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# Discrete Morse Theory and Extended $L^2$ Homology

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A brief overview of Forman's discrete Morse theory is presented, from which analogues of the main results of classical Morse theory can be derived for discrete Morse functions, these being functions mapping the set of cells of a CW complex



using a Witten deformation technique. This deformation argument is adapted to provide strong Morse inequalities for infinite CW complexes which have a finite cellular domain under the free cellular action of a discrete group. The inequalities derived are analogous to the $L^2$ Morse inequalities of Novikov and Shubin and the asymptotic $L^2$ Morse inequalities of an inexact Morse 1-form as derived by Mathai and Shubin. We also obtain quantitative lower bounds for the Morse numbers whenever the spectrum of the Laplacian contains zero, using the extended category of Farber. 

*Key Words:* discrete Morse functions; $L^2$ homology; von Neumann algebras; discrete $L^2$ Morse inequalities; asymptotic discrete $L^2$ Morse inequalities; extended category.

## INTRODUCTION

Robin Forman proposed and developed a discrete Morse theory for finite cell complexes, which is closely analogous to the classical, smooth Morse theory, but has the benefit of being simple and easy to calculate with. Forman gives a very readable summary of the theory in [Form1].

The key difference between discrete Morse theory and the smooth theory is that it deals with *discrete Morse functions*, which are functions from the set of cells of a CW complex to the real numbers, instead of a smooth function on a manifold. Analogous to the concept of a critical point is that of a critical cell of a complex, and one can go on to define analogues of the gradient vector field and the Morse complex.

Why examine discrete Morse theory? The chief benefit from a practical point of view is ease of computation; that is, it allows the algorithmic



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construction of discrete Morse functions on CW complexes (see, for example, the flat Morse–Witten function construction in [Form2]). Further, the combinatorial nature of the theory allows the discrete analogues of Morse theory results to be proven with little or no difficult analysis.

Forman has used the theory to provide a proof of a key element of the $s$-cobordism theorem for PL-manifolds, allowing a completely Morse-theoretical proof of the result.

*Discrete Morse Theory.* Consider a function $f$ from the set of cells of a regular CW complex to the real numbers. $f$ is a *discrete Morse function* if for any $p$-cell $\sigma$ there is at most one $(p+1)$-cell $\tau$ of which $\sigma$ is a face for which $f(\tau) \leqslant f(\sigma)$, and at most one $(p-1)$-cell $v$ which is a face of $\sigma$ for which $f(\sigma) \leqslant f(v)$.

The requirement that the complex be regular can be waived with only a slight increase in complexity of the definition, but this will be deferred to later sections for the purposes of exposition.

The critical $p$-cells are those $p$-cells $\sigma$, such that for all $(p-1)$-cells $v$ and $(p+1)$-cells $\tau$, $v$ a face of $\sigma$ implies $f(v) < f(\sigma)$ and $\sigma$ a face of $\tau$ implies $f(\sigma) < f(\tau)$. Given this definition, one can derive a number of results analogous to those of classical Morse theory.

Consider the classical result: if $g: M \to \mathbb{R}$ is a Morse function on a manifold and $g^{-1}([a, b])$ contains no critical points (for a non-empty interval $[a, b]$) then the submanifold $M(a) = g^{-1}((-\infty, a])$ is a deformation retract of the submanifold $M(b)$. In the discrete case, with a discrete Morse function $f: K \to \mathbb{R}$ on a CW complex $X$, one can define subcomplexes $X(a)$ consisting of all the cells whose image under $f$ is less than or equal to $a$, together with all their faces. If $f^{-1}([a, b])$ contains no critical cells, then $X(a)$ is a deformation retract of $X(b)$.

One of the most important parallels is the strong Morse inequality. Recall that there is a relationship between the Morse numbers $m_k$ of a Morse function on a manifold and the Betti numbers of that manifold $b_k$, given by

$$m_k - m_{k-1} + \cdots \pm m_0 \geqslant b_k - b_{k-1} + \cdots \pm b_0$$

with equality when $k$ is the dimension of the manifold. An identical relation holds in the discrete case on a finite cell complex, when $m_k$ is defined to be the number of cells of order $k$ that are critical under the discrete Morse function.

This inequality for discrete Morse theory was proved by Forman using both a geometrical argument, and by the use of the Witten deformation technique. It is this deformation technique that is applied in later sections to derive strong Morse inequalities for discrete Morse theory on certain classes of infinite CW complex.

Novikov and Shubin in [NoSh1] extended the standard strong Morse inequalities to an $L^2$ Morse inequality, which relates the Morse numbers to $L^2$ Betti numbers: von Neumann dimensions of the homology spaces of a covering space. Mathai and Shubin [MaSh] extended this result to produce asymptotic $L^2$ Morse inequalities for Morse 1-forms.

The analogous discrete case considers an infinite CW complex $X$ with a finite cellular domain under the free and cellular action of a group $\Gamma$, and a discrete Morse function $f$ on the cells of $X$ which is *almost $\Gamma$-invariant*, that is, the associated gradient vector field of $f$ commutes with the $\Gamma$-action and $f$ provides a representation of $\Gamma$ into $\mathbb{R}_+$ by $\gamma \mapsto \exp k_\gamma$ where $k_\gamma = f(\gamma\sigma) - f(\sigma)$ is constant for all cells $\sigma$.

The $L^2$ Morse numbers $m_i^{(2)}$ are defined to be the von Neumann dimensions of the spaces generated by the critical $i$-cells in the complex, while the $L^2$ Betti numbers $b_i^{(2)}$ of $X$ are the von Neumann dimensions of the homology spaces of the $L^2$ chain complex $C_\bullet^{(2)}(X)$. As the exponential map $e^{tf}$ used in the Witten deformation technique is not in general a bounded operator, the derived inequalities relate the Morse numbers to *deformed* $L^2$ Betti numbers $b_k^{(2)}(t)$ for $t \gg 0$. The deformed $L^2$ Betti numbers are the $L^2$ Betti numbers of the chain complex $C^t_\bullet(X)$, which is identical to $C_\bullet^{(2)}(X)$ save for the boundary operator, which is the normal boundary conjugated with $e^{tf}$. The inequalities obtained are

$$m_k^{(2)} - m_{k-1}^{(2)} + \cdots \pm m_0^{(2)} \geqslant b_k^{(2)}(t) - b_{k-1}^{(2)}(t) + \cdots \pm b_0^{(2)}(t), \qquad t \gg 0,$$

with equality when $k = \dim X$ (Proposition 3.8.) When $f$ is a $\Gamma$-invariant function, $e^{tf}$ induces a chain isomorphism and the deformed $L^2$ Betti numbers are constant, $b_i^{(2)}(t) = b_i^{(2)}$ (Proposition 3.3).

In the last section, we study a discrete analogue of a question due to Gromov: If $f$ is a $\Gamma$-invariant discrete Morse function on $X$, then can one give quantitative lower bounds for $m_p^{(2)}$? We will give an answer to this question in this section, using the extended category of Farber [Farb1] (see also [Luck]). In particular, we will show that if zero is in the spectrum of the Laplacian on $L^2$ $p$-chains on $X$, then $m_p^{(2)} > 0$ for any $\Gamma$-invariant discrete Morse function $f$ on $X$, even if $b_p = 0$ and $b_p^{(2)} = 0$, so that the positivity of $m_p^{(2)}$ does not follow from the classical discrete Morse inequalities or from its $L^2$ analogue from the earlier sections. We also study the same question for almost $\Gamma$-invariant discrete Morse functions $f$, obtaining an asymptotic lower bound for $m_p^{(2)}$ in this case.

The results of this paper form an extension of results in the thesis of the second author [Yates] under the supervision of the first author.

## 1. PRELIMINARIES

1.1. *CW Complexes.* Let $X$ be a CW complex. We will generally denote the set of cells by $K$, and in particular the set of $p$-cells by $K^p$. $X^{(p)}$ denotes the subspace $X^{(p)} = \bigcup_{j \leqslant p} K^j$.

The characteristic map $(D^k, S^{k-1}) \to (X^{(k-1)} \cup \sigma, S^{(k-1)})$ will be written as $\Phi_\sigma$.

Cells will usually be denoted by $\sigma$, $\tau$ or $\eta$. A bracketed superscript will denote the dimension of the cell; for example $\sigma^{(p)}$ refers to a $p$-cell.

A CW complex grants a partial ordering to its cells: one writes $\sigma < \tau$ if $\sigma$ is a proper face of $\tau$, that is, $\sigma$ is a subset of the boundary of $\bar{\tau}$.

1.2. *The Associated Chain Complex.* Let $C_\bullet(K)$ denote the chain complex associated with the oriented CW complex $X$ with cells $K$. We will write the boundary operator as

$$\partial \tau^{(p+1)} = \sum_{\sigma \in K^p} [\tau : \sigma]\, \sigma,$$

where $[\tau : \sigma]$ denotes the incidence number of the cell $\tau$ with $\sigma$. Note that $[\tau : \sigma]$ is zero if $\sigma \nless \tau$.

The chain complex can be given the structure of an inner product space by taking coefficients in $\mathbb{R}$ and taking the $p$-cells as an orthogonal basis for each $C_p$. One then has $\langle \partial \tau, \sigma \rangle = [\tau : \sigma]$. One can also then look at the adjoint of the boundary operator $\partial^*$: $\langle \partial^* \sigma^{(p)}, \tau^{(p+1)} \rangle = \langle \sigma, \partial \tau \rangle$. This allows the definition of the *combinatorial Laplacian*

$$\begin{aligned} \Delta &= \partial \partial^* + \partial^* \partial \\ &= (\partial + \partial^*)^2. \end{aligned}$$

It is notationally convenient to regard $\partial$ as a matrix operator, acting by left multiplication on chains, considered as elements of a vector space with basis $K$. Elements of the matrix will be indicated through the use of subscripts, such that

$$\langle \partial \tau, \sigma \rangle = \partial_{\sigma, \tau} = [\tau : \sigma].$$

In terms of this notation, one has

$$\begin{aligned} \partial^*_{\tau, \sigma} &= \partial_{\sigma, \tau} \\ \Delta_{\sigma, \sigma'} &= \sum_{\nu} \partial_{\nu, \sigma} \partial_{\nu, \sigma'} + \sum_{\tau} \partial_{\sigma, \tau} \partial_{\sigma', \tau}. \end{aligned}$$

Different choices of orientation for the CW complex give rise to isomorphic chain complexes [CoFi].

1.3. *Regular Faces.* It is useful to distinguish between regular and irregular faces; regular faces enjoy some properties that are crucial to the definition later of a discrete Morse function. Forman [Form3] makes the following definition:

DEFINITION 1.1. A face $\sigma^{(p-1)} < \tau^{(p)}$ is *regular* if

(1) $\overline{\Phi_\tau^{-1}(\sigma)}$ is a closed $(p-1)$-ball, and

(2) $\Phi_\tau|_{\overline{\Phi_\tau^{-1}(\sigma)}}$ is a homeomorphism,

where $\Phi_\tau$ is the characteristic map for $\tau$.

If all the faces in a CW complex are regular, it is termed a *regular CW complex*. In this case all the characteristic maps are homeomorphisms.

Two properties of regular faces are of particularly useful:

LEMMA 1.2. *If $\sigma^{(p)}$ is a regular face of $\tau^{(p+1)}$, then $[\tau : \sigma] = \pm 1$.*

LEMMA 1.3. *Let $v^{(p-1)} < \sigma^{(p)} < \tau^{(p+1)}$ with $\sigma$ a regular face of $\tau$ and $v$ a regular face of $\sigma$. Then there exists a cell $\tilde{\sigma}^{(p)} \neq \sigma$ such that $v < \tilde{\sigma} < \tau$.*

This second property is Theorem 1.2 of [Form3].

## 2. DISCRETE MORSE THEORY

In [Form3, Form2], Forman develops a combinatorial analogue of Morse theory for CW complexes. The concept of a Morse function is replaced by a discrete (or "combinatorial") Morse function, which assigns a real number to each cell of the complex and satisfies some combinatorial conditions. Results are obtained for these functions which correspond to the fundamental theorems of Morse theory.

This section presents a brief summary of the elementary part of Forman's development of discrete Morse theory, so that it might subsequently be extended to an $L^2$ version.

2.1. *Discrete Morse Functions.* The fundamental definition of the theory is that of a discrete or combinatorial Morse function.

DEFINITION 2.1. A *discrete Morse function* (see Fig. 1) on a CW complex $X$ with set of cells $K$ is a map $f\colon K \to \mathbb{R}$ satisfying

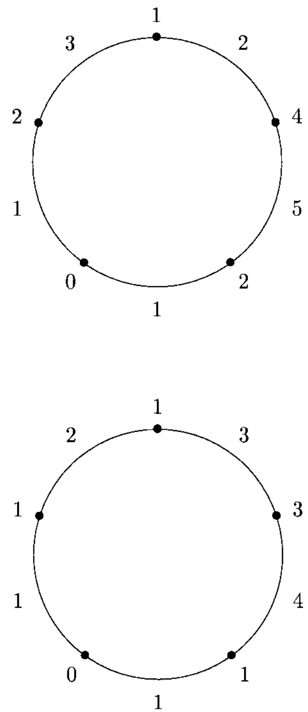


**FIG. 1.** Examples of discrete Morse functions on $S^1$. Top, a discrete Morse function. Bottom, a combinatorial Witten–Morse function.

(1) $\sigma^{(p)}$ an irregular face of $\tau^{(p+1)} \Rightarrow f(\sigma) < f(\tau)$,

(2) for each cell $\sigma^{(p)}$, there is at most one $\tau^{(p+1)} > \sigma$ such that $f(\tau) \leqslant f(\sigma)$, and at most one $\nu^{(p-1)} < \sigma$ such that $f(\nu) \geqslant f(\sigma)$.

The concept of a critical cell corresponds to that of a critical point in Morse theory.

DEFINITION 2.2. A cell $\sigma^{(p)}$ is *critical* for a discrete Morse function $f$ if

$$f(\nu) < f(\sigma) < f(\tau) \qquad \forall \nu^{(p-1)} < \sigma < \tau^{(p+1)}.$$

A critical $p$-cell is also termed a *critical cell of index p*.

An elementary result proved in [Form3] follows directly from the definition.

LEMMA 2.3. *Let $f$ be a discrete Morse function. Then for each cell $\sigma^{(p)}$ there is at most one cell $\nu^{(p-1)}$ or $\tau^{(p+1)}$ such that $f(\tau) \leqslant f(\sigma)$ or $f(\nu) \geqslant f(\sigma)$.*

This essentially shows that at most one cell can prevent a cell from being critical.

For every CW complex there is at least one discrete Morse function, $f\colon \sigma^{(p)} \mapsto p$, for which every cell is critical.

2.2. *Equivalence of Discrete Morse Functions.* The geometry associated with a discrete Morse function lies in the ordering it imposes on the cells. This leads to the definition of an equivalence relation on functions from the set of cells of a CW complex to the reals.

DEFINITION 2.4. Two functions $f$, $g$: $K \to \mathbb{R}$ are *equivalent* if for every pair of cells $\sigma^{(p)} < \tau^{(p+1)}$, $f(\sigma) < f(\tau) \Leftrightarrow g(\sigma) < g(\tau)$.

It is easily seen that any such function that is equivalent to a Morse function is itself a Morse function, and that the relation preserves critical points.

A particular variety of discrete Morse functions—the Witten–Morse functions as defined by Forman in [Form2]—are crucial to constructions in later sections.

DEFINITION 2.5. A discrete Morse function $f$ is a *Witten–Morse* function if

(1) $\sigma^{(p)}, \sigma'^{(p)} < \tau^{(p+1)} \Rightarrow f(\sigma) + f(\sigma') < 2f(\tau)$, and
(2) $\sigma^{(p)}, \sigma'^{(p)} > \nu^{(p-1)} \Rightarrow f(\sigma) + f(\sigma') > 2f(\nu)$.

Forman demonstrates a construction [Form2, Theorem 1.4] that provides an equivalent Witten–Morse function for any discrete Morse function on a finite dimensional CW complex.

2.3. *Combinatorial Vector Fields.* Associated with every Morse function is a *combinatorial vector field.*

DEFINITION 2.6. A *combinatorial vector field* on a cellular complex $X$ with cells $K$ is a chain map $V$: $C_\bullet(K) \to C_\bullet(K)$ of degree one, satisfying

(1) $V(\sigma^{(p)}) = 0$ or $-[\tau : \sigma]\tau^{(p+1)}$ for some cell $\tau$,
(2) $V(\sigma) = k\tau$ for some $k \neq 0 \Rightarrow \sigma$ is a regular face of $\tau$,
(3) $V^2 = 0$.

Note that conditions (1) and (2) imply that $V\sigma \neq 0 \Rightarrow V\sigma = \pm\tau$ for some $\tau$, as $[\tau^{(p+1)} : \sigma^{(p)}] = \pm 1$ if $\sigma$ is a regular face of $\tau$ (Lemma 1.2).

Combinatorial vector fields can be defined equivalently as maps $V'$ from $K$ to $K \cup \{0\}$ which are 1-1 on $K \backslash \ker V'$, satisfying similar conditions: $V'^2 = 0$, $V'(\sigma^{(p)}) \in K^{p+1} \cup \{0\}$, and $V'(\sigma) = \tau \Rightarrow \sigma$ is a regular face of $\tau$. Every combinatorial vector field $V$ defines a unique map $V'$ by $V'(\sigma)=0$ if $V(\sigma)=0$ and $V'(\sigma)=\tau$ if $V(\sigma)=k\tau$. Similarly, given a map $V'$: $K \to K \cup \{0\}$ satisfying the preceding conditions, one can derive a unique combinatorial vector field $V$ by $V(\sigma)=0$ if $V'(0)=0$ and $V(\sigma)=-[\tau:\sigma]\, V'(\sigma)$ otherwise, and extending linearly.

As a notational convenience, $v < V\eta$ will indicate that $\exists \sigma$: $v<\sigma=\pm V\eta$.

The combinatorial vector field $V_f$ corresponding to a discrete Morse function $f$ has

$$V_f(\sigma^{(p)}) = \begin{cases} 0, & \text{if} \quad f(\sigma) < f(\tau) \quad \forall \tau^{(p+1)} > \sigma \\ -[\tau:\sigma]\, \tau, & \text{where} \quad \sigma < \tau^{(p+1)} \quad \text{and} \quad f(\sigma) \geqslant f(\tau). \end{cases}$$

Such a map is called the *gradient* of $f$.

The map $V_f$ captures the geometrical properties of the discrete Morse function $f$:

LEMMA 2.7. *Two discrete Morse functions $f$ and $g$ are equivalent if and only if $V_f = V_g$.*

*Proof.* If $f$ and $g$ are equivalent, then the equality of $V_f$ and $V_g$ follows directly from the definition above.

Suppose $V_f = V_g$, and let $\sigma^{(p)}$ and $\tau^{(p+1)}$ be two faces with $\sigma < \tau$. Then

$$\begin{aligned} f(\sigma) \geqslant f(\tau) &\Rightarrow V_f(\sigma) = \pm\tau \\ &\Rightarrow V_g(\sigma) = \pm\tau \\ &\Rightarrow g(\sigma) \geqslant g(\tau). \end{aligned}$$

Similarly $g(\sigma) \geqslant g(\tau) \Rightarrow f(\sigma) \geqslant f(\tau)$, so $f(\sigma) < f(\tau) \Leftrightarrow g(\sigma) < g(\tau)$. ∎

DEFINITION 2.8. A cell $\sigma$ is a *rest point* of $V$ if $V\sigma = 0$ and $\sigma \notin \operatorname{im} V$.

The rest points of $V_f$ are exactly the critical cells of $f$.

Note that not every combinatorial vector field corresponds to a discrete Morse function; Forman [Form3, Sect. 9] provides a necessary and sufficient condition for a combinatorial vector field to have an associated discrete Morse function:

THEOREM 2.9. *Let $V$ be a combinatorial vector function. $V$ is the gradient of a discrete Morse function if and only if there exists no sequence of $p$-cells $\sigma_0, \sigma_1, ..., \sigma_r$ (for some $r > 0$ and some $p$) satisfying*

(1) $\sigma_{i+1} < \tau$ *where* $V\sigma_i = \pm\tau$ *for some* $(p+1)$*-cell* $\tau$,

(2) $\sigma_i \neq \sigma_{i+1}$,

(3) $\sigma_0 = \sigma_r$.

DEFINITION 2.10. A combinatorial vector field that is the gradient of a discrete Morse function is called a *gradient vector field.*

Sequences of cells satisfying the first two conditions of Theorem 2.9 will be referred to as *V-paths.* A $V$-path $\sigma_0, \sigma_1, ..., \sigma_r$ is *closed* if there is some $i$ and $j \neq i$ such that $\sigma_i = \sigma_j$. (Note that this definition is slightly different from that of Forman.)

The condition on $V$ in Theorem 2.9—and thus the necessary and sufficient condition for $V$ to be a gradient vector field—is then that $V$ admits no closed $V$-paths.

Using this language, one can restate Forman's Witten–Morse function equivalence result in terms of combinatorial vector fields.

THEOREM 2.11. *Let $V$ be a gradient vector field on a CW complex $X$ with cells $K$. If there is an upper bound on the length of $V$-paths, there exists a Witten–Morse function $f: K \to \mathbb{R}$ such that $V = V_f$. In particular, if $V$ is a gradient vector field on a finite CW complex, it is the gradient of a Witten–Morse function.*

## 3. $L^2$ DISCRETE MORSE THEORY

3.1. *The $L^2$ Chain Complex.* The strong Morse inequality can be stated as

$$m_n - m_{n-1} + \cdots \pm m_0 \geqslant b_n - b_{n-1} + \cdots \pm b_0, \tag{1}$$

where $m_i$ is the $i$th Morse number and $b_i$ is the $i$th Betti number.

For discrete Morse functions on a finite CW complex, the Morse numbers correspond to the dimensions of the spaces generated by the critical cells. However the dimensions of these spaces need not be finite if the CW complex is not finite.

*We suppose in this paper that we have an infinite CW complex upon which a group $\Gamma$ acts freely and cellularly such that the fundamental domain under this action is finite.* If the discrete Morse function is $\Gamma$-invariant, one can determine the von Neumann dimensions of the homology spaces and of the spaces generated by the critical cells. The von Neumann dimension acts analogously to the familiar dimension on vector spaces, save that it takes values that are non-negative real numbers instead of integers. This allows a sensible definition of the Morse and Betti numbers, which leads to the statement of a strong Morse inequality for such Morse functions on this class of infinite CW complexes.

Let $X$ be a CW complex with cells $K$. The space of $L^2$ chains on $X$, denoted by $C_\bullet^{(2)}(X, \mathbb{R})$, is a graded Hilbert space with the elements of order $p$ being formal sums

$$\sum_{\sigma \in K^p} c_\sigma \sigma, \qquad \text{such that} \quad \sum_{\sigma \in K^p} |c_\sigma|^2 < \infty.$$

The inner product is defined in the usual manner, with $\langle \sigma, \sigma' \rangle = \delta_{\sigma\sigma'}$ and extending linearly. The boundary operator on chains on $X$ restricts to a bounded operator $\partial$ on $L^2$ chains on $X$.

Let $\Gamma$ be a discrete group acting freely and cellularly on $X$, and such that there exists a finite cellular fundamental domain for the action of $\Gamma$ on $X$. Then the $C_i^{(2)}$ form examples of $L^2\Gamma$-modules, and moreover $(C_\bullet^{(2)}(X, \mathbb{R}), \partial)$ is a example of an $L^2\Gamma$-complex:

Definition 3.1 (from [Cohen]). Given a discrete group $\Gamma$, let $\mathscr{H}_\Gamma$ be the category of Hilbert spaces with a $\Gamma$ action, with morphisms being continuous $\Gamma$-equivariant maps. Let $L^2\Gamma$ be the object of $\mathscr{H}_\Gamma$ which has basis $\Gamma$ and on which $\Gamma$ acts by left-multiplication.

A *free $L^2\Gamma$-module* is an object of $\mathscr{H}_\Gamma$ isomorphic to the completion of $L^2\Gamma \otimes A$ for some vector space $A$. In this paper, we will always assume that $A$ is *finite dimensional*.

An *$L^2\Gamma$-module* is a closed $\Gamma$-invariant subspace of a free $L^2\Gamma$-module. The full subcategory of $\mathscr{H}_\Gamma$ consisting of $L^2\Gamma$-modules is denoted $\mathscr{L}^2\Gamma$.

An *$L^2\Gamma$-complex* $(C, \partial)$ is a complex in $\mathscr{H}_\Gamma$, i.e., the spaces $C$ are $L^2\Gamma$-modules and the differentials $\partial$ are continuous $\Gamma$-equivariant maps such that $\partial^2 = 0$.

In the category $\mathscr{H}_\Gamma$ the image object of a morphism is the closure of the set-image. In the following, im will refer to this categorical image

$$\operatorname{im} f = \overline{\operatorname{im}_{\mathbf{Set}} f}.$$

Armed with this terminology, the *reduced homology* $H_\bullet^{(2)}(X; \mathbb{R})$ is defined as

$$H_\bullet^{(2)}(X; \mathbb{R}) = \ker \partial / \operatorname{im} \partial = \ker \partial / \overline{\operatorname{im}_{\mathbf{Set}} \partial}.$$

As shown in [Cohen], one can define the *von Neumann dimension* $\dim_\Gamma$ of $L^2\Gamma$-modules as the trace $\operatorname{tr}_\Gamma$ of the corresponding projection. This allows the definition of the $L^2$ Betti and Morse numbers.

Definition 3.2. Let $X$ be a CW complex $X$ with cells $K$, and with a free cellular group action of a discrete group $\Gamma$.

The $L^2$ *Betti numbers* are given by

$$b_p^{(2)} = \dim_\Gamma H_p^{(2)}(X; \mathbb{R}).$$

Let $f$ be a discrete Morse function on $X$ whose critical cells form a $\Gamma$-invariant subset of $K$. Then the $L^2$ *Morse numbers* are given by

$$m_p^{(2)} = \dim_\Gamma R_p^{(2)},$$

where $R_p^{(2)}$ is the $L^2\Gamma$-module generated by the critical $p$-cells of $X$.

This allows us to state the discrete $L^2$ analogue of the strong Morse inequality.

PROPOSITION 3.3 (Combinatorial $L^2$ Strong Morse Inequality). *Let $X$ be a CW complex with a finite cellular domain under a free cellular $\Gamma$-action, and let $f$ be a $\Gamma$-invariant discrete Morse function on $X$. Then the strong Morse inequality holds*

$$m_n^{(2)} - m_{n-1}^{(2)} + \cdots \pm m_0^{(2)} \geqslant b_n^{(2)} - b_{n-1}^{(2)} + \cdots \pm b_0^{(2)},$$

*where $b_i^{(2)}$ are the $L^2$ Betti numbers of the complex and $m_i^{(2)}$ are the $L^2$ Morse numbers of $f$ as defined in* 3.2, *with equality when $n = \dim X$.*

3.2. *Almost $\Gamma$-Invariant Discrete Morse Functions.* Mathai and Shubin in [MaSh] introduce *twisted* $L^2$ invariants of a manifold associated with a flat Hilbertian $\mathscr{A}$-bundle $\mathscr{E}$ over the manifold and a closed 1-form $\theta$. A twisted complex is constructed by deforming the differential with $\theta$ and looking at the space of $L^2\mathscr{E}$ valued forms with this deformed differential. The twisted $L^2$ Betti numbers are defined as the von Neumann dimensions of the reduced cohomology of this complex.

With this framework they derive the asymptotic strong Morse inequalities for a Morse 1-form $\theta$ (this being a 1-form which lifts to the differential of a Morse function on the universal cover) which relate the Morse numbers of $\theta$ to the twisted $L^2$ Betti numbers associated with $s\theta$, with $s$ a large parameter.

A similar (but more elementary) construction can be applied in the discrete case by examining discrete Morse functions that are not $\Gamma$-invariant and comparing their $L^2$ Morse numbers with the $L^2$ Betti numbers of a deformed complex $C^t_\bullet(K)$.

Consider a combinatorial Witten–Morse function $f: K \to \mathbb{R}$ which while not being $\Gamma$-invariant itself, has its associated gradient vector field $V_f$

commuting with the group action. We define the deformed chain complexes $C^t_\bullet(K)$ as

$$C_i^t = C_i,$$
$$\partial^t = e^{tf} \circ \partial \circ e^{-tf},$$

where $e^{tf}$ is the cellular map that takes $\sigma$ to $e^{tf(\sigma)}\sigma$.

For this to be well defined, it is necessary that the deformed boundary operator $\partial^t$ commute with the group action. This commutation condition can be expressed as

$$\partial^t_{\sigma, \tau} = \partial^t_{\gamma\sigma, \gamma\tau} \qquad \forall \gamma \in \Gamma.$$

Examining the components of $\partial^t$ gives

$$\partial^t_{\sigma, \tau} = \partial_{\sigma, \tau} e^{t(f(\tau) - f(\sigma))}.$$

So one has

$$\begin{aligned} \partial_{\sigma, \tau} e^{t(f(\tau) - f(\sigma))} &= \partial_{\gamma\sigma, \gamma\tau} e^{t(f(\gamma\tau) - f(\gamma\sigma))} \\ &= \partial_{\sigma, \tau} e^{t(f(\gamma\tau) - f(\gamma\sigma))}. \end{aligned}$$

Thus for all $\sigma^{(p)} < \tau^{(p+1)}$, $f(\gamma\sigma) - f(\sigma) = f(\gamma\tau) - f(\tau)$. Given that $X$ is connected, this implies $f(\gamma\sigma) - f(\sigma)$ is constant for all $\sigma \in K$.

Let $k$: $\Gamma \to \mathbb{R}$ denote the map $k$: $\gamma \mapsto f(\gamma\sigma) - f(\sigma) = k_\gamma$. Observe that

$$\begin{aligned} k_{\gamma\gamma'} &= f(\gamma\gamma'\sigma) - f(\sigma) \\ &= f(\gamma\gamma'\sigma) - f(\gamma'\sigma) + f(\gamma'\sigma) - f(\sigma) \\ &= k_\gamma + k_{\gamma'}. \end{aligned}$$

This defines a group homomorphism $\rho$: $\Gamma \to \mathbb{R}_+$ by $\rho$: $\gamma \mapsto \exp k_\gamma$.

The required conditions on $f$ can then be summarized thus:

(1) $f$: $K \to \mathbb{R}$ is a Witten–Morse function,

(2) $V_f\gamma = \gamma V_f$ for all $\gamma \in \Gamma$,

(3) $f(\gamma\sigma) - f(\sigma) = k_\gamma$ is constant for all $\sigma$, and defines a representation

$$\exp k_\gamma\colon \Gamma \to \mathbb{R}_+.$$

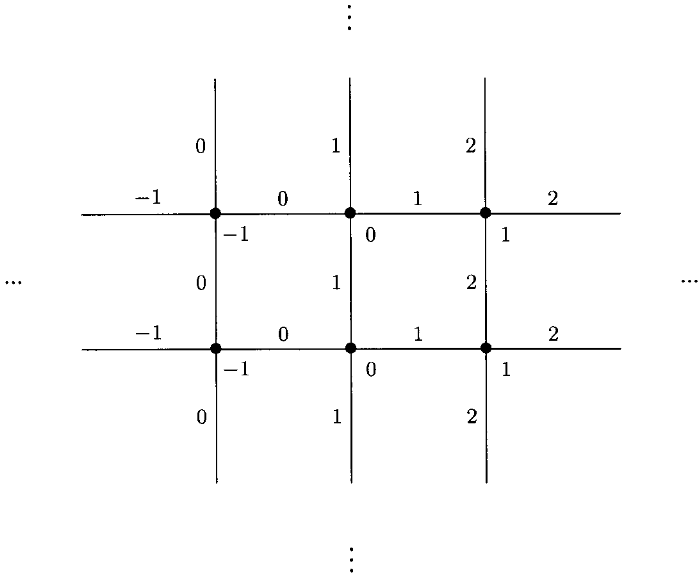


**FIG. 2.** An almost $\Gamma$-invariant discrete Morse function.

For the sake of brevity in the following, Morse functions that satisfy the last two conditions will be termed *almost $\Gamma$-invariant*.

A simple non-trivial example of an $f$ with this property is the infinite grid as in Fig. 2 with $\Gamma = \mathbb{Z} \oplus \mathbb{Z}$ acting as translations.

It is worth noting the parallel between this representation and that arising in the smooth case with a Morse 1-form $\theta$. If $\alpha_\gamma$ is a loop on the manifold representing an element $\gamma$ of the group, then the integral $\int_{\alpha_\gamma} \theta$ depends only on $\gamma$, and defines a representation into $\mathbb{R}_+$

$$\rho_\theta \colon \gamma \mapsto \exp\left( - \int_{\alpha_\gamma} \theta \right).$$

When $k_\gamma = 0\ \forall \gamma$, $f$ is simply $\Gamma$-invariant. In this case, $f$ takes only a finite number of values and so the chain map $e^{tf}$ is bounded and further, is a chain isomorphism from $C^t_\bullet$ to $C_\bullet$.

When $f$ is not $\Gamma$-invariant, it is not immediately clear that $\partial^t$ is bounded.

LEMMA 3.4. *Let $X$ be a CW complex with cells $K$ which has a finite cellular domain $G$ under the action of a $\Gamma$, and suppose that the function $f \colon K \to \mathbb{R}$ has the property $f(\gamma\sigma) - f(\sigma) = k_\gamma$, $\forall \sigma \in K$. Then $\{ f(\tau) - f(\sigma) \colon \sigma, \tau \in K, \partial_{\sigma,\tau} \neq 0 \}$ is a finite set, and thus bounded.*

*Proof.* $\partial$ commutes with the Gamma action, so

$$
\begin{aligned}
&\{f(\tau)-f(\sigma)\colon \sigma,\tau\in K, \partial_{\sigma,\tau}\neq 0\}\\
&\quad=\{f(\gamma\tau)-f(\sigma)\colon \sigma\in K, \tau\in G, \gamma\in\Gamma, \partial_{\sigma,\gamma\tau}\neq 0\}\\
&\quad=\{f(\gamma\tau)-f(\gamma\sigma)\colon \sigma\in K, \tau\in G, \gamma\in\Gamma, \partial_{\gamma\sigma,\gamma\tau}\neq 0\}\\
&\quad=\{f(\tau)-f(\sigma)\colon \sigma\in K, \tau\in G, \partial_{\sigma,\tau}\neq 0\}.
\end{aligned}
$$

The number of cells $\sigma$, such that $\partial_{\sigma,\tau}\neq 0$ is finite (Lemma 3.16). So the set is finite. ▮

As $\partial^t_{\sigma,\tau}=\partial_{\sigma,\tau}e^{t(f(\tau))}$, this demonstrates that $\partial^t$ is a bounded operator.

With a well-defined deformed complex $C^t_\bullet$ we can now define the analogue of the twisted $L^2$ Betti numbers of Mathai and Shubin.

DEFINITION 3.5. Given an almost $\Gamma$-invariant Witten–Morse function, define the *deformed $L^2$ Betti numbers* as

$$b_p^{(2)}(t)=\dim_\Gamma H_p(C^t),$$

where $C^t_\bullet$ is as described above.

LEMMA 3.6. *The map $t\mapsto b_p^{(2)}(t)$ is upper semi-continuous.*

*Proof.* The proof follows the same argument as applies in the smooth case [MaSh, Lemma 5.2]. ▮

*Conjecture* 3.7. $b_p^{(2)}(t)$ is constant for all but at most countably many $t$, when the group $\Gamma$ is finitely presented.

It is known that for any $\Gamma$-invariant operator $A$ acting on $l^2(\Gamma)\otimes\mathbb{C}^k$ that comes from the group algebra $\mathbb{C}(\Gamma)$, $\dim_\Gamma \ker A$ is rational when $\Gamma$ is an elementary amenable group, or the extension of such a group by right orderable groups. There is a conjecture (see for example Linnell [Linn]) that every finitely presented group has this property. For such a $\Gamma$, the deformed Betti numbers $b_p^{(2)}(t)$ will be rational. Coupled with their upper semi-continuity in $t$, one has that the $b_p^{(2)}(t)$ are constant except at countably many points.

The asymptotic Morse inequality can be stated in terms of these $b_p^{(2)}(t)$.

PROPOSITION 3.8 (Combinatorial Strong Asymptotic Morse Inequality). *Let $X$ be a CW complex with a finite cellular domain under a free cellular $\Gamma$-action, and let $f$ be a combinatorial Witten–Morse function on $X$ that is also almost $\Gamma$-invariant. Then the strong asymptotic Morse inequality holds*

$$m_n^{(2)}-m_{n-1}^{(2)}+\cdots\pm m_0^{(2)}\geqslant b_n^{(2)}(t)-b_{n-1}^{(2)}(t)+\cdots\pm b_0^{(2)}(t),\qquad t\gg 0,$$

*where* $b_i^{(2)}(t)$ *are the deformed* $L^2$ *Betti numbers of the complex and* $m_i^{(2)}$ *are the* $L^2$ *Morse numbers of* $f$ *as defined in Definitions* 3.5 *and* 3.2, *with equality when* $n = \dim X$.

3.3. *Derivation of the Combinatorial Morse Inequalities.* The von Neumann dimension used to define the $L^2$ Morse and Betti numbers behaves similarly to the usual dimension defined for vector spaces, the chief difference being that it is real or infinite valued. Quoting [Cohen, Proposition 5]:

PROPOSITION 3.9. *The von Neumann dimension* $\dim_\Gamma$ *has the following properties*:

(i) $\dim_\Gamma(\bigoplus_{k=1}^n M_k) = \sum_{k=1}^n \dim_\Gamma M_k$.

(ii) *If* $M \subsetneq N$ *then* $\dim_\Gamma M < \dim_\Gamma N$.

(iii) *If* $M \cong N$ *then* $\dim_\Gamma M = \dim_\Gamma N$.

(iv) *If* $f: M \to N$, *then* $\dim_\Gamma M = \dim_\Gamma \ker f + \dim_\Gamma \operatorname{im} f$ *and* $\dim_\Gamma N = \dim_\Gamma \operatorname{im} f + \dim_\Gamma \operatorname{coker} f$.

These properties allow the direct application of a well-known result of linear algebra to these $\Gamma$-invariant subspaces (see [Shub].)

LEMMA 3.10. *If a chain complex* $(V, d)$: $0 \xrightarrow{d} V^n \xrightarrow{d} \cdots \xrightarrow{d} V^0 \xrightarrow{d} 0$ *is such that* $\dim_\Gamma V^i = m_i$ *and* $\dim_\Gamma H^i(V) = b_i$, *then*

$$\sum_{i=0}^{k} (-1)^{k-i} m_i \geqslant \sum_{i=0}^{k} (-1)^{k-i} b_i$$

*with equality when* $V^n = 0$ *for all* $n > k$.

This relation provides the key to investigating the Morse inequalities in this context. If for every almost $\Gamma$-invariant Witten–Morse function $f$ with Morse numbers $m_i^{(2)}$ we can find an $L^2\Gamma$-complex $W_\bullet$ with $\dim_\Gamma W_i = m_i^{(2)}$ and $\dim_\Gamma H_i(W) = b_i^{(2)}(t)$ for $t \gg 0$, the combinatorial strong asymptotic Morse inequality (Proposition 3.8) follows.

In the following we construct such a chain complex using an analogue of Forman's argument in [Form2]. *For notational convenience, the* $^{(2)}$ *superscript on chain complexes will be dropped, it being understood that the chain complexes discussed are* $L^2\Gamma$-*complexes.*

Given a normal operator $\phi$ on a Hilbert space $C$, the spectral decomposition of $\phi$ refers to the unique resolution of the identity $E$ such that $\phi = \int_{\sigma(\phi)} \lambda \, dE(\lambda)$. Recall these results on spectral decompositions:

LEMMA 3.11. *Suppose $E$ is the spectral decomposition of a normal operator $\phi$: $C \to C$. Then*

(1) $\omega \cap \rho = \emptyset \Rightarrow E(\omega)C \oplus E(\rho)C = E(\omega \cup \rho)C$ *and* $\omega \subseteq \rho \Rightarrow E(\omega)C \subseteq E(\rho)C$,

(2) *if $\phi$ commutes with a continuous linear operator $f$: $C \to C$, then $E(\omega)$ commutes with $f$,*

(3) $E(\{0\})$ *is the projection onto* $\ker \phi$.

In order to more easily apply the machinery of spectral analysis, it is convenient to regard $L^2\Gamma$-complexes as Hilbert spaces.

LEMMA 3.12. *Let $(C, \partial)$ be an $L^2\Gamma$-complex, and $f$: $C \to C$ a continuous linear operator that commutes with $\partial$ and commutes with the projections $p_i$: $C \to C_i$. Then* $(\operatorname{im} f, \partial|_{\operatorname{im} f})$ *is also an $L^2\Gamma$-complex.*

*Proof.* All that needs to be shown is that $\operatorname{im} f$ is a graded space and that $\operatorname{im} \partial|_{\operatorname{im} f} \subseteq \operatorname{im} f$.

As the $p_i$ commute with $f$, one has $\operatorname{im} f = \oplus p_i \operatorname{im} f$ and $p_i \operatorname{im} f \cap p_j \operatorname{im} f = \emptyset$ for $i \neq j$.

Let $\pi_f$ denote the projection from $C$ to $\operatorname{im} f$. Then as $\partial$ and $f$ commute, $\partial$ and $\pi_f$ commute. One thus has $\pi_f \circ \partial \circ \pi_f = \pi_f^2 \circ \partial = \pi_f \circ \partial = \partial \circ \pi_f$, and so ensuring that $\operatorname{im} \partial|_{\operatorname{im} f} \subseteq \operatorname{im} f$. ▮

Recall that two $L^2\Gamma$-complexes $(C, \partial)$, $(C', \partial')$ are said to be *chain homotopy equivalent* if there are chain morphisms $f$: $C \to C'$, $g$: $C' \to C$ and morphisms $T$: $C_i \to C_{i-1}$ $\forall i \geqslant 0$, $S$: $C'_i \to C'_{i-1}$ $\forall i \geqslant 0$ such that $g \circ f - I = \partial T + T\partial$ and $f \circ g - I = \partial' S + S\partial'$. Thus armed, we arrive at the following result.

LEMMA 3.13. *Let $E$ be the spectral decomposition of $\Delta$: $C \to C$, and let $\omega \subset \mathbb{R}$ with $0 \in \omega$. Then $E(\omega)C$ forms a chain subcomplex of $C$ which is chain homotopy equivalent to $C$. In particular, $H_\bullet(E(\omega)C) \cong H_\bullet(C)$.*

*Proof.* Let

$$i_\omega: E(\omega)C \to C$$

$$P_\omega: C \to E(\omega)C$$

denote the inclusion and the projection morphisms. Observe that $P_\omega \circ i_\omega = I$ and that $i_\omega \circ P_\omega = P_\omega$. Define $G = \Delta^{-1}(I - P_\omega)$: $C \to C$, where we observe that $\Delta$ is invertible on the range of the projection $(I - P_\omega)$, and we set $G = 0$ on the range of the projection $P_\omega$. Then $G$ is a morphism which commutes with $\partial$ and $\partial^*$ and we compute

$$\begin{aligned} I - P_\omega &= \Delta\Delta^{-1}(I - P_\omega) \\ &= \partial\partial^* G + \partial^*\partial G \\ &= \partial(\partial^* G) + (\partial^* G)\,\partial. \end{aligned}$$

This establishes the claimed chain homotopy equivalence. ∎

Suppose now that $C_\bullet$ is the $L^2\Gamma$-complex associated with a CW complex $X$, and that $X$ has a finite cellular domain under the $\Gamma$ action. Let $f$ be an almost $\Gamma$-invariant Witten–Morse function on $X$ and construct the deformed chain complexes $C^t_\bullet$ as before.

The deformed Laplacian on $C^t_\bullet$ is defined in the expected way:

$$\Delta^t = \partial^t \partial^{t*} + \partial^{t*} \partial^t.$$

In terms of matrix elements, one has

$$\begin{aligned} \partial^t_{\sigma,\tau} &= \partial_{\sigma,\tau} e^{t(f(\tau)-f(\sigma))} \\ \Delta^t_{\sigma,\sigma'} &= \sum_\nu \partial^t_{\nu,\sigma} \partial^t_{\nu,\sigma'} + \sum_\tau \partial^t_{\sigma,\tau} \partial^t_{\sigma',\tau} \\ &= \sum_\nu \partial_{\nu,\sigma} \partial_{\nu,\sigma'} e^{-2t(f(\nu)-f(\sigma)-f(\sigma'))} \\ &\quad + \sum_\tau \partial_{\sigma,\tau} \partial_{\sigma',\tau} e^{2t(f(\tau)-f(\sigma)-f(\sigma'))}. \end{aligned}$$

If $E^t$ is the spectral decomposition of the deformed Laplacian, then the Witten complex $W^t_\bullet(\lambda)$ is defined to be the chain complex

$$W^t_\bullet(\lambda) = E^t((-\infty, \lambda])\, C^t_\bullet.$$

From Lemma 3.13, $H_\bullet(W^t(\lambda))$ is isomorphic to $H_\bullet(C^t)$ for $\lambda \geqslant 0$; in particular

$$\dim_\Gamma H_i(W^t(\lambda)) = b_i^{(2)}(t).$$

So if $\lambda$, $t$ can be found such that $\dim_\Gamma W^t_i(\lambda) = m_i^{(2)}$ (where $m_i^{(2)}$ is the $i$th $L^2$ Morse number of $f$) then we will have the required chain complex for the application of Lemma 3.10.

In order to show that such a $t$ and $\lambda$ exist, an appeal will be made to the variational principle of Efremov and Shubin [EfSh, Theorem 3.1]:

THEOREM 3.14. *Let $A$ be a self-adjoint $\Gamma$-invariant operator, semi-bounded below. Let $E(\lambda) = E((-\infty, \lambda])$ be the corresponding spectral projection.*

*Then* $\mathrm{tr}_\Gamma E(\lambda) = \sup \mathrm{tr}_\Gamma P$ *where the supremum is taken over all affiliated projections such that* $\mathrm{im}\, P \subseteq D(A)$ *and* $P(A-\lambda)\, P \leqslant 0$.

When the operator $A$ is bounded (as it is in this case, with $A = \Delta^t$), the condition on $P$ becomes

$$\begin{aligned} P(A-\lambda)\, P \leqslant 0 &\Leftrightarrow \langle P(A-\lambda)\, Px, x\rangle \leqslant 0 \qquad && \forall x \\ &\Leftrightarrow \langle (A-\lambda)\, Px, Px\rangle \leqslant 0 && \forall x \\ &\Leftrightarrow \langle (A-\lambda)\, y, y\rangle \leqslant 0 && \forall y \in \mathrm{im}\, P \\ &\Leftrightarrow \langle Ay, y\rangle \leqslant \lambda\, \|y\|^2 && \forall y \in \mathrm{im}\, P. \end{aligned}$$

Given that $\mathrm{tr}_\Gamma P = \dim_\Gamma \mathrm{im}\, P$ we get

LEMMA 3.15. *Let* $W_i^t(\lambda)$ *be the Witten complex as defined above. Then*

$$\begin{aligned} \dim_\Gamma W_i^t(\lambda) &= \mathrm{tr}_\Gamma E^t((-\infty, \lambda])|_{C_i^t} \\ &= \sup \{\dim_\Gamma S \mid S \subset C_i^t \textit{ is a closed } \Gamma\textit{-invariant subspace} \\ &\qquad \textit{and } \langle \Delta^t c, c\rangle \leqslant \lambda\, \|c\|^2\ \forall c \in S\}. \end{aligned}$$

As before, let $R$ refer to the subspace of $C$ that is generated by the critical cells of $X$. Let $R'$ denote its orthogonal complement. We seek bounds on $\langle \Delta^t c, c\rangle$ for $c$ in $R$ and $R'$.

First, some bounds on the incidence numbers and numbers of faces within the CW complex need to be determined.

The closure-finiteness condition for a CW complex ensures that for any $p$-cell $\sigma$, $\{\tau^{(p+1)} : \tau > \sigma\}$ is finite. The existence of a finite cellular domain is sufficient to ensure that $\{\nu^{(p-1)} : \nu < \sigma\}$ is also finite, as shown in the following lemma.

LEMMA 3.16. *Let $X$ be a CW complex with cells $K$ and a free cellular $\Gamma$-action with a finite cellular domain $G \subset K$. Then every $p$-cell has a finite number of $(p-1)$-cell faces.*

*Proof.* Let $\sigma$ be a $p$-cell of $X$, and let $N = \{\nu_i : i \in I\}$ be the set of $(p-1)$-cell faces of $\sigma$. For each $\nu_i$ there exists $\gamma_i \in \Gamma$ such that $\gamma_i \nu_i \in G$. For each $(p-1)$-cell $\nu$ in $G$, let $I_\nu = \{i : \gamma_i \nu_i = \nu\} \subset I$. It follows then that $\nu < \gamma_i \sigma$ $\forall i \in I_\nu$.

$\Gamma$ acts freely, so the closure-finiteness condition implies that $I_\nu$ is finite for all $\nu \in G$. $\# N = \sum_{\nu \in G} \# I_\nu$, so $N$ is also finite. ∎

For each $p$-cell then, there is a bound on the number of $(p-1)$-cell faces and the number of $(p+1)$-cells of which it is a face. By virtue of the cellular domain being finite, a global bound can be found by taking the maximum of these values over the cells in the cellular domain. This global bound will be termed $B$,

$$B = \max_{\sigma \in K} \#\{\tau : \partial_{\sigma,\tau} \neq 0 \text{ or } \partial_{\tau,\sigma} \neq 0\}.$$

As one would expect, the incidence numbers $\partial_{\sigma,\tau}$ are also bounded.

LEMMA 3.17. *Let $X$ be a CW complex with cells $K$ and a free cellular $\Gamma$-action, with finite cellular domain $G$. Then $\{[\sigma : \nu] : \sigma, \nu \in K\}$ is bounded.*

*Proof.* Closure finiteness implies $\{\sigma : [\sigma : \nu] \neq 0\}$ is finite. So the set $A = \{[\sigma : \nu] : \nu \in G, \sigma \in K\}$ is finite.

For any cell $\nu \in K$ there exists some $\gamma \in \Gamma$ such that $\gamma\nu \in G$, and so

$$[\sigma : \nu] = [\gamma\sigma : \gamma\nu] \in A.$$

So $\{[\sigma : \nu] : \sigma, \nu \in K\}$ is finite, and hence bounded. ∎

Let the bound on the incidence numbers be denoted by $\zeta$,

$$\zeta = \sup\{[\tau : \sigma] : \sigma, \tau \in K\}.$$

Lastly, as the set of differences $\{f(\tau) - f(\sigma) : \partial_{\sigma,\tau} \neq 0\}$ is finite for almost $\Gamma$-invariant $f$ (Lemma 3.4), the set $\{f(\tau) - f(\sigma) : f(\tau) > f(\sigma), \partial_{\sigma,\tau} \neq 0\}$ has a positive infinum. Further, as $f$ is Witten–Morse, the sets $\{2f(\tau^{(p+1)}) - f(\sigma^{(p)} - f(\sigma'^{(p)}) : \sigma, \sigma' < \tau\}$ and $\{f(\sigma^{(p)}) + f(\sigma'^{(p)}) - 2f(\nu^{(p-1)}) : \nu < \sigma, \sigma'\}$ also have a positive infinum. These bounds are captured in a constant $\kappa$,

$$\begin{aligned}\kappa = \max\{&\exp -\inf\{f(\tau) - f(\sigma) : f(\tau) > f(\sigma), \partial_{\sigma,\tau} \neq 0\},\\ &\exp -\inf\{2f(\tau^{(p+1)}) - f(\sigma^{(p)}) - f(\sigma'^{(p)}) : \sigma, \sigma' < \tau\},\\ &\exp -\inf\{f(\sigma^{(p)}) + f(\sigma'^{(p)}) - 2f(\nu^{(p-1)}) : \nu < \sigma, \sigma'\}\},\end{aligned}$$

$\kappa$ is a positive number less than one.

With these bounds at our disposal, examine $|\partial^t_{\sigma,\tau}|$.

From $\partial^t_{\sigma,\tau} = \partial_{\sigma,\tau} e^{t(f(\tau) - f(\sigma))}$, and that $V_f\sigma = \tau$ implies $[\tau : \sigma] = \pm 1$ ($\sigma$ must be a regular face of $\tau$) one gets

$$|\partial^t_{\sigma,\tau}| = \begin{cases} 0 & \text{if } \sigma \nless \tau \text{ or } \deg \tau \neq \deg \sigma + 1 \\ e^{-t(f(\tau) - f(\sigma))} \geqslant 1 & \text{if } V_f\sigma = \tau \text{ (and thus } f(\sigma) \geqslant f(\tau)) \\ |[\tau : \sigma]|\, e^{-t(f(\tau) - f(\sigma))} \leqslant \zeta\kappa^t & \text{otherwise.} \end{cases}$$

As we are interested in the situation where $t$ becomes large, there is no danger in assuming that $t$ is sufficiently great that $\zeta\kappa^t < 1$.

Examining $\langle \Delta^t c, c\rangle$,

$$\begin{aligned}\langle \Delta^t c, c\rangle &= \sum_{\sigma, \sigma'} \langle c, \sigma'\rangle\langle \sigma, c\rangle\langle \Delta^t\sigma, \sigma'\rangle\\ &= \sum_{\sigma, \sigma'} c_\sigma c_{\sigma'} \Delta^t_{\sigma, \sigma'}\\ &= \sum_{\sigma} c_\sigma^2 \Delta^t_{\sigma, \sigma} + \sum_{\sigma \neq \sigma'} c_\sigma c_{\sigma'} \Delta^t_{\sigma, \sigma'}.\end{aligned}$$

The components $\Delta^t_{\sigma, \sigma'}$ are given by

$$\Delta^t_{\sigma, \sigma'} = \sum_{v} \partial^t_{v, \sigma}\partial^t_{v, \sigma'} + \sum_{\tau} \partial^t_{\sigma, \tau}\partial^t_{\sigma', \tau}.$$

Consider the first term when $\sigma \neq \sigma'$. If $V_f v = \sigma$ or $\sigma'$, one of $[\sigma : v]$ and $[\sigma' : v]$ will be 1, while the other is at most $\zeta$. One then has

$$\begin{aligned}|\partial^t_{v, \sigma}\partial^t_{v, \sigma'}| &\leqslant \zeta e^{-t(f(\sigma) + f(\sigma') - 2f(v))}\\ &\leqslant \zeta\kappa^t.\end{aligned}$$

If this is not the case, both $|\partial^t_{v, \sigma}|$ and $|\partial^t_{v, \sigma}|$ are at most $\zeta\kappa^t$, so their product is at most $(\zeta\kappa^t)^2 < \zeta\kappa^t$. Similarly the terms $\partial^t_{\sigma, \tau}\partial^t_{\sigma', \tau}$ of the second sum have an absolute value of at most $\zeta\kappa^t$ when $\sigma \neq \sigma'$.

When $\sigma = \sigma'$ is critical, both $|\partial^t_{v, \sigma}|$ and $|\partial^t_{\sigma, \tau}|$ are at most $\zeta\kappa^t$. Each term in the sums is a square and thus greater than or equal to zero and at most $\zeta\kappa^{2t} < \zeta\kappa^t$.

If $\sigma = \sigma'$ is not critical, then there will exist exactly one $v^{(p-1)} < \sigma^{(p)}$ or $\tau^{(p+1)} > \sigma^{(p)}$ such that $V_f v = \sigma$ or $V_f\sigma = \tau$. Hence there will be exactly one term in the expression for $\Delta^t_{\sigma, \sigma'}$ that is greater than or equal to one. As each term is a square, the other terms will have values between zero and $\zeta\kappa^{2t}$, as in the critical case.

Note that in each of the sums, the number of non-zero terms must be bounded by $B$.

Let $d$ denote the simplicial metric. Two $p$-cells $\sigma$, $\sigma'$ have $d(\sigma, \sigma') = 0$ if they are identical, and $d(\sigma, \sigma') = 1$ if there exists a $(p-1)$-cell $v$ or $(p+1)$-cell $\tau$ with $\sigma > v < \sigma'$ or $\sigma < \tau > \sigma'$. $\Delta^t_{\sigma, \sigma'}$ must be zero if $d(\sigma, \sigma') > 1$, or if $\sigma \neq \deg \sigma'$.

Bounds on the elements $\Delta^t_{\sigma, \sigma'}$ are thus

$$\begin{aligned}&\Delta^t_{\sigma, \sigma'} = 0 && \text{when} \quad \deg \sigma \neq \deg \sigma' \text{ or } d(\sigma, \sigma') > 1,\\ &\Delta^t_{\sigma, \sigma'} \geqslant 1 && \text{when} \quad \sigma = \sigma' \text{ is non-critical},\\ &|\Delta^t_{\sigma, \sigma'}| \leqslant 2B\zeta\kappa^t && \text{otherwise}.\end{aligned}$$

Examine the second term in the expansion of $\langle \Delta^t c, c\rangle$,

$$\begin{aligned}\left|\sum_{\sigma\neq\sigma'} c_\sigma c_{\sigma'}\,\Delta^t_{\sigma,\sigma'}\right| &= \left|\sum_{d(\sigma,\sigma')=1} c_\sigma c_{\sigma'}\,\Delta^t_{\sigma,\sigma'}\right|\\ &\leqslant \sum_{d(\sigma,\sigma')=1} |c_\sigma c_{\sigma'}\,\Delta^t_{\sigma,\sigma'}|\\ &\leqslant 2B\zeta\kappa^t \sum_{d(\sigma,\sigma')=1} |c_\sigma c_{\sigma'}|\\ &= 2B\zeta\kappa^t\,|\langle F\bar{c}, \bar{c}\rangle|,\end{aligned}$$

where $\bar{c} = \sum |c_\sigma|\,\sigma$ and $F\colon C_\bullet \to C_\bullet$ is the chain map that takes a cell $\sigma$ to the sum of all its neighbours $\sigma'$ with $d(\sigma', \sigma) = 1$:

$$Fc = \sum_\sigma \left(\sum_{d(\sigma,\sigma')=1} c_{\sigma'}\right)\sigma.$$

As each $p$-cell has at most $B$ $(p-1)$-cell faces and $(p+1)$-cells of which it is a face, the norm of $F$ is crudely bounded by $B^2$. Noting that the norm of $\bar{c}$ is the same as that of $c$, one has

$$\left|\sum_{\sigma\neq\sigma'} c_\sigma c_{\sigma'}\,\Delta^t_{\sigma,\sigma'}\right| \leqslant 2B\,\|F\|\,\|\bar{c}\|^2\,\zeta\kappa^t \leqslant 2B^3\,\|c\|^2\,\zeta\kappa^t.$$

If $c\in R$, $c_\sigma$ is zero for all non-critical cells $\sigma$. One then has

$$\begin{aligned}|\langle \Delta^t c, c\rangle| &\leqslant \left|\sum_\sigma c_\sigma^2\,\Delta^t_{\sigma,\sigma}\right| + \left|\sum_{\sigma\neq\sigma'} c_\sigma c_{\sigma'}\,\Delta^t_{\sigma,\sigma'}\right|\\ &\leqslant 2B\zeta\kappa^t\,\|c\|^2 + 2B^3\zeta\kappa^t\,\|c\|^2.\end{aligned}$$

If $c\in R'$, $c_\sigma$ is zero for all critical cells $\sigma$, giving

$$\begin{aligned}\langle \Delta^t c, c\rangle &= \sum_\sigma c_\sigma^2\,\Delta^t_{\sigma,\sigma} + \sum_{\sigma\neq\sigma'} c_\sigma c_{\sigma'}\,\Delta^t_{\sigma,\sigma'}\\ &\geqslant \|c\|^2 + \sum_{\sigma\neq\sigma'} c_\sigma c_{\sigma'}\,\Delta^t_{\sigma,\sigma'}\\ &\geqslant \|c\|^2 - 2B^3\zeta\kappa^t\,\|c\|^2.\end{aligned}$$

So letting $A = 2B\zeta(1+B^2)$ be a positive constant,

$$\begin{aligned}&\langle \Delta^t c, c\rangle \leqslant A\kappa^t\,\|c\|^2, && c\in R\\ &\langle \Delta^t c, c\rangle > (1 - A\kappa^t)\,\|c\|^2, && c\in R'.\end{aligned}$$

Choosing sufficiently large $T$, $t > T$ implies

$$|\langle \Delta^t c, c \rangle| \leqslant \tfrac{1}{3} \|c\|^2 \qquad \text{if} \quad c \in R,$$
$$|\langle \Delta^t c, c \rangle| \geqslant \tfrac{2}{3} \|c\|^2 \qquad \text{if} \quad c \in R'.$$

Suppose $S$ is a $\Gamma$-invariant subspace of $C_i^t$ with $c \in S \Rightarrow |\langle \Delta^t c, c \rangle| \leqslant \frac{1}{3} \|c\|^2$. Then $S \cap R_i' = \{0\}$, and so $\dim_\Gamma S \leqslant \dim_\Gamma R_i$.

This allows the application of Lemma 3.15 to give for $\lambda = \frac{1}{3}$ and $t > T$

$$\begin{aligned} \dim_\Gamma W_i^t(\lambda) &= \dim_\Gamma E^t([-\infty, \lambda])\, C_i^t \\ &= \sup \{\dim_\Gamma S \mid S \subset C_i^t \text{ is a closed } \Gamma\text{-invariant subspace} \\ &\qquad \text{and } \langle \Delta^t c, c \rangle \leqslant \tfrac{1}{3} \|c\|^2\ \forall c \in S\} \\ &= \dim_\Gamma R_i = m_i^{(2)}. \end{aligned}$$

Recalling that $\dim_\Gamma H_i(W^t(\lambda)) = b_i^{(2)}(t)$, the strong asymptotic Morse inequality for discrete Morse functions (Proposition 3.8) follows from Lemma 3.10.

Recall Theorem 2.11, which states that every gradient vector field $V$ with bounded $V$-paths is the gradient of a Witten–Morse function. Let $f$ be a $\Gamma$-invariant discrete Morse function. As $f$ can take only a finite number of values, and as the value of $f$ must decrease strictly along $V_f$-paths, it follows that all the $V_f$-paths are bounded. Therefore $f$ is equivalent to a Witten–Morse function $f'$, which by Forman's construction [Form2, Theorem 1.4] is also $\Gamma$-invariant. Recall further that $f'$ $\Gamma$-invariant implies that the chain map $e^{tf'}$ is a chain isomorphism from $C^t_\bullet$ to $C_\bullet$, and thus the deformed $L^2$ Betti numbers $b_p^{(2)}(t)$ are simply the $L^2$ Betti numbers $b_p^{(2)}$.

Applying the strong asymptotic Morse inequality to the equivalent Witten–Morse function $f'$ of a $\Gamma$-invariant discrete Morse function $f$ therefore gives us the non-asymptotic inequality (Proposition 3.3) as a corollary.

## 4. THE DISCRETE MORSE–FARBER TYPE INEQUALITIES

Our goal in this section is to study a discrete analogue of a question due to Gromov. If $f$ is a discrete $\Gamma$-invariant Morse function on $X$, can one give quantitative lower bounds for $m_p^{(2)}$? We will give an answer to this question in this section. In particular, we will show that if zero is in the spectrum of the Laplacian on $p$-forms on $X$, then $m_p^{(2)} > 0$ for any discrete $\Gamma$-invariant Morse function $f$ on $X$, even if $b_p = 0$ and $b_p^{(2)} = 0$, where the positivity of $m_p^{(2)}$ does not follow from the classical discrete Morse inequalities or from its $L^2$ analogue from the earlier sections.

4.1. *The Extended Category of Farber.* Farber in [Farb1, Farb2] constructs an abelian category called the *extended category of Hilbertian modules*, which is constructed from the morphisms of the category of Hilbertian modules through a technique first suggested by Freyd in [Freyd]. (Lück constructs an equivalent category using a different technique in [Luck].) This extended category naturally breaks down into projective and torsion components; the projective part has the von Neumann dimension as an invariant, while a Novikov–Shubin invariant is associated with the torsion part. Further, a new independent integer invariant—the "minimal number of generators"—can be associated with an object in this extended category.

Apart from its utility in allowing a homological treatment of the Novikov–Shubin invariants, the minimal number of generators invariant gives a new Morse-like inequality for smooth Morse functions on a closed manifold. Let $\mathscr{A}$-be a finite von Neumann algebra.

Definition 4.1. A *Hilbertian $\mathscr{A}$-module* is a space with continuous left $\mathscr{A}$ action such that there exists an inner product $\langle\ ,\ \rangle$ compatible with the topology of the space, and gives the space the structure of a Hilbert $\mathscr{A}$-module. Such inner products are called *admissible*.

A Hilbert $\mathscr{A}$-module is what was called a $L^2\Gamma$-module in the earlier sections, where $\mathscr{A}$ is in this case the von Neumann algebra generated by right translations on $L^2\Gamma$. It is useful to deal with Hilbertian modules in preference to Hilbert modules when the topology of a Hilbert module is important, but the choice of a particular inner product is unnecessary.

Using Farber's notation, denote the category of Hilbertian $\mathscr{A}$-modules by $\mathscr{H}(\mathscr{A})$. Recalling the definition of a Hilbert $\mathscr{A}$-module, the choice of a particular finite, normal and faithful trace on $\mathscr{A}$ is essential. Such a choice of trace, while not explicitly mentioned in the notation, is associated with the category. The morphisms of $\mathscr{H}(\mathscr{A})$ are continuous linear maps that commute with the $\mathscr{A}$ action. Note that the categorical image of a morphism in $\mathscr{H}(\mathscr{A})$ corresponds to the closure of the set image.

Definition 4.2. Denote the *extended category* of Hilbertian $\mathscr{A}$-modules by $\mathscr{E}(\mathscr{A})$. The objects of $\mathscr{E}(\mathscr{A})$ (termed *virtual Hilbertian modules*) are morphisms $(\alpha\colon A' \to A)$ in $\mathscr{H}(\mathscr{A})$.

The morphisms of $\mathscr{E}(\mathscr{A})$ are equivalence classes of morphisms in $\mathscr{H}(\mathscr{A})$, defined as follows. Let $(\alpha\colon A' \to A)$ and $(\beta\colon B' \to B)$ be objects of $\mathscr{E}(\mathscr{A})$, and consider the set $M$ of morphisms $f\colon A \to B$ in $\mathscr{H}(\mathscr{A})$ such that for each $f$ in $M$ there is some $\mathscr{H}(\mathscr{A})$ morphism $g\colon A' \to B'$ with $f \circ \alpha = g \circ \beta$. Then the set of morphisms from $(\alpha\colon A' \to A)$ to $(\beta\colon B' \to B)$ is $M/\sim$, the equivalence relation given by $f \sim f'$ if there exists some $\mathscr{H}(\mathscr{A})$-morphism $h$ with $f - f' = \beta \circ h$.

There is a full embedding of $\mathscr{H}(\mathscr{A})$ into $\mathscr{E}(\mathscr{A})$, taking an object $A$ of $\mathscr{H}(\mathscr{A})$ to the zero morphism $(0 \to A)$ in $\mathscr{E}(\mathscr{A})$, and taking $\mathscr{H}(\mathscr{A})$ morphisms $f\colon A \to B$ to $[f]\colon (0 \to A) \to (0 \to B)$ in $\mathscr{E}(\mathscr{A})$. Farber goes on to prove that an object of $\mathscr{E}(\mathscr{A})$ is projective if and only if it is isomorphic to a Hilbertian module (under this embedding).

The torsion objects of $\mathscr{E}(\mathscr{A})$ are the objects $(\alpha\colon A' \to A)$ with $A = \overline{\alpha(A')}$. These generate a full subcategory $\mathscr{T}(\mathscr{A})$ of $\mathscr{E}(\mathscr{A})$. There exist two contravariant functors $\mathscr{T}$ and $\mathscr{H}$ from $\mathscr{E}(\mathscr{A})$ to $\mathscr{T}(\mathscr{A})$ and $\mathscr{H}(\mathscr{A})$, respectively.

Chain complexes of Hilbertian $\mathscr{A}$-modules can then be regarded as complexes in $\mathscr{E}(\mathscr{A})$. The homology $\mathscr{H}_i$ of such chains in $\mathscr{E}(\mathscr{A})$ is well defined as $\mathscr{E}(\mathscr{A})$ is an abelian category. One can look at the torsion part of the homology $\mathscr{T}(\mathscr{H}_i(C))$ and at the projective part $\mathscr{P}(\mathscr{H}_i(C))$, which is identical to the familiar reduced homology $H_i^{(2)}(C)$.

Definition 4.3 (Definition 7.2 of [Farb]). Let $\mathscr{X}$ be an object of $\mathscr{E}(\mathscr{A})$. Then the *minimal number of generator* of $\mathscr{X}$ is denoted $\mu(\mathscr{X})$ and is the least integer for which there is an epimorphism from $\bigoplus_{i=1}^{\mu(\mathscr{X})} l^2(\mathscr{A})$ onto $\mathscr{X}$.

It has the following properties:

(1) $\mu(\mathscr{X}) = 0$ if and only if $\mathscr{X} = 0$; besides, $\mu(\mathscr{X}) = \mu(P(\mathscr{X}))$ if and only if $T(\mathscr{X}) = 0$.

(2) If $\mathscr{X}$ is projective, then $\mu(\mathscr{X}) \geqslant \dim_\Gamma \mathscr{X}$.

(3) if $\mathscr{X}'$ is another virtually Hilbertian $\mathscr{A}$-module, then

$$\max\{\mu(\mathscr{X}), \mu(\mathscr{X}')\} \leqslant \mu(\mathscr{X} \oplus \mathscr{X}') \leqslant \mu(\mathscr{X}) + \mu(\mathscr{X}').$$

(4) If $\mathscr{A}$ is a factor, then $\mu(\mathscr{X}) = 1$ for any non-trivial torsion object.

The first three properties together with calculations of this invariant can be found in [Farb1, Sect. 7]. The last property was explained in [MaSh].

4.2. *Discrete Morse–Farber Inequalities.* The following abstract theorem is due to Farber [Farb1, Theorem 8.1].

Theorem 4.4. *Let $C_\bullet$ be a chain complex of free finitely generated Hilbertian $\mathscr{A}$-modules in $\mathscr{E}(\mathscr{A})$. Then $\dim_\Gamma C_i \geqslant \mu[\mathscr{H}_i(C) \oplus \mathscr{T}(\mathscr{H}_{i-1}(C))]$ for all $i$.*

We will use this to construct the discrete $L^2$ Morse–Farber inequalities and their asymptotic counterpart. These are the discrete analogues of [Farb1, Theorem 8.2]; [MaSh, Theorem 4.2], and we follow the strategy of proof as in those papers. From now on $\mathscr{A}$ will denote the von Neumann algebra generated by the right action of $\Gamma$ on $\ell^2(\Gamma)$, or equivalently by the commutant of the left action of $\Gamma$ on $\ell^2(\Gamma)$.

Theorem 4.5. *Let $X$ be a CW complex with a finite cellular domain under a free cellular $\Gamma$-action, and let $f$ be a combinatorial Witten–Morse function on $X$ with $L^2$ Morse numbers $m_i^{(2)}$ as defined in Definition* 3.2.

(1) (*Discrete Morse–Farber inequalities*) *Suppose that $f$ is $\Gamma$-invariant. Then the discrete Morse–Farber inequality holds*

$$m_i^{(2)} \geqslant \mu[\mathscr{H}_i(C(X)) \oplus \mathscr{T}(\mathscr{H}_{i-1}(C(X)))],$$

*for all $i \geqslant 0$, where the $\mathscr{H}_i(C(X))$ are objects of $\mathscr{E}(\mathscr{A})$, being the homology of the chain complex $C(X)$.*

(2) (*Asymptotic discrete Morse–Farber inequalities*) *Suppose now that $f$ is almost $\Gamma$-invariant* (*see Subsection* 3.2). *Then the asymptotic discrete Morse–Farber inequality holds*

$$m_i^{(2)} \geqslant \mu[\mathscr{H}_i(C^t(X)) \oplus \mathscr{T}(\mathscr{H}_{i-1}(C^t(X)))], \qquad t \gg 0,$$

*for all $i \geqslant 0$, where the $\mathscr{H}_i(C^t(X))$ are objects of $\mathscr{E}(\mathscr{A})$, being the homology of the chain complex $C^t(X)$.*

*Proof.* For $\lambda$, $t > 0$, let $W^t_\bullet(\lambda)$ denote the Witten complex. Then we know that $\dim_\Gamma W_i^t(\lambda) = m_i^{(2)}$ for $t \gg 0$ (see Subsection 3.3). Now we apply Theorem 4.4 with $C = W^t(\lambda)$; in the case when the conditions in part (2) are satisfied by $f$, we can make use of the fact that the complexes $C^t(X)$ and $W^t(\lambda)$ are bounded $\Gamma$-chain homotopy equivalent for $t \gg 0$ (Lemma 3.13). To obtain the asymptotic discrete Morse–Farber inequalities, we need to also show that the $L^2\Gamma$-modules $W^t_\bullet(\lambda)$ are free for $t \gg 0$. In the case when $\mathscr{A}$ is a factor, since $\dim_\Gamma W_i^t(\lambda) = m_i^{(2)}$ for $t \gg 0$ is an *integer*, we see that $W_i^t(\lambda)$ is a free $L^2\Gamma$-module. One can use the factor decomposition of $\mathscr{A}$ to establish this fact in the general case.

If $f$ is also $\Gamma$-invariant, then $C^t(X)$ and $C(X)$ are bounded $\Gamma$-chain isomorphic by the morphism $e^{tf}$ for all $t \geqslant 0$, therefore we obtain the discrete Morse–Farber inequalities. ∎

The following corollary can be viewed as giving an answer to the discrete version of the question of Gromov that was mentioned at the beginning of the section. We give a quantitative lower bound for $m_i^{(2)}$ for any $\Gamma$-invariant discrete Morse function $f$ on $X$ when the spectrum of the Laplacian on $L^2$ $i$-chains on $X$ contains zero.

Corollary 4.6. *Let $X$ be a CW complex with a finite cellular domain under a free cellular $\Gamma$-action, and let $f$ be a combinatorial Witten–Morse* *function on X. Let $\lambda_0(\Delta_i^t)$ denote the bottom of the $L^2$ spectrum of the Witten Laplacian $\Delta_i^t$ acting on the complement of its $L^2$ kernel.*

(a) *Suppose that $\lambda_0(\Delta_i^t)=0$, i.e., there is no spectral gap at zero, and also that $f$ is $\Gamma$-invariant. Then*

$$m_i^{(2)} > \mu[P(\mathscr{H}_i(C(X)))] \geqslant b_i^{(2)}.$$

(b) *Suppose that for $t \gg 0$, $\lambda_0(\Delta_i^t)=0$, i.e. there is no spectral gap at zero, and also that $f$ is almost $\Gamma$-invariant* (*see Subsection* 3.2). *Then*

$$m_i^{(2)} > \mu[P(\mathscr{H}_i(C^t(X)))] \geqslant b_i^{(2)}(t) \qquad \forall t \gg 0.$$

*Proof.* Since $\lambda_0(\Delta_i^t)=0$, it follows that $T(\mathscr{H}_i(C^t(X)))$ is a non-trivial virtual Hilbertian $\mathscr{A}$-module. By property (1) of the minimal number of generators, one has $\mu(\mathscr{H}_i(C^t(X))) > \mu(P(\mathscr{H}_i(C^t(X))))$. By Theorem 4.5, part (1), and property (3) of the minimal number of generators invariant $\mu$, one has $m_i^{(2)} \geqslant \mu[\mathscr{H}_i(C^t(X))] > \mu[P(\mathscr{H}_i(C^t(X)))]$. The last inequality in part (b) follows from property (2) of this invariant. Part (a) is proved similarly, once we observe that since $f$ is $\Gamma$-invariant, that the complexes $C^t(X)$ and $C(X)$ are bounded $\Gamma$-chain isomorphic by the morphism $e^{tf}$ for all $t \geqslant 0$. ∎

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