COBORDISM, HOMOTOPY AND HOMOLOGY OF GRAPHS IN $R^3$

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§1. INTRODUCTION

Throughout this paper we work in the piecewise linear category. All maps in this paper are piecewise linear maps.

Our graphs are finite, consisting of finite number of vertices and finite number of edges. We consider a graph as a topological space i.e. as a one-dimensional CW complex.

By a spatial embedding we mean an embedding of a graph into the three-dimensional Euclidean space $R^3$.

The purpose of this paper is to study spatial embeddings of a graph under various topological equivalence relations.

The most natural topological equivalence relation is ambient isotopy. The classification of links up to ambient isotopy is the main theme of knot theory. On the other hand various important topological equivalence relations including isotopy, link cobordism or link concordance, link homotopy and so on are defined and studied for knots and links [5, 4, 14, 15]. In particular links in $R^3$ are classified up to link homotopy in [8]. But almost all study of spatial embeddings of graphs has been done only about ambient isotopy.

Definitions. Let $f, g : G \to R^3$ be spatial embeddings of a graph $G$. Let $I = [0, 1]$ be the unit closed interval. We say that a map $\Phi : G \times I \to R^3 \times I$ is

(a) level preserving iff there is a map $\phi_t : G \to R^3$ for each $t \in I$ such that $\Phi(x, t) = (\phi_t(x), t)$ for all $x \in G$, $t \in I$.

(b) locally flat iff each point of the image of $\Phi$ has a neighborhood $N$ such that the pair $(N, N \cap \Phi(G \times I))$ is homeomorphic to the standard disk pair $(D^3, D^2)$ or $(D^3 \times I, X_n \times I)$ for some non-negative integer $n$ where $(D^3, X_n)$ is shown in Fig. 1.

(c) between $f$ and $g$ iff there is a real number $\varepsilon > 0$ such that $\Phi(x, t) = (f(x), t)$ for all $x \in G$, $0 \leq t \leq \varepsilon$ and $\Phi(x, t) = (g(x), t)$ for all $x \in G$, $1 - \varepsilon \leq t \leq 1$.

Fig. 1.
We say that \( f \) and \( g \) are

1. **ambient isotopic** \( \iff \) there is a continuous family \( h_t: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \), \( 0 \leq t \leq 1 \) of self-homeomorphisms such that \( h_0 = id_{\mathbb{R}^3} \) and \( h_1 \circ f = g \). Or equivalently, there is a level preserving locally flat embedding \( \Phi: G \times I \rightarrow \mathbb{R}^3 \times I \) between \( f \) and \( g \);

2. **cobordant** \( \iff \) there is a locally flat embedding \( \Phi: G \times I \rightarrow \mathbb{R}^3 \times I \) between \( f \) and \( g \);

3. **isotopic** \( \iff \) there is a level preserving embedding \( \Psi: G \times I \rightarrow \mathbb{R}^3 \times I \) between \( f \) and \( g \);

4. **I-equivalent** \( \iff \) there is an embedding \( \Phi: G \times I \rightarrow \mathbb{R}^3 \times I \) between \( f \) and \( g \);

5. **homotopic** \( \iff \) \( g \) is obtained from \( f \) by a series of self-crossing changes (Fig. 2) and ambient isotopy;

6. **weakly homotopic** \( \iff \) \( g \) is obtained from \( f \) by a series of crossing changes of adjacent edges (Fig. 3) and ambient isotopy;

7. **homologous** \( \iff \) there is a locally flat embedding \( \Phi: (G \times I) \# \bigcup_{i=1}^{n} S_i \rightarrow \mathbb{R}^3 \times I \) between \( f \) and \( g \) where \( n \) is a natural number, \( S_i \) is a closed orientable surface and \( \# \) means the connected sum. More precisely, there is an edge \( e \) of \( G \) for each \( i \) such that \( S_i \) is attached to an open disk \( int(e \times I) \) by the usual connected sum of surfaces;

8. **\( Z_n \)-homologous** \( \iff \) there is a locally flat embedding \( \Phi: (G \times I) \# \bigcup_{i=1}^{n} S_i \rightarrow \mathbb{R}^3 \times I \) between \( f \) and \( g \) where \( n \) is a natural number and \( S_i \) is a closed (possibly non-orientable) surface.

We have the following fundamental theorem which establishes the relations of these equivalence relations.

**Fundamental Theorem.**

\[
(2) \quad (1) \rightarrow (4) \rightarrow (6) \rightarrow (7) \rightarrow (8).
\]

Thus, for example, if \( f \) and \( g \) are cobordant, then they are I-equivalent and therefore homotopic and so on. Thus if there is a homotopy invariant of spatial embeddings of a graph \( G \), then it is automatically an I-equivalence invariant and so forth.

**Remarks.**

(a) The definitions (1), (2), (3) and (4) are natural generalizations of the concepts ambient isotopy, link cobordism (or link concordance), isotopy and piecewise linear I-equivalence of links. The definition (5) is a graph version of Milnor’s link homotopy [14]. The definition (6)
makes its proper sense only for graphs. The definitions (7) and (8) are natural generalizations of the concepts link homology and \( Z_2 \)-link homology respectively. But link homology and \( Z_2 \)-link homology are not well-known concepts because they are completely determined by linking number and \( Z_2 \)-linking number respectively, cf. [16].

(b) Fundamental theorem is already established for links [2, 6, 7].

c) These eight equivalence relations are indeed different equivalence relations.

Two links of Fig. 4(a) are cobordant but not isotopic because a non-split link is not isotopic to a split link. Two knots of Fig. 4(b) are isotopic but not cobordant because they have different signatures. These two examples show that the converses of the first four implications of Fundamental theorem does not hold. Two links of Fig. 4(c) are homotopic but not \( I \)-equivalent because they have different Milnor’s \( \mu \)-invariants. Two spatial embeddings of Fig. 4(d) are weakly homotopic but not homotopic that is shown in §3. Two links of Fig. 4(e) are homologous but not weakly homotopic because weak homotopy implies homotopy for links and these links are not link homotopic detected by Milnor’s \( \mu \)-invariant. Finally, two links of Fig. 4(f) are \( Z_2 \)-homologous but not homologous detected by linking number.

d) The cobordism classes of spatial embeddings of the theta curve form a group under the vertex connected sum [18]. This is an evidence of the naturality and the advantage of the concept of cobordism of spatial graphs. We will discuss on homotopy and homology of spatial graphs in [19] and [20].

Fig. 4.
Let $A$ be one of ambient isotopy, cobordism, . . . , and $Z_2$-homology. We note here that $A$-equivalence behaves naturally under the subdivision of graphs. That is:

**Proposition 1.1.** Let $f, g : G \to R^3$ be spatial embeddings of a graph $G$. Let $G'$ be a subdivision of $G$. Let $f', g' : G' \to R^3$ be spatial embeddings of $G'$ defined naturally by $f$ and $g$ respectively. Then $f$ and $g$ are $A$-equivalent if and only if $f'$ and $g'$ are $A$-equivalent.

Before starting knot theory, we must examine whether or not the knotting phenomenon really exists. Namely we next decide whether or not a graph has spatial embeddings that are not $A$-equivalent.

**Definitions.** A graph $G$ is unique up to $A$-equivalence iff any two spatial embeddings of $G$ are $A$-equivalent.

A graph $G$ is called a generalized bouquet iff there is a vertex $v$ of $G$ such that $G - v$ is acyclic i.e. the first Betti number $\beta_1(G - v) = 0$.

A graph $H$ is called a minor of a graph $G$ iff $H$ is obtained from $G$ by a series of taking a subgraph and edge contraction.

For an integer $n \geq 3$, an $n$-wheel $W_n$ is a graph that is the join of an $n$-cycle $C_n$ and a vertex $v$. See Fig. 5(a). An edge $e$ of $W_n$ is called a spoke iff $e$ is incident to $v$.

A loopless graph $G$ is called a multi-spoke $n$-wheel iff the underlying simple graph of $G$ is $W_n$ and only spokes may have multi-edges.

A loopless graph $G$ is called a double-trident iff the underlying simple graph of $G$ is the graph of Fig. 5(b) and only the edges joining the vertices both of which have valence four may have multi-edges.

For a graph $G$, let $G^*$ be the maximal subgraph of $G$ that has no vertices of valence one and no isolated vertices. We call $G^*$ the reduced graph of $G$.

**Theorem A.** For a graph $G$, the following conditions are mutually equivalent:

(i) $G$ is unique up to ambient isotopy.

(ii) $G$ is unique up to cobordism.

(iii) $G$ is acyclic.

(iv) $G$ does not contain any subdivision of the loop ($G_1$ of Fig. 6).

(v) The loop is not a minor of $G$.

**Theorem B.** For a graph $G$, the following conditions are mutually equivalent:

(vi) $G$ is unique up to isotopy.

(vii) $G$ is unique up to $I$-equivalence.

(viii) $G$ is unique up to homotopy.

(ix) $G$ is a generalized bouquet.

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![Fig. 5](image-url)
COBORDISM, HOMOTOPY AND HOMOLOGY OF GRAPHS IN $R^3$

Fig. 6.

$G_1, G_2, G_3 = K_4, G_4, G_5 = K_5, G_6 = K_{2, 2}$

(x) $G$ does not contain any subdivision of the graphs $G_2, G_3$ and $G_4$ of Fig. 6.

(xi) None of $G_2, G_3$ and $G_4$ is a minor of $G$.

**THEOREM C.** For a graph $G$, the following conditions are mutually equivalent.

(xii) $G$ is unique up to weak homotopy.

(xiii) $G$ is unique up to homotopy.

(xiv) $G$ is unique up to $Z_2$-homology.

(xv) $G$ is a planar graph which does not contain disjoint cycles.

(xvi) $G$ is a generalized bouquet or $G^*$ is a subdivision of a multi-spoke wheel or a subgraph of a double-trident.

(xvii) $G$ does not contain any subdivision of the graphs $G_2, G_5$ and $G_6$ of Fig. 6.

(xviii) None of $G_2, G_5$ and $G_6$ is a minor of $G$.

§2. PROOF OF FUNDAMENTAL THEOREM

The first four implications of the fundamental theorem follow directly from the definitions. In order to prove that $I$-equivalence implies homotopy, we prepare some lemmas.

**Lemma 2.1.** Let $t_n$ be a graph shown in Fig. 7. Let $f: t_n \to D^3$ be an embedding of $t_n$ into the unit 3-ball $D^3$ such that $f(t_n) \cap \partial D^3 = \bigcup_{i=1}^{n} f(v_i)$. Then we can deform $f$ into the standard embedding, which maps each edge onto a straight line segment, by a series of self-crossing changes and ambient isotopy.

**Proof.** We will deform $f$ step by step by a series of self-crossing changes and ambient isotopy such that the first $k$ edges $f(uv_1), f(uv_2), \ldots, f(uv_k)$ are straight line segments for $k = 1, 2, \ldots, n$. The first step is clear.

Suppose that $f(uv_j)$ is a straight line segment for $1 \leq j \leq k - 1$. We fix a regular projection and trace the image $f(uv_k)$ of the edge $uv_k$ from $f(u)$ to $f(v_k)$. If $f(uv_k)$ has crossings with $f(uv_j)$, $1 \leq j \leq k - 1$, then we can eliminate them one by one by ambient isotopy and self-crossing changes of $f(uv_k)$. See Fig. 8. Thus we complete the proof.

We note here that $f$ is isotopic to the standard embedding by Alexander's trick. Conversely, suppose that $\Phi: G \to I \to R^3 \times I$ is an isotopy between $g$ and $h$. Since we are working in the piecewise linear category, $\Phi$ has only finitely many non-locally flat points.
Fig. 8.

The embedding changes only at the level that has non-locally flat points. Therefore $h$ is obtained from $g$ by a series of birth or death of local knots and local change near a vertex like $f$ and the standard embedding of Lemma 2.1. Therefore Lemma 2.1 proves that 'isotopy implies homotopy'. Then, roughly speaking, 'equivalence implies homotopy' follows from 'cobordism implies homotopy', which will be proved by the same method that is used for links in [7]. See also [6].

For this purpose we next state the normalization of $I$-equivalence. This is a natural generalization of the normalization of link cobordism given in [11] and [21].

**Lemma 2.2.** Let $f, g : G \to R^3$ be $I$-equivalent spatial embeddings of a graph $G$. Then there is an embedding $\Phi : G \times I \to R^3 \times I$ between $f$ and $g$ with the following properties:

(a) The composition $\pi \circ \Phi_{|v \times 1} : v \times I \to I$ is a homeomorphism for each vertex $v$ of $G$, where $\pi : R^3 \times I \to I$ is a natural projection.

(b) The image of $\Phi$ has only finitely many non-locally flat points. All of them lie in $R^3 \times \{\frac{1}{4}\}$.

(c) The map $\pi \circ \Phi_{|e \times 1} : e \times I \to I$ has only finitely many critical points in $\text{int}(e \times I)$ for each edge $e$ of $G$, consisting of minimal points, saddle points and maximal points.

(d) All of the minimal points lie in $R^3 \times \{\frac{1}{4}\}$ and all of the maximal points lie in $R^3 \times \{\frac{3}{4}\}$.

(e) All of the saddle points lie in $R^3 \times \{\frac{3}{4}\}$ and $R^3 \times \{\frac{1}{4}\}$ such that the cross-section $\Phi(G \times I) \cap R^3 \times \{\frac{3}{4}\}$ is isomorphic to $G$.

The following proof of this lemma is essentially same as that of the normalization of link cobordism given in Section 3 of [11]. We give a sketch proof here. We refer [11] or Section 1 of [21] for detailed discussions.

**Proof.** By the assumption we have an embedding $\Phi : G \times I \to R^3 \times I$ between $f$ and $g$. Then it is easy to deform $\Phi$ so that $\Phi$ satisfies the conditions (a), (b), (c), (d) and that all of the saddle points lie in $R^3 \times [\frac{3}{4}, \frac{1}{4}]$. For (e) we first gather all saddle points in $R^3 \times \{\frac{3}{4}\}$. Then we may consider that $\Phi(G \times I) \cap R^3 \times \{\frac{3}{4}\}$, is homotopy equivalent to a graph obtained from $\Phi(G \times I) \cap R^3 \times \{\frac{3}{4}\}$ by adding some edges that correspond to the saddle points. Let $J$ be a graph whose vertices correspond to the components of $\Phi(G \times I) \cap R^3 \times \{\frac{3}{4}\}$ and whose edges correspond to the saddle points. Thus $J$ may have some loops. Let $T$ be a maximal acyclic subgraph of $J$. We pull down the saddle points that correspond to the edges of $T$ to the level of $R^3 \times \{\frac{3}{4}\}$ and pull up other saddle points to the level $R^3 \times \{\frac{1}{4}\}$. Then we have a desired embedding. 

\[\square\]
Remark. If \( \Phi \) satisfies the conditions of Lemma 2.2, then the number of minimal points equals the number of saddle points in \( \mathbb{R}^3 \times \{ \frac{1}{2} \} \) and the number of maximal points equals the number of saddle points in \( \mathbb{R}^3 \times \{ \frac{3}{2} \} \). This follows by counting the Euler characteristic of \( G \times I \).

Proof of (4) \( \rightarrow \) (5). Let \( f, g : G \rightarrow \mathbb{R}^3 \) be \( I \)-equivalent embeddings. Let \( \Phi : G \times I \rightarrow \mathbb{R}^3 \times I \) be an embedding between \( f \) and \( g \) that satisfies the conditions of Lemma 2.2. Let \( f' : G \rightarrow \mathbb{R}^3 \) be an embedding defined by the section \( \Phi(G \times I) \cap \mathbb{R}^3 \times \{ \frac{1}{2} \} \). Then we have that \( f \) and \( f' \) are isotopic embeddings and \( f' \) and \( g \) are cobordant embeddings. Then by Lemma 2.1 we have that \( f \) is homotopic to \( f' \). Let \( f'' : G \rightarrow \mathbb{R}^3 \) be an embedding defined by the section \( \Phi(G \times I) \cap \mathbb{R}^3 \times \{ \frac{3}{2} \} \). Then the same argument of [7] shows that \( f' \) is homotopic to \( f'' \) and \( g \) is homotopic to \( f'' \). Thus we have that \( f \) is homotopic to \( g \).

Proof of (5) \( \rightarrow \) (6). A self-crossing change may be replaced by crossing changes of adjacent edges as illustrated in Fig. 9. This proves that homotopy implies weak homotopy.

Proof of (6) \( \rightarrow \) (7). A crossing change of adjacent edges is realized by an orientable one-handle in \( \mathbb{R}^3 \times I \) as illustrated in Fig. 10. This shows that weak homotopy implies homology.

The last implication of Fundamental Theorem follows directly from the definitions.

§3. PROOF OF THEOREM A AND THEOREM B

We first note the following proposition.

Proposition 3.1. Let \( G \) and \( H \) be graphs.

(a) Suppose that \( G \) is a subdivision of \( H \). Then \( G \) is unique up to \( A \)-equivalence if and only if \( H \) is unique up to \( A \)-equivalence.

(b) If \( H \) is a subgraph of \( G \) and \( H \) is not unique up to \( A \)-equivalence, then \( G \) is not unique up to \( A \)-equivalence.

Theorem A follows from the previous remark that the loop has spatial embeddings that are not cobordant and hence not ambient isotopic (Fig. 4(b)).

Next we prove Theorem B.

Proof of (vi) \( \rightarrow \) (vii) \( \rightarrow \) (viii). This is a direct consequence of Fundamental Theorem.
For the proof of (viii) $\rightarrow$ (x), it is sufficient to show that the graphs $G_2$, $G_3 = K_4$, and $G_4$ of Fig. 6, respectively have different embeddings up to homotopy as illustrated in Fig. 11 (a), (b) and (c).

The first one is detected by linking number. For (b) and (c), we define a homotopy invariant of spatial embeddings of $G_3$ and $G_4$.

Let $\mathcal{C}(K) \in \{0, 1\}$ be the Robertello-Arf invariant of a knot $K$ [1], [17]. Let $G = G_3$ or $G_4$, and let $\Gamma_G$ be the set of all 3-cycles and 4-cycles of $G$. For an embedding $f: G \rightarrow \mathbb{R}^3$, we define $\alpha(f) \in \{0, 1\}$ by

\[ \alpha(f) \equiv \sum_{\gamma \in \Gamma_G} \alpha(f(\gamma)) \quad (\text{mod } 2). \]

In [3], Conway and Gordon has defined this invariant for the complete graph $K_7$ with $\Gamma_K$, being the set of all Hamiltonian cycles of $K_7$, and showed that this invariant is invariant under any crossing changes. They calculated this invariant for a particular embedding of $K_7$ and showed that the value is 1. As $\alpha(\text{unknot}) = 0$, they could conclude that every spatial embedding of $K_7$ contains a nontrivially knotted (Hamiltonian) cycle.

Now we assert that:

**Theorem 3.2.** $\alpha(f)$ is a homotopy invariant

**Proof.** We will show that $\alpha(f)$ is invariant under any self-crossing changes of $G_3$ and $G_4$. The proof is considerably simpler than that of $K_7$ in [3]. The key fact here is the following equality [9, 3, 13]:

\[ \alpha(K_+) - \alpha(K_-) \equiv \text{lk}(L_0) \quad (\text{mod } 2) \]
where $K_+, K_-$ and $L_0$ are knots and a two-component link as illustrated in Fig. 12 and $lk$ denotes the linking number.

Let $G = G_3$ or $G_4$. Suppose that an embedding $g: G \to R^3$ is obtained from an embedding $f: G \to R^3$ by a self-crossing change on an edge $e$ of $G$. Then we have

\[ \alpha(f) - \alpha(g) = \sum_{\gamma \in \Gamma_0} \alpha(f(\gamma)) - \sum_{\gamma \in \Gamma_0} \alpha(g(\gamma)) = \sum_{\gamma \in \Gamma_0, \gamma \neq e} \text{lk}(f(\gamma)) \]

where $f(\gamma) = f(\gamma) \cup K_j$ is the two-component link that forms a triple $L_0, K_+, K_-$ of Fig. 12 with the knots $f(\gamma)$ and $g(\gamma)$. We remark here that the component $K_j$ is common for all cycles in $\Gamma_0$ that contain the edge $e$. It is easy to check that the homological sum $\sum_{\gamma \in \Gamma_0, \gamma \neq e} f(\gamma)$ is always zero modulo 2. Therefore we have

\[ \alpha(f) - \alpha(g) = \sum_{\gamma \in \Gamma_0, \gamma \neq e} \text{lk}(f(\gamma), K_j) \equiv \text{lk}(0, K_j) = 0 \pmod{2}. \]

This completes the proof.

Proof of (viii) $\Rightarrow$ (x). It is easy to check that the spatial embeddings of Fig. 11(b) have different $c$-invariants, hence they are not homotopic. The embeddings of Fig. 11(c) also have different $c$-invariants and they are not homotopic. Therefore we conclude that the graphs $G_2, G_3$ and $G_4$ are not unique up to homotopy.

**Proof of (x) $\Rightarrow$ (xi).** Let $G$ be a graph which does not contain any subdivision of $G_2$, $G_3$ and $G_4$. If $G$ has at most one cycle, then $G$ is a generalized bouquet. If $G$ has at least two cycles, then they intersect. Therefore $G$ contains a subdivision of $H_1$ or $H_2$ of Fig. 13. If $G$ contains a subdivision of $H_1$ and does not contain any subdivision of $H_3$, then every cycle of $G$ must contain both $v_1$ and $v_2$ and $G$ is a generalized bouquet. If $G$ contains a subdivision of $H_2$ and there is a cycle of $G$ which does not contain the vertex $v$, then $G$ must contain a subdivision of $H_3$ of Fig. 13.

But then every cycle of $G$ must contain the vertex $v$ and $G$ is a generalized bouquet.

**Proof of (xi) $\Rightarrow$ (vi).** Let $G$ be a generalized bouquet such that $G - v$ is acyclic. Then for any two embeddings of $G$, we can deform them by ambient isotopy so that they are identical except a small neighborhood of $v$. Since isotopy kills such a difference, we have that they are isotopic.
Proof of (ix) \(\rightarrow\) (xi). The set of all generalized bouquets is closed under minor reduction i.e. every minor of a generalized bouquet is also a generalized bouquet. It is clear that none of \(G_2, G_3\) and \(G_4\) is a generalized bouquet. These facts imply the conclusion. \(\square\)

Proof of (xii) \(\rightarrow\) (x). It is clear. \(\square\)

\section{Proof of Theorem C}

Proof of (xii) \(\rightarrow\) (xiii) \(\rightarrow\) (xiv). This follows from Fundamental theorem. \(\square\)

In order to prove (xiv) \(\rightarrow\) (xvii), we must show that the graphs \(G_2, G_5 = K_5\) and \(G_6 = K_{3,3}\) respectively have different embeddings up to \(Z_2\)-homology. It is clear that the linking number modulo 2 is a \(Z_2\) homology invariant. Therefore the embeddings of Fig. 11(a) are not \(Z_2\)-homologous.

In the summer of 1990, J. Simon gave a lecture at Tokyo. In the lecture he defined an invariant for spatial embeddings of \(K_5\) and \(K_{3,3}\) as follows.

We give an orientation of the edges as illustrated in Fig. 14.

Let \(G = K_5\) or \(K_{3,3}\). For two disjoint edges \(x, y\) we define the sign \(\varepsilon(x, y) = \varepsilon(y, x)\) as follows:

\[
\varepsilon(e_i, e_j) = 1, \varepsilon(d_i, d_j) = -1 \quad \text{and} \quad \varepsilon(e_i, d_j) = -1 \quad \text{for} \quad i, j \in \{1, 2, 3, 4, 5\}.
\]

\[
\varepsilon(c_i, c_j) = 1, \varepsilon(b_k, b_l) = 1 \quad \text{and}
\]

\[
\varepsilon(c_i, b_k) = \begin{cases} 1 & \text{if} \quad c_i \text{ and } b_k \text{ are parallel in Fig. 14} \\ -1 & \text{if} \quad c_i \text{ and } b_k \text{ are anti-parallel in Fig. 14} \end{cases}
\]

for \(i, j \in \{1, 2, 3, 4, 5, 6\}, k, l \in \{1, 2, 3\}\).

Let \(f: G \to \mathbb{R}^3\) be an embedding and let \(\pi: \mathbb{R}^3 \to \mathbb{R}^2\) be a projection defined by \(\pi(x, y, z) = (x, y)\). Suppose that \(\pi \circ f\) is a regular projection. For two disjoint oriented edges \(x\) and \(y\) of \(G\), let \(l(f(x), f(y))\) be the sum of the signs of the mutual crossings \(\pi \circ f(x) \cap \pi \circ f(y)\) where the sign of a crossing is defined by Fig. 15.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig14}
\caption{Fig. 14.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig15}
\caption{Fig. 15.}
\end{figure}
Now we define an integer $L(f)$ by

$$L(f) = \sum_{x \cap y = 0} c(x, y) l(f(x), f(y))$$

where the summation is taken over all disjoint edge pairs of $G$.

**Theorem 4.1** (Simon). $L(f)$ is a well-defined ambient isotopy invariant. That is, if $f, g : G \to R^3$ are ambient isotopic embeddings and both $\pi \circ f$ and $\pi \circ g$ are regular projections then $L(f) = L(g)$.

**Proof.** It is known that two regular projections represent ambient isotopic embeddings if and only if they are connected by a sequence of Reidemeister moves (I ~ V) illustrated in Fig. 16 and ambient isotopy on $R^2$ [10].

Then it is easy to check that $L(f)$ is invariant under these moves. 0

Simon also proved that $L(f)$ is always an odd number. This follows from the observation that the change of $L(f)$ under a crossing change equals $-2, 0$ or $2$, and there is an embedding $f$ with $L(f) = 1$. See Fig. 19.

Thus we can define $L_2(f) \in \{-1, 1\}$ by

$$L_2(f) \equiv L(f) \pmod{4}.$$
We will prove:

**Theorem 4.2.** $L_2(f)$ is a $Z_2$-homology invariant.

The following lemma follows easily from the definition of $Z_2$-homology.

**Lemma 4.3.** Let $f, g: G \to R^3$ be spatial embeddings of a graph $G$. Suppose that both $\pi \circ f$ and $\pi \circ g$ are regular projections. Then $f$ and $g$ are $Z_2$-homologous if and only if $\pi \circ f$ and $\pi \circ g$ are connected by a series of Reidemeister moves (I $\sim$ V), ambient isotopy on $R^2$ and the following operations (VI) and (VII):

(VI) A birth or death of a trivial circle which belongs to one of the edges of $G$.

(VII) A hyperbolic transformation on an edge of $G$.

See Fig. 17.

Proof of Theorem 4.2. We will show that $L_2(f)$ is invariant under (VI) and (VII). For this purpose we must give an orientation to each trivial circle. But since the surfaces in the definition of $Z_2$-homology may be non-orientable, we cannot decide the orientation. But we assert that $L_2(f)$ is invariant under any choice of the orientations of the circles. This follows from the following facts:

(a) For any edge $e$ of $G = K_5$ or $K_{3,3}$, the edges of $G$ that are disjoint from $e$ forms a cycle in $G$.

(b) The number of intersection points of two immersed circles in $R^2$ in general position is always even.

Concerning the move (VII) we may have the necessity of changing the orientation as illustrated in Fig. 18.

But this does not change $L_2(f)$ by the same reason. This completes the proof.

We will discuss more on homology in [20].

Proof of (xiv) $\rightarrow$ (xvii). Let $f$, $g$, $h$ and $i$ be spatial embeddings as illustrated in Fig. 19. Then we have $L_2(f) = 1$, $L_2(g) = -1$, $L_2(h) = 1$ and $L_2(i) = -1$. Therefore $f$ is not $Z_2$-homologous to $g$ and $h$ is not $Z_2$-homologous to $i$. Thus we can conclude that the graphs $G_2$, $G_5 = K_5$ and $G_6 = K_{3,3}$ are not unique up to $Z_2$-homology.
Proof of (xvii) → (xv). The Kuratowski theorem [12] states that a graph $G$ is planar if and only if $G$ does not contain any subdivision of $K_5$ and $K_{3,3}$. Therefore the result follows.

For the proof of (xv) → (xvi), we prepare some lemmas.

**Lemma 4.4.** Let $G$ be a planar graph without disjoint cycles. If $G$ contains a subdivision of the 4-wheel $W_4$, then the reduced graph $G^*$ is a subdivision of a multi-spoke wheel or a double-trident.

**Proof.** The reduced graph $G^*$ is obtained from a subdivision of $W_4$ by a series of attaching edges. Therefore, if $G^*$ is not a subdivision of a multi-spoke 4-wheel, then we have either $G^*$ contains a subdivision of $W_4$ or $G^*$ contains a subdivision of the graph of Fig. 5(b). Then it is easy to see that the first case yields a multi-spoke wheel and the second case yields a double-trident.

**Lemma 4.5.** Let $G$ be a planar graph without disjoint cycles. Suppose that $G$ does not contain any subdivision of $W_4$ but contains a subdivision of $W_5 = K_5$. Then $G^*$ is a subdivision of a multi-spoke 3-wheel or a graph whose underlying simple graph is $W_5$ and only non-spoke edges may have multi-edges.

The proof of Lemma 4.5 is similar to that of Lemma 4.4 and we omit it.

**Lemma 4.6.** Let $G$ be a planar graph without disjoint cycles. Suppose that $G$ does not contain any subdivision of $W_3$ but contains a subdivision of the theta curve (H1 of Fig. 13). Then we have either $G$ is a generalized bouquet or $G^*$ is a subdivision of a loopless graph whose underlying simple graph is the 3-cycle $C_3$.

**Proof.** If $G^*$ has a cut vertex or a loop, then we easily have that $G$ is a generalized bouquet. Therefore we may suppose that $G^*$ has no cut vertices and loops. If $G$ does not
contain any subdivision of $H_3$ of Fig. 13, then $G^*$ is a subdivision of order $n$ theta curve and thus $G$ is a generalized bouquet. Suppose that $G$ contains a subdivision of $H_4$. If $G$ contains a cycle that does not contain the vertex $v$ of $H_3$, then we have that $G$ contains a subdivision of the graph $G_4$ of Fig. 6. Then we must have that $G^*$ is a desired graph.

Proof of (xv) $\rightarrow$ (xvi). By Lemmas 4.4, 4.5 and 4.6 it is sufficient to check the case that $G$ does not contain any subdivision of the theta curve. Then we easily have that $G$ is a generalized bouquet.

Proof of (xvi) $\rightarrow$ (xii). If $G$ is a generalized bouquet, then by Theorem B $G$ is unique up to isotopy hence unique up to weak homotopy. Suppose that $G^*$ is a multi-spoke wheel or a subgraph of a double-trident. It is sufficient to show that a crossing change of any two edges of $G^*$ can be replaced by crossing changes of adjacent edges of $G^*$. The proof repeatedly uses the technic of Fig. 9. We note here that we can use any one of four ends as illustrated in Fig. 20.

Let $G^*$ be a multi-spoke wheel. We first show that a crossing change between a spoke edge and a non-spoke edge can be replaced by crossing changes of adjacent edges. This is easily proved by the induction on the 'distance' of these two edges measured on the cycle of non-spoke edges. Next we show that a crossing change between two non-spoke edges can be replaced by crossing changes of adjacent edges. This is also proved by the induction on the 'distance' of these two edges. Thus we have the desired conclusion. The case that $G^*$ is a subgraph of a double trident is similar and we omit it.

Proof of (xv) $\rightarrow$ (xviii). It is easy to check that the set of planar graphs without disjoint cycles are closed under minor reduction. Clearly none of $G_2$, $G_3$ and $G_6$ belongs to this set. This completes the proof.

Proof of (xviii) $\rightarrow$ (xvii). This is clear.

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