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Convergence theorems for implicit iteration process for a finite family of continuous pseudocontractive mappings ☆

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Abstract

In this paper, a necessary and sufficient conditions for the strong convergence to a common fixed point of a finite family of continuous pseudocontractive mappings are proved in an arbitrary real Banach space using an implicit iteration scheme recently introduced by Xu and Ori [H.K. Xu, R.G. Ori, An implicit iteration process for nonexpansive mappings, Numer. Fuct. Anal. Optim. 22 (2001) 767–773] in condition $\alpha_n \in (0, 1]$, and also strong and weak convergence theorem of a finite family of strictly pseudocontractive mappings of Browder–Petryshyn type is obtained. The results presented extend and improve the corresponding results of M.O. Osilike [M.O. Osilike, Implicit iteration process for common fixed points of a finite family of strictly pseudocontractive maps, J. Math. Anal. Appl. 294 (2004) 73–81].

Keywords: Continuous pseudocontractive mappings; Strictly pseudocontractive mappings; Implicit iteration process; Common fixed points

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1. Introduction and preliminaries

In this paper, let E be a real Banach space and let J denote the normalized duality mapping from E into 2^{E^*} given by $J(x) = \{f \in E^*;\ \langle x, f\rangle = \|x\|\|f\|,\ \|x\| = \|f\|\}$, $\forall x \in E$, where E^* denotes the dual space of E and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. If E^* is strictly convex, then J is single-valued. In the sequel, we shall denote the single-valued duality mapping by j, and denote $F(T) = \{x \in E;\ Tx = x\}$.

Definition 1.1.

(i) A mapping T with domain $D(T)$ and range $R(T)$ in E is called *pseudocontractive*, if for all $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$\big\langle Tx - Ty, j(x-y)\big\rangle \leqslant \|x-y\|^2. \tag{1}$$

(ii) A mapping T with domain $D(T)$ and range $R(T)$ in E is called *strictly pseudocontractive* in the terminology of Browder–Petryshyn, if for all $x, y \in D(T)$, there exists $k \in (0,1)$ and $j(x-y) \in J(x-y)$ such that

$$\big\langle Tx - Ty, j(x-y)\big\rangle \leqslant \|x-y\|^2 - k\big\|x - y - (Tx - Ty)\big\|^2. \tag{2}$$

Remark 1.1. If I denotes the identity operator, then (2) can be written in the form

$$\big\langle (I-T)x - (I-T)y, j(x-y)\big\rangle \geqslant k\big\|(I-T)x - (I-T)y\big\|^2. \tag{3}$$

Equation (1) can be written in the form

$$\big\langle (I-T)x - (I-T)y, j(x-y)\big\rangle \geqslant 0. \tag{4}$$

It is easy to know that every strictly pseudocontractive map is L-Lipschitzian and continuous. Indeed, it follows from (3) that

$$\begin{aligned} k\big\|(x-y) - (Tx-Ty)\big\|^2 &\leqslant \big\|(x-y) - (Tx-Ty)\big\|\,\big\|j(x-y)\big\|, \\ k\|Tx - Ty\| - k\|x-y\| &\leqslant k\big\|(x-y) - (Tx-Ty)\big\| \\ &\leqslant \|x-y\|, \end{aligned}$$

i.e.,

$$\|Tx - Ty\| \leqslant L\|x-y\|, \qquad L = \frac{k+1}{k}.$$

Let K be a nonempty convex subset of E, T is continuous pseudocontractive map. For every $u \in K$ and $t \in (0,1)$, the operator $S_t : K \to K$ defined by $S_t x = tu + (1-t)Tx$, $\forall x \in K$, satisfies $\forall x, y \in K$, $\exists j(x-y) \in J(x-y)$ such that

$$\big\langle S_t x - S_t y, j(x-y)\big\rangle = (1-t)\big\langle Tx - Ty, j(x-y)\big\rangle \leqslant (1-t)\|x-y\|^2.$$

Thus S_t is strongly pseudocontractive. Since S_t is also continuous, so that S_t has unique fixed point $x_t \in K$ (see [4]), i.e., $x_t = tu + (1-t)Tx_t$. Let $\{T_i\}_{i=1}^N$ be a finite family of continuous pseudocontractive self-mappings of K. This implies a finite family of continuous pseudocontractive maps employed the following implicit iteration process recently introduced by Xu and Ori [3]. For $x_0 \in K$ and $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$, the sequence $\{x_n\}_{n=1}^{\infty}$ is generated as follows:

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + (1-\alpha_1)T_1 x_1,\\
x_2 &= \alpha_2 x_1 + (1-\alpha_2)T_2 x_2,\\
&\ \vdots\\
x_N &= \alpha_N x_{N-1} + (1-\alpha_N)T_N x_N,\\
x_{N+1} &= \alpha_{N+1} x_N + (1-\alpha_{N+1})T_1 x_{N+1},\\
&\ \vdots
\end{aligned}$$

The scheme is expressed in a compact form as

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n, \quad n \geqslant 1, \tag{5}$$

where $T_n = T_{n \bmod N}$.

In this paper, we extend the result of strictly pseudocontractive mappings of Browder–Petryshyn type [1, Theorem 2] to continuous pseudocontractive mappings using condition $\alpha_n \in (0,1]$, i.e., we prove a necessary and sufficient condition for the strong convergence to a common fixed point of a finite family of continuous pseudocontractive mappings in an arbitrary real Banach space using implicit iteration scheme (5), and also obtain strong convergence theorem of a finite family of strictly pseudocontractive mappings in an arbitrary real Banach space, meanwhile also extend weak convergence theorem of a finite family of strictly pseudocontractive mappings of Browder–Petryshyn type from Hilbert space [1, Theorem 1] to q-uniformly smooth Banach space which is also uniformly convex. The results presented extend and improve some corresponding results in [1–3].

In the sequel, we shall need the following

Definition 1.2. Let K be a closed subset of a Banach space E. A mapping $T : K \to K$ is said to be semicompact, if for any bounded sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$ $(n \to \infty)$, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \to x^* \in K$ $(i \to \infty)$.

Definition 1.3. A Banach space E is said to satisfy *Opial's condition*, if whenever $\{x_n\}$ is a sequence in E which converge weakly to x, as $n \to \infty$, then

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad \forall y \in E,\ y \neq x.$$

Definition 1.4. A Banach space E is said to be *q-uniformly smooth* ($q > 1$), if exists a constant $c > 0$, such that

$$\rho_E(t) \leqslant ct^q,$$

where $\rho_E(t)$ is *modulus of smoothness of* E defined by

$$\rho_E(t) = \sup\left\{\frac{1}{2}\big(\|x+y\| + \|x-y\|\big) - 1;\ \|x\| = 1,\ \|y\| = t\right\}, \quad t > 0.$$

Theorem OU [2]. *Let E be a real q-uniformly smooth Banach space which is also uniformly convex. Let K be a nonempty closed convex subset of E and $T: K \to K$ a strictly pseudocontractive mapping in the terminology of Browder–Petryshyn. Then $(I - T)$ is demiclosed at zero, i.e., $\{x_n\} \subset D(T)$ such that $\{x_n\}$ converges weakly to $x \in D(T)$ and $\{(I - T)x_n\}$ converges strongly to 0, then $x - Tx = 0$.*

Lemma 1.1. *If $J: E \to 2^{E^*}$ is a normalized duality mapping, then for all $x, y \in E$,*

$$\|x + y\|^2 \leqslant \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall j(x + y) \in J(x + y).$$

2. Main results

Lemma 2.1. *Let E be a real Banach space and let K be a nonempty closed convex subset of E. Let $T_i: K \to K$, $i = 1, 2, \dots, N$, be continuous pseudocontractive map such that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let $\{\alpha_n\}_{n=1}^\infty$ be a real sequence satisfying the conditions: $\alpha_n \in (0, 1]$. Let $x_0 \in K$ and let $\{x_n\}$ be defined by*

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n, \quad n \geqslant 1,$$

where $T_n = T_{n \bmod N}$. Then

(i) $\lim_{n\to\infty} \|x_n - p\|$ *exists for all $p \in F$,*
(ii) $\lim_{n\to\infty} d(x_n, F)$ *exists, where $d(x_n, F) = \inf_{p\in F} \|x_n - p\|$.*

Proof. Let $\forall p \in F$, $\forall n \geqslant 1$, $\exists j(x_n - p) \in J(x_n - p)$ such that (using (1))

$$\begin{aligned}\|x_n - p\|^2 &= \langle \alpha_n x_{n-1} + (1 - \alpha_n)(T_n x_n - p), j(x_n - p)\rangle \\ &= (1 - \alpha_n)\langle T_n x_n - p, j(x_n - p)\rangle + \alpha_n\langle x_{n-1} - p, j(x_n - p)\rangle \\ &\leqslant (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n \|x_{n-1} - p\| \|j(x_n - p)\| \\ &= (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n \|x_{n-1} - p\| \|x_n - p\|.\end{aligned}$$

So

$$\|x_n - p\|^2 \leqslant \|x_{n-1} - p\| \|x_n - p\|. \tag{6}$$

If $\|x_n - p\| = 0$, the result is apparent. Next let $\|x_n - p\| > 0$, from (6) we have

$$\|x_n - p\| \leqslant \|x_{n-1} - p\|. \tag{7}$$

Taking infimum over all $p \in F$, we have

$$d(x_n, F) \leqslant d(x_{n-1}, F),$$

hence

$$\lim_{n\to\infty} \|x_n - p\| \text{ exists}, \qquad \lim_{n\to\infty} d(x_n, F) \text{ exists}.$$

The proof is complete. □

Lemma 2.2. *Let E be a real Banach space and let K be a nonempty closed convex subset of E. Let $T_i : K \to K$, $i = 1, 2, \dots, N$, be strictly pseudocontractive mapping in the terminology of Browder–Petryshyn such that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, and let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence satisfying the conditions*:

$$0 < a \leqslant \alpha_n \leqslant b < 1.$$

Let $x_0 \in K$ and let $\{x_n\}$ be defined by

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n, \quad n \geqslant 1,$$

where $T_n = T_{n \bmod N}$. Then

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0, \quad \forall l \in I = \{1, 2, \dots, N\}.$$

Proof. Since $T_i : K \to K$, $i \in I$, be strictly pseudocontractive, from (3) we have $\forall x, y \in K$, there exists $j(x - y) \in J(x - y)$ such that

$$\big\langle (I - T_i)x - (I - T_i)y, j(x - y)\big\rangle \geqslant k_i \big\|(I - T_i)x - (I - T_i)y\big\|^2, \\ i \in I,\ k_i \in (0, 1).$$

Let $k = \min_{1\leqslant i\leqslant N}\{k_i\}$, then

$$\big\langle (I - T_i)x - (I - T_i)y, j(x - y)\big\rangle \geqslant k \big\|(I - T_i)x - (I - T_i)y\big|^2, \\ i \in I,\ k \in (0, 1). \tag{8}$$

By $x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n$, $n \geqslant 1$, we have

$$x_{n-1} = \frac{1}{\alpha_n} x_n + \left(1 - \frac{1}{\alpha_n}\right) T_n x_n. \tag{9}$$

It now follows from (9) that

$$x_n - x_{n-1} = \left(1 - \frac{1}{\alpha_n}\right)(x_n - T_n x_n),$$

$$\begin{aligned} \big\langle x_n - x_{n-1}, j(x_n - p)\big\rangle &= \left(1 - \frac{1}{\alpha_n}\right)\big\langle x_n - T_n x_n, j(x_n - p)\big\rangle \\ &= -\frac{1 - \alpha_n}{\alpha_n}\big\langle x_n - T_n x_n, j(x_n - p)\big\rangle. \end{aligned} \tag{10}$$

It now follows from (10) and (8) that $\forall p \in F$, $\forall n \geqslant 1$, $\exists j(x_n - p) \in J(x_n - p)$ such that (using Lemma 1.1)

$$\begin{aligned} \|x_n - p\|^2 &= \|x_{n-1} - p + x_n - x_{n-1}\|^2 \\ &\leqslant \|x_{n-1} - p\|^2 + 2\big\langle x_n - x_{n-1}, j(x_n - p)\big\rangle \\ &= \|x_{n-1} - p\|^2 - 2\frac{1 - \alpha_n}{\alpha_n}\big\langle x_n - T_n x_n - (p - Tp), j(x_n - p)\big\rangle \\ &\leqslant \|x_{n-1} - p\|^2 - 2k\frac{1 - \alpha_n}{\alpha_n}\|x_n - T_n x_n\|^2. \end{aligned} \tag{11}$$

Thus, from (11) and condition $0 < a \leqslant \alpha_n \leqslant b < 1$, we obtain that

$$2k\frac{1-\alpha_n}{\alpha_n}\|x_n - T_n x_n\|^2 \leqslant \|x_{n-1} - p\|^2 - \|x_n - p\|^2,$$
$$\frac{2k(1-b)}{b}\|x_n - T_n x_n\|^2 \leqslant \|x_{n-1} - p\|^2 - \|x_n - p\|^2,$$
$$\sum_{i=1}^{n}\frac{2k(1-b)}{b}\|x_i - T_i x_i\|^2 \leqslant \|x_0 - p\|^2 - \|x_n - p\|^2.$$

Thus

$$\sum_{n=1}^{\infty}\frac{2k(1-b)}{b}\|x_n - T_n x_n\|^2 < +\infty.$$

Thus

$$\lim_{n\to\infty}\|x_n - T_n x_n\|^2 = 0,$$

i.e.,

$$\lim_{n\to\infty}\|x_n - T_n x_n\| = 0.$$

Therefore,

$$\|x_{n-1} - T_n x_n\| = \frac{1}{\alpha_n}\|x_n - T_n x_n\| \leqslant \frac{1}{a}\|x_n - T_n x_n\| \to 0 \quad (n \to \infty)$$

and

$$\|x_n - x_{n-1}\| = (1-\alpha_n)\|x_{n-1} - T_n x_n\| \to 0 \quad (n \to \infty),$$

thus

$$\|x_{n+i} - x_n\| \to 0 \quad (n \to \infty),\ \forall i \in I.$$

Since every T_i is L_i-Lipschitz, if we choose $L = \max_{1\leqslant i\leqslant N}\{L_i\}$, then

$$\|T_i x - T_i y\| \leqslant L\|x - y\|, \quad \forall i \in I.$$

Therefore,

$$\begin{aligned}\|x_n - T_{n+i}x_n\| &\leqslant \|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| + \|T_{n+i}x_{n+i} - T_{n+i}x_n\| \\ &\leqslant \|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| + L\|x_{n+i} - x_n\| \\ &= (1+L)\|x_{n+i} - x_n\| + \|x_{n+i} - T_{n+i}x_{n+i}\| \to 0 \quad (n \to \infty).\end{aligned}$$

Then

$$\lim_{n\to\infty}\|x_n - T_{n+i}x_n\| = 0, \quad \forall i = 1, 2, \ldots, N. \tag{12}$$

Without loss of generality, we can assume that $n_k = j \pmod N$ for all k and some $j \in \{1, 2, 3, \ldots, N\}$. For any fixed $l \in \{1, 2, 3, \ldots, N\}$, we can find an $i \in \{1, 2, 3, \ldots, N\}$, independent of k, such that $n_k + i = l \pmod N$ for all k. It then follows from (12) that

$$\lim_{n_k\to\infty}\|x_{n_k} - T_l x_{n_k}\| = 0, \quad \forall l = 1, 2, \ldots, N.$$

Thus

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0, \quad \forall l = 1, 2, \ldots, N.$$

The proof is complete. □

Theorem 2.3. *Let E be a real Banach space and let K be a nonempty closed convex subset of E. Let $T_i : K \to K$, $i = 1, 2, \ldots, N$ be continuous pseudocontractive map such that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, and let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence satisfying the conditions: $\alpha_n \in (0, 1]$. Let $x_0 \in K$ and let $\{x_n\}$ be defined by*

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n, \quad n \geqslant 1,$$

where $T_n = T_{n \bmod N}$. Then $\{x_n\}$ strongly converges to a common fixed point of the mappings $\{T_i\}_{i=1}^{N}$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof. At first, the necessity is apparent, secondly we show the sufficiency. Suppose $\liminf_{n\to\infty} d(x_n, F) = 0$, then our Lemma 2.1 implies that

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

It follows from (7) that for all $n \geqslant 1$, $\forall p \in F$ we have

$$\|x_{n+m} - x_n\| \leqslant \|x_{n+m} - p\| + \|x_n - p\| \leqslant 2\|x_n - p\|. \tag{13}$$

Taking infimum over all $p \in F$, from (13) we obtain

$$\|x_{n+m} - x_n\| \leqslant 2d(x_n, F) \to 0 \quad (n \to \infty).$$

Thus $\{x_n\}_{n=1}^{\infty}$ is Cauchy sequence. Suppose $\lim_{n\to\infty} x_n = u$, then

$$d(u, F) = \lim_{n\to\infty} d(x_n, F) = 0.$$

As T_i is continuous pseudocontractive mapping, we claim that $F(T_i)$ is closed $\forall i \in I$. In fact, $F(T_i) \neq \emptyset$, let $\forall \{p_n\} \subset F(T_i)$, $n \geqslant 1$ such that $\lim_{n\to\infty} p_n = p$, then we have

$$T_i p = \lim_{n\to\infty} T_i p_n = \lim_{n\to\infty} p_n = p.$$

Thus $p \in F(T_i)$. Therefore $\forall i \in I$, $F(T_i)$ is closed, so that F is closed. Hence $u \in F$. The proof is complete. □

Corollary 2.4. *Let E be a real Banach space and let K be a nonempty closed convex subset of E. Let $T_i : K \to K$, $i = 1, 2, \ldots, N$ be continuous pseudocontractive map such that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, and let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence satisfying the conditions: $\alpha_n \in (0, 1]$. Let $x_0 \in K$ and let $\{x_n\}$ be defined by*

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n, \quad n \geqslant 1,$$

where $T_n = T_{n \bmod N}$. Then $\{x_n\}$ strongly converges to a common fixed point of the mappings $\{T_i\}_{i=1}^{N}$ if and only if $\{x_n\}$ has a subsequence which converges to some $u \in F$.

Theorem 2.5. *Let E be a real Banach space and let K be a nonempty closed convex subset of E. Let $T_i : K \to K$, $i = 1, 2, \dots, N$, be strictly pseudocontractive mapping in the terminology of Browder–Petryshyn such that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and there exists one map $T \in \{T_i; i \in I\}$ to be semicompact. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence satisfying the conditions*:

$$0 < a \leqslant \alpha_n \leqslant b < 1.$$

Let $x_0 \in K$ and let $\{x_n\}$ be defined by

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n, \quad n \geqslant 1,$$

where $T_n = T_{n \bmod N}$.Then $\{x_n\}$ strongly converges to a common fixed point of the mappings $\{T_i\}_{i=1}^{N}$.

Proof. It follows from Lemmas 2.1 and 2.2 that

$$\lim_{n\to\infty} \|x_n - p\| \text{ exists}, \quad \forall p \in F,$$
$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0, \quad \forall l \in I = \{1, 2, \dots, N\}.$$

Thus $\{x_n\}$ is bounded, then by hypothesis that there exists one map $T \in \{T_i;\ i \in I\}$ to be semicompact, we may assume that T_1 is semicompact without loss of generality. Therefore,

$$\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0$$

and by the definition of semicompact there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that

$$x_{n_i} \to x^* \in K \quad (i \to \infty).$$

Thus

$$\|x^* - T_l x^*\| = \lim_{i\to\infty} \|x_{n_i} - T_l x_{n_i}\| = 0, \quad \forall l \in I,$$

i.e., $x^* \in F$. Therefore,

$$\liminf_{n\to\infty} d(x_n, F) = 0,$$

so that by Theorem 2.3 we have that

$$\lim_{n\to\infty} x_n = x^* \in F.$$

The proof is complete. □

Theorem 2.6. *Let E be a real q-uniformly smooth Banach space which is also uniformly convex and satisfies Opial's condition. Let K be a nonempty closed convex subset of E and $T_i : K \to K$, $i = 1, 2, \dots, N$ be strictly pseudocontractive mapping in the terminology of Browder–Petryshyn such that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, and let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence satisfying the conditions*:

$$0 < a \leqslant \alpha_n \leqslant b < 1.$$

Let $x_0 \in K$ and let $\{x_n\}$ be defined by

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n, \quad n \geqslant 1,$$

where $T_n = T_{n \bmod N}$. Then $\{x_n\}$ weakly converges to a common fixed point of the mappings $\{T_i\}_{i=1}^N$.

Proof. It follows from Lemmas 2.1 and 2.2 that

$$\lim_{n\to\infty} \|x_n - p\| \text{ exists}, \quad \forall p \in F,$$
$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0, \quad \forall l \in I = \{1, 2, \ldots, N\}.$$

Then $\{x_n\}$ is bounded. Since E is uniformly convex, $\{x_n\}$ has a subsequence $\{x_{n_k}\}_{k=1}^\infty$ which converges weakly to some $u \in K$, and hence we have

$$\lim_{k\to\infty} \|x_{n_k} - T_l x_{n_k}\| = 0.$$

From Theorem OU, we obtain that $u = T_l u$, i.e., $u \in F(T_l)$, $l \in I$. Since $l \in I$ is arbitrary, then $u \in F$. Secondly, we prove $\{x_n\}$ converges weakly to u, supposed that $\{x_n\}$ does not converge weakly to u, then there exists another subsequence $\{x_{n_j}\}_{j=1}^\infty$ of $\{x_n\}$ which is weakly convergent to some $y \neq u$, $y \in K$. We also have $y \in F$. Because $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$, and E satisfies Opial condition, thus

$$\begin{aligned} \lim_{n\to\infty} \|x_n - u\| &= \lim_{k\to\infty} \|x_{n_k} - u\| < \lim_{k\to\infty} \|x_{n_k} - y\| \\ &= \lim_{j\to\infty} \|x_{n_j} - y\| < \lim_{j\to\infty} \|x_{n_j} - u\| \\ &= \lim_{n\to\infty} \|x_n - u\|. \end{aligned}$$

This is a contradiction, we must have $y = u$. Thus $\{x_n\}$ converges weakly to $u \in F$. The proof is complete. □

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