) *the associated rectangle.*

Definition. A set of complex numbers is said to be *well-spaced* if the difference between the imaginary parts of any two numbers is bounded below. It is assumed that there is more than one complex number.

Theorem 4. *Let $\tau \geq \tau_0(\varepsilon, \delta)$ and let $\alpha_0 + i\tau (\alpha_0 = \alpha_0(\tau))$ be a Titchmarsh point for a* GDS *$F(s)$. Then the associated rectangle contains $\gg \log \tau$ zeros provided $F(s)$ is continuable analytically in the associated rectangle and there* $\max |F(s)|$ *is less than t^A for some constant $A > 0$. Next if $\alpha_0(\tau)$ are bounded below by say α, a constant independent of τ and $F(s)$ can be continued analytically in $(\sigma \geq \alpha - \delta$, $T \leq t \leq 2T)$ and there* $\max |F(s)| < T^A$ *and further if there are $\gg T$ well-spaced Titchmarsh points in $(\sigma \geq \alpha - \delta, T \leq t \leq 2T)$, then $F(s)$ has $\gg T \log T$ zeros in $(\sigma \geq \alpha - \delta, T \leq t \leq 2T)$, i.e. $N(\alpha - \delta, T) \gg_{\delta} T \log T$.*

Proof. See Theorem 3 on page 311 of our earlier paper III[1] of this series. □

Remark. If $\alpha + i\tau$ is a Titchmarsh point for $F(s)$, all derivatives $F^{(l)}(s)$ $(l \geq 0)$ have Titchmarsh points $\alpha^{(l)} + i\tau$ where $\alpha^{(l)} \geq \alpha$. This follows from

$$|F(\alpha + i\tau) - F(\alpha_1 + i\tau)| \leq \int_{\alpha}^{\alpha_1} |F'(u + i\tau)| \, d\tau$$

(where $\alpha_1 \geq \alpha$) and the iterations of this inequality.

In the remainder of the proof of Theorem 2 we concentrate on producing $\gg T$

well-spaced Titchmarsh points on ($\sigma = \frac{1}{2} - \delta, T \le t \le 2T$), ($0 < \delta < \frac{1}{2}$) for the GDS in question namely $F(s)$. We first consider the case $k > 1$.

Lemma 4. *Let* $s = \frac{1}{2} - \delta + it, 0 < \delta < \frac{1}{2}$. *Then given any* $\varepsilon > 0$ *there exist* $\gg_\varepsilon T$ *disjoint unit t-intervals I all contained in* $[T, 2T]$ *such that*

$$(25)\qquad \int_I |Z(s)|\,\mathrm{d}t \gg_{\varepsilon,\delta} T^{k\delta-\varepsilon}.$$

Proof. By Theorem 1 on page 1 of our paper [4], we have

$$(26)\qquad \int_{|t-t_0|\le C(\varepsilon)} |Z_1(1-s)|\,\mathrm{d}t \gg_{\varepsilon,\delta} t_0^{-\varepsilon}$$

for a suitable constant $C(\varepsilon) > 0$. Hence (26) is true with $|t - t_0| \le C(\varepsilon)$ replaced by some unit interval contained in $(t_0 - C(\varepsilon) - 1, t_0 + C(\varepsilon) + 1)$. Functional equation gives (25). □

Lemma 5. *We have,*

$$(27)\qquad \sum_I \int_I |Z(s) - F(s)|\,\mathrm{d}t \ll_{\varepsilon,\delta} T^{1+\delta+\varepsilon}$$

where the sum is over the unit intervals I of Lemma 4.

Proof. By a method similar to the proof of (15), we have

$$(28)\qquad \int_T^{2T} |Z(s) - F(s)|^2\,\mathrm{d}t \ll_{\varepsilon,\delta} T^{1+2\delta+2\varepsilon}.$$

This proves Lemma 5 by a simply application of Hölder's inequality. □

Lemma 6. *The number of intervals I with*

$$(29)\qquad \int_I |Z(s) - F(s)|\,\mathrm{d}t \ge T^{\delta+2\varepsilon}$$

is $\ll_{\varepsilon,\delta} T^{1-\varepsilon}$.

Proof. Follows from Lemma 5. □

Lemma 7. *For* $\gg T$ *unit intervals* I, *we have,*

$$(30)\qquad \int_I |F(s)|\,\mathrm{d}t \gg_{\varepsilon,\delta} T^{k\delta-\varepsilon}.$$

Proof. From the intervals I of Lemma 4, we exclude the intervals which satisfy (29). The remaining intervals are $\gg T$ in number and for these, we have,

$$\int_I |F(s)|\,\mathrm{d}t \ge \int_I |Z(s)|\,\mathrm{d}t - \int_I |Z(s) - F(s)|\,\mathrm{d}t \ge C_{\varepsilon,\delta} T^{k\delta-\varepsilon} - T^{\delta+2\varepsilon}$$

for some $C_{\varepsilon,\delta} > 0$. This proves Lemma 7 on choosing ε to be small enough. □

Remark 1. Lemma 7 gives $\gg T$ well-spaced Titchmarsh points on $(\sigma = \frac{1}{2} - \delta,\ T \leq t \leq 2T)$ provided $k > 1$. If FE is perfect (note that the condition $1 < k \leq 2$ is already in the definition of PFE) it is possible to have $k\delta$ in place of $k\delta - \varepsilon$ in (30) as follows. By the FE and standard convexity arguments

$$(31) \qquad \frac{1}{T} \int_{T}^{2T} |Z(s)|\, \mathrm{d}t \gg T^{k\delta}$$

and so by (28) we have

$$(32) \qquad \frac{1}{T} \int_{T}^{2T} |F(s)|\, \mathrm{d}t \gg T^{k\delta}.$$

Using the PFE (and hence Lemma 2) and the result (28), we have

$$(33) \qquad \frac{1}{T} \int_{T}^{2T} |F(s)|^2\, \mathrm{d}t \ll T^{2k\delta}.$$

By a simple but important principle due to us (see our earlier papers III[1] and IV[2] of this series) (32) and (33) imply (30) with $k\delta$ in place of $k\delta - \varepsilon$.

Remark 2. The remaining case $k = 1$ is delicate and restricted. Of course FE gives

$$(34) \qquad \frac{1}{T} \int_{T}^{2T} |Z(s)|\, \mathrm{d}t \gg T^{\delta} \quad \text{and} \quad \frac{1}{T} \int_{T}^{2T} |Z(s)|^2\, \mathrm{d}t \ll T^{2\delta}.$$

But to pass to

$$(35) \qquad \frac{1}{T} \int_{T}^{2T} |F(s)|\, \mathrm{d}t \gg T^{\delta} \quad \text{and} \quad \frac{1}{T} \int_{T}^{2T} |F(s)|^2\, \mathrm{d}t \ll T^{2\delta},$$

we need some restrictions. To prove the second inequality of (35) we need that

$$(36) \qquad \frac{1}{T} \int_{T}^{2T} |Z(s) - F(s)|^2\, \mathrm{d}t \ll T^{2\delta}.$$

This needs that $Z(s)$ should have an SFE and that $|\alpha_n| \ll 1$. But to prove the first of (35) we need that LHS of (36) should be less than a certain constant times $T^{2\delta}$. For this it is necessary to have $|\alpha_n|$ less than a small positive constant. It is possible to make this constant independent of δ by first taking $\delta = \frac{1}{4}$ and then applying some convexity arguments. The proof of the inequalities (35) will be illustrated in the next section by taking the special case $Z(s) = \zeta(s)$.

§5. A THEOREM ON ALGEBRAIC NUMBER FIELDS

In this section we prove

Theorem 5. *Let K be an algebraic number field of degree d and let $f(n)$ denote the number of integral ideals of norm n. Let $\zeta_K(s)$ denote the zeta-function of K and*

$\{\alpha_n\}$ $(n = 1, 2, 3, \ldots)$ *be any sequence of real numbers with* $|\alpha_n| \leq 10^{-5}$. *Then for the function* $N(\alpha, T)$ *associated with the function*

(37) $$F(s) = \zeta_K(s) + \sum_{n=1}^{\infty} f(n)((n + \alpha_n)^{-s} - n^{-s}), \quad (\sigma > 0),$$

we have the following inequalities.

(38) (i) $N(\frac{1}{2} + \delta, T) \ll_\delta T$ *if* $d \leq 2$.

(39) (ii) $N(\frac{1}{2} - \delta, T) \gg_\delta T \log T$ *for all* d.

Remark. Note that $k = d$. If $d \geq 2$ then the theorem follows from Theorems 1 and 2 even with the restriction $|\alpha_n| \ll_\varepsilon n^\varepsilon$ for every $\varepsilon > 0$, provided $F(s)$ is a GDS. Hence it remains to consider the case $d = 1$ i.e. the case when $\zeta_K(s)$ is the Riemann's zeta-function. Here (38) follows from Theorem 1 (even with the restrictions $|\alpha_n| \ll_\varepsilon n^\varepsilon$ and that $F(s)$ shall be a GDS) and so it remains to prove only (39) in case $d = 1$.

Proof. We begin with the result

(40) $$\lim_{T \to \infty} \left(\frac{1}{T} \int_T^{2T} \left| \zeta\left(\frac{3}{4} + it\right) \right| \mathrm{d}t \right) \geq \frac{17}{16}$$

which follows by a result (see Theorem 7.11 on page 155 of [9]) due to A.E. Ingham. From this result (and the functional equation for $\zeta(s)$) it follows that for $T \geq T_0$ (a large positive constant), we have,

(41) $$\frac{1}{T} \int_T^{2T} \left| \zeta\left(\frac{1}{4} + it\right) \right| \mathrm{d}t \geq \left(\frac{T}{2\pi}\right)^{1/4}.$$

Let $|\alpha_n| \leq 10^{-5}$ and

(42) $$F(s) = \zeta(s) + \sum_{n=1}^{\infty} ((n + \alpha_n)^{-s} - n^{-s}), \quad (\sigma > 0).$$

Then

(43) $$|F(s)| \geq |\zeta(s)| - |F(s) - \zeta(s)|.$$

We first check that, with $s = \frac{1}{4} + it$,

(44) $$\frac{1}{T} \int_T^{2T} |F(s) - \zeta(s)| \, \mathrm{d}t \leq \left(\frac{T}{2\pi}\right)^{1/4} - \eta T^{1/4}$$

for a small positive constant η. This would prove that

(45) $$\frac{1}{T} \int_T^{2T} |F(s)| \, \mathrm{d}t \geq \eta T^{1/4},$$

and by convexity (see for example Theorem 7 on page 7 of [4])

(46) $$\frac{1}{T} \int_T^{2T} |F(\tfrac{1}{2} - \delta + it)| \, \mathrm{d}t \gg T^\delta, \quad (0 < \delta < \tfrac{1}{4}),$$

since we have by an easy computation

(47) $$\frac{1}{T}\int_T^{2T} |F(\sigma_1+it)|^2 \ll T^{1-2\sigma_1}, \quad (0<\sigma_1<\tfrac{1}{2}).$$

We now come back to proving (44). We write $s=\frac{1}{4}+it$ till the end of the proof. Let D (>0) be a small constant (to be fixed later). Let

(48) $$F(s)-\zeta(s)=\Sigma_1+\Sigma_2+\Sigma_3$$

where

(49) $$\Sigma_1=-\sum_{n\le DT} n^{-s}, \quad \Sigma_2=\sum_{n\le DT}(n+\alpha_n)^{-s} \quad \text{and} \quad \Sigma_3=\text{the rest}.$$

For any two complex numbers z_1 and z_2 we have $|z_1+z_2|\le\sqrt{2}(|z_1|^2+|z_2|^2)^{1/2}$. We now apply Montgomery-Vaughan Theorem. Note that $\Delta_n^{-1}=n+O(1)$ and so

(50) $$\frac{1}{T}\int_T^{2T}|\Sigma_1+\Sigma_2|\,\mathrm{d}t\le 2^{1/2}\left\{\sum_{n\le DT}\frac{2+6\pi nT^{-1}}{n^{1/2}}\right\}^{1/2}+o(T^{1/4}).$$

From now on we ignore terms like $o(T^{1/4})$. If $|\alpha_n|\le C\le\frac{1}{3}$, we have

(51) $$\frac{1}{T}\int_T^{2T}|\Sigma_3|\,\mathrm{d}t\le\frac{1}{T}\int_T^{2T}\int_{-C}^{C}\left|s\sum_{n\ge DT}ch(u,\alpha_n)(n+u)^{-s-1}\,\mathrm{d}u\right|\mathrm{d}t$$

(with a meaning for $ch(u,\alpha_n)$ similar to that in (21)),

$$\le 2\int_{-C}^{C}\int_T^{2T}\left|\sum_{n\ge DT}ch(u,\alpha_n)(n+u)^{-s-1}\right|\mathrm{d}t\,\mathrm{d}u$$

$$\le 2(2C)^{1/2}\left(\int_{-C}^{C}T\int_T^{2T}\left|\sum_{n\ge DT}ch(u,\alpha_n)(n+u)^{-s-1}\right|^2\mathrm{d}t\,\mathrm{d}u\right)^{1/2}$$

$$\le 4CT\left(\sum_{n\ge DT}(1+3\pi nT^{-1})n^{-5/2}\right)^{1/2}$$

$$\le 4CT\left(\frac{2}{3}(DT)^{-3/2}+\frac{3\pi}{T}\cdot 2\cdot(DT)^{-1/2}\right)^{1/2}.$$

Hence by ignoring terms which are $o(1)$, we have,

$$\left(\frac{1}{T}\int_T^{2T}|F(s)-\zeta(s)|\,\mathrm{d}t\right)T^{-1/4}$$

$$\le 2^{3/2}(D^{1/2}+\pi D^{3/2})^{1/2}+4C(\tfrac{2}{3}D^{-3/2}+6\pi D^{-1/2})^{1/2}$$

$$\le D^{1/4}(8(\pi+D))^{1/2}+4\sqrt{\tfrac{2}{3}}\,(1+9\pi D)^{1/2}D^{1/4}$$

(by choosing $C=D$),

$$\le\left(\frac{1}{2\pi}\right)^{1/4}-\eta$$

and this completes the proof of (44) and hence Theorem 5 is completely proved.

On examining the paper one finds that (apart from equation (25)) the crucial equations are (15) and (28). We have assumed the FE condition

$$E_1 : \sum_{n\leq x} |a_n|^2 \ll_\varepsilon x^{1+\varepsilon} \text{ for all } x \geq 2 \text{ and all } \varepsilon > 0.$$

The extra condition which we assumed was $|\alpha_n| \ll_\varepsilon n^\varepsilon$ for every $\varepsilon > 0$. This latter condition can be replaced by the following two conditions

$$E_2 : \sum_{n\leq x} |a_n \alpha_n|^2 \ll x^{1+\varepsilon}.$$

and

$$E_3 : |\alpha_n| \leq (1-\eta)\lambda_n \text{ for some constant } \eta > 0 \text{ and all } n \geq n_0.$$

Under the conditions E_1, E_2 and E_3 we can prove (15) and (28) as we shall see below. (From this it follows that (15), (25) and (28) are true for

$$F^*(s) = (C^*)^{-s} Z(s) + \sum_{n=1}^{\infty} a_n((\lambda_n + \alpha_n)^{-s} - \lambda_n^{-s})(C^*)^{-s} + Z_0(s)$$

where C^* (> 0) is any constant, $Z_0(s) = \sum_{n\leq C_0^*} d_n \nu_n^{-s}$ is any finite generalised Dirichlet series, and $F^*(s)$ is a generalised Dirichlet series, $Z(s)$ being the same as before). Hence we have the following

Theorem 6. *If $N(\alpha, T)$ denotes the number of zeros of $F^*(s)$ in ($\sigma \geq \alpha$, $T \leq t \leq 2T$) then (subject to one at least of the conditions* (i) *and* (ii) *of Theorem* 1) *we have $N(\frac{1}{2}+\delta, T) \ll_\delta T$, and further if $k > 1$, $N(\frac{1}{2}-\delta, T) \gg_\delta T \log T$ where $0 < \delta \leq \frac{1}{4}$, provided $Z(s)$ satisfies* FE *(note the condition E_1) and $\{\alpha_n\}$ satisfies E_2 and E_3.*

To prove this theorem we have only to prove the following two Theorems 7 and 7′.

Theorem 7. *Put $Q(s) = \sum_{n\geq T} a_n((\lambda_n+\alpha_n)^{-s} - \lambda_n^{-s})$. Then*

$$\frac{1}{T}\int_T^{2T} |Q(\sigma+it)|^2 \, dt \ll_\varepsilon T^{1-2\sigma+\varepsilon}$$

(uniformly in $\frac{1}{4} \leq \sigma \leq \frac{3}{4}$) holds for every $\varepsilon > 0$, provided the conditions E_1, E_2 and E_3 are satisfied.

Proof. The contribution to $Q(s)$ from $n \geq T^{100}$ is negligible, and so it suffices to prove the assertion for $Q_1(s) = \sum_{T\leq n\leq T^{100}} W(s)$ (where $W(s) = a_n((\lambda_n+\alpha_n)^{-s} - \lambda_n^{-s})$) in place of $Q(s)$. The range $|\alpha_n| \leq T^\varepsilon$ is covered already. We divide the range for n into $O(\log T)$ intervals of the type (one of them being possibly smaller) $U \leq n \leq (1 + (\eta/100))U = U_1$ say. Then for every fixed U we divide the range for $|\alpha_n|$ viz. (T^ε, T^{100}) into $O(\log T)$ intervals of the type (one of them being possibly smaller) $V \leq |\alpha_n| \leq (1 + (\eta/100))V = V_1$ say. Hence it suffices to prove that

$$\frac{1}{T}\int_T^{2T} |Q(U, V, s)|^2 \, dt \ll TV^2 \sum_{U \leq n \leq U_1}{}^{(V)} |a_n|^2 n^{-2\sigma - 1}$$

where $Q(U, V, s) = \sum_{U \leq n \leq U_1}{}^{(V)} W(s)$ and the range for $|\alpha_n|$ is as explained above. To prove this we write

$$Q(U, V, s) = s \int_{-V_1}^{V_1} \sum_{U \leq n \leq U_1}{}^{(V)} \frac{a_n ch(u, \alpha_n)}{(\lambda_n + u)^{s+1}} \, du$$

with the notation for $ch(u, \alpha_n)$ same as in (21). A simple application of Montgomery-Vaughan theorem gives the required result stated above. Hence we obtain, for any fixed U,

$$\sum_V \frac{1}{T}\int_T^{2T} |Q(U, V, s)|^2 \, dt \ll \sum_V TV^2 \sum_{U \leq n \leq U_1}{}^{(V)} |a_n|^2 n^{-2\sigma - 1}$$
$$\ll T \sum_{U \leq n \leq U_1} |a_n, \alpha_n|^2 n^{-2\sigma - 1}.$$

Now summing over U, we have,

$$\frac{1}{T}\int_T^{2T} |Q_1(\sigma + it)|^2 \, dt \ll_\varepsilon T^\varepsilon \sum_U \sum_V \frac{1}{T}\int_T^{2T} |Q(U, V, s)|^2 \, dt$$
$$\ll_\varepsilon T^{1+\varepsilon} \sum_{n \geq T} |a_n \alpha_n|^2 n^{-2\sigma - 1} \ll_\varepsilon T^{1 - 2\sigma + 2\varepsilon}.$$

This proves Theorem 7. □

Theorem 7′. *Put* $Q_2(s) = \sum_{n<T} a_n(\lambda_n + \alpha_n)^{-s}$. *Then*

$$\frac{1}{T}\int_T^{2T} |Q_2(\sigma + it)|^2 \, dt \ll_\varepsilon \max(T^{1 - 2\sigma + \varepsilon}, T^\varepsilon)$$

(uniformly in $\frac{1}{4} \leq \sigma \leq \frac{3}{4}$*) holds for every* $\varepsilon > 0$ *provided the conditions* E_1, E_2 *and* E_3 *are satisfied.*

Proof. The range $n \leq \exp(\sqrt{\log T})$ presents no difficulty. We split the range $\exp(\sqrt{\log T}) < n < T$ introducing as before U and V (similar to what we did in the proof of Theorem 7). Next we write (in an obvious notation)

$$Q_3(s) = \sum_{\alpha_n}{}^{(V)} \sum_{\lambda_n}{}^{(U)} a_n(\lambda_n + \alpha_n)^{-s}.$$

It suffices to prove

$$\sum_V \sum_U \frac{1}{T}\int_T^{2T} |Q_3(\sigma + it)|^2 \, dt \ll_\varepsilon \max(T^{1 - 2\sigma + \varepsilon}, T^\varepsilon).$$

To prove this we write

$$Q_3(s) = \sum_{n_0 = 1}^{\infty}{}' \frac{A_{n_0}}{\nu_{n_0}^s}$$

where we have arranged the distinct $\lambda_n + \alpha_n$, say ν_{n_0} $(n_0 = 1, 2, 3, \ldots)$, in the increasing order and the sum is over those n_0 for which $A_{n_0} \neq 0$. Since $F(s)$ is a GDS, $Q_3(s)$ is a finite generalised Dirichlet series in a certain obvious sense and we have $U \ll \nu_1$ and $1 \ll \nu_{n_0+1} - \nu_{n_0}$. Since the distances between consecutive λ_n are bounded below it is easy to see that

$$|A_{n_0}| \leq \sum_{|j_\nu| \ll V} |a_{n_1 + j_\nu}|$$

where $\nu_{n_0} = \lambda_{n_1} + \alpha_{n_1}$ is a certain fixed representation of ν_{n_0}. (Observe that if $\lambda_{n_1} + \alpha_{n_1} = \lambda_{n_2} + \alpha_{n_2}$ with $n_1 \neq n_2$ then $|n_1 - n_2| \ll V$ and also if $\alpha_{n_1} = \alpha_{n_2}$ then $n_1 = n_2$). Thus

$$|A_{n_0}|^2 \ll V \sum_{|j_\nu| \ll V} |a_{n_1 + j_\nu}|^2.$$

Now a simple application of Theorem 3 proves our assertion regarding the sum $\sum_V \sum_U \ldots$, in view of the conditions E_1, E_2 and E_3. Thus Theorem 7′ is proved.

Recently K. Ramachandra and A. Sankaranarayanan have proved a general theorem which implies that for every $\delta > 0$ we have $N(\frac{1}{2} + \delta, T)$, the number of zeros of the function (3) in $(\sigma \geq \frac{1}{2} + \delta, T \leq t \leq 2T)$, is $O_\delta(T)$. It also implies similar new results on upper bounds for the function $N(\frac{1}{2} + \delta, T)$ for some functions (37) even when the degree of K is > 2. These results will be published as paper XVI with the same title and the remarks made above in the profilieration are also relevant in paper XVI.

§7. IMPOSSIBILITY OF $0 < k < 1$

Let k be the real constant involved in the functional equation (5). *We prove that* $0 < k < 1$ *is impossible.* Let ε (> 0) be a small constant and put $X = T^{1-\varepsilon}$. Let h (> 0) be a large constant. We have with $w = u + iv$ and $s = \sigma + it$ $(\sigma > 0$, $T \leq t \leq 2T, T \geq T_0)$ the identity

$$(52) \qquad \sum_{n=1}^{\infty} a_n \lambda_n^{-s} \exp\left(-\left(\frac{\lambda_n}{X}\right)^h\right) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} Z(s+w) X^w \Gamma\left(\frac{w}{h} + 1\right) \frac{dw}{w}.$$

The contribution to the integral from $|v| \geq (\log T)^2$ is negligible. In the rest of the integral we move the line of integration to $u = -(h/2)$. We see that

$$(53) \qquad Z(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s} \exp\left(-\left(\frac{\lambda_n}{X}\right)^h\right) + o(1).$$

By Montgomery-Vaughan theorem we obtain for $\sigma \geq 0$, $T \geq T_0$,

$$(54) \qquad M(Z, \sigma, T) = \sum_{n=1}^{\infty} \frac{|a_n|^2}{\lambda_n^{2\sigma}} \exp\left(-2\left(\frac{\lambda_n}{X}\right)^h\right)\left(1 + O\left(\frac{n}{T}\right)\right)$$

where

$$M(Z, \sigma, T) = \frac{1}{T} \int_T^{2T} |Z(\sigma + it)|^2 \, dt.$$

Let $M(Z_1, \sigma, T)$ denote the mean-value of $Z_1(s)$ in place of $Z(s)$. Since $\sum_{n \leq x} |a_n|^2 \ll_\varepsilon x^{1+\varepsilon}$ for every $\varepsilon > 0$, we have, $M(Z, \frac{1}{2}, T) \asymp M(Z_1, \frac{1}{2}, T) \ll_\varepsilon T^\varepsilon$. So it is not hard to see that

$$(55) \qquad 1 \ll M(Z_1, \tfrac{1}{2} + \delta, T) \ll_\varepsilon T^\varepsilon, \quad (0 < \delta \leq \tfrac{1}{2}),$$

and so by FE, we have,

$$(56) \qquad T^{2k\delta} \ll M(Z, \tfrac{1}{2} - \delta, T) \ll T^{2k\delta+\varepsilon}.$$

This with (54) gives (since $k > 0$) a sequence of numbers $U = U_1, U_2, U_3, \ldots \to \infty$ where

$$(57) \qquad \sum_{U \leq \lambda_n \leq 2U} |a_n|^2 \lambda_n^{2\delta-1} \gg 1 \quad \text{and hence} \sum_{\lambda_n \leq 2U} |a_n|^2 \geq U^{1-2\delta}.$$

Now, since $1 \ll M(Z_1, 1, T) \ll_\varepsilon T^\varepsilon$, we have,

$$(58) \qquad M(Z, 0, T) \ll_\varepsilon T^{k+\varepsilon}.$$

We now use (54) and (57) and choose $X = U$, $T = U^{1/1-\varepsilon}$, and we obtain

$$(59) \qquad U^{1-2\delta} \leq \sum_{\lambda_n \leq 2U} |a_n|^2 \ll T^{k+\varepsilon} = U^{k+\varepsilon/1-\varepsilon}.$$

This leads to a contradiction if $0 < k < 1$, provided ε and δ are small enough. Thus we state

Theorem 8. *$k \leq 1$ implies $k = 0$ or $k = 1$.*

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