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Unique Graph Homomorphisms onto Odd Cycles, II

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A natural generalization of graph colorings is graph homomorphisms. Let G and H be simple graphs. A map θ: $V(G) \to V(H)$ is called a *homomorphism* if θ preserves adjacency. The set of all homomorphism from G to H is denoted by $\text{Hom}(G, H)$. A graph G is *uniquely* H-colorable if $\text{Hom}(G, H) \neq \emptyset$, and if for $\theta_1, \theta_2 \in \text{Hom}(G, H)$, there is an automorphism φ of H such that $\varphi\theta_1 = \theta_2$. In this paper, we investigate some necessary conditions of unique C^{2k+1}-colorings and prove a best possible sufficient condition involving $\delta(G)$ for G to be uniquely C^{2k+1}-colorable under some necessary conditions. This generalizes a result of Bollobás on unique C^3-colorings [*J. Combin. Theory Ser. B* **25** (1978), 55–61]. We also find best possible conditions on the connectedness of the subgraphs of G induced by the preimages of θ, for any $\theta \in \text{Hom}(G, C^{2k+1})$. © 1989 Academic Press, Inc.

We shall use the notation of Bollobás [1]. Let G be a graph. We use $\delta(G)$ to denote the minimum degree of G. For simple graphs G and H, a map θ: $V(G) \to V(H)$ is called a *homomorphism* if θ preserves adjacency. The set of all homomorphisms from G to H is denoted by $\text{Hom}(G, H)$. If $\varphi \in \text{Hom}(G, G)$ is bijective, then φ is called an *automorphism* of G. Cycles of length m are denoted by C^m and occasionally by $\mathbb{Z}_m$, the set of integers modulo m, where i and j are adjacent if and only if $i - j \equiv \pm 1 \pmod{m}$.

A graph G is *H-colorable* if $\text{Hom}(G, H) \neq \emptyset$, and G is said to be *uniquely H-colorable* if $\text{Hom}(G, H) \neq \emptyset$ and if $\forall \theta_1, \theta_2 \in \text{Hom}(G, H)$, there exists an automorphism φ of H such that $\varphi\theta_1 = \theta_2$.

Graph homomorphisms are regarded as a generalization of graph colorings. If a graph G is K^k-colorable, then G is k-colorable in the usual meaning. For homomorphisms into odd cycles, see [4] and [5].

In 1978, Bollobás proved two theorems on unique graph colorings.

THEOREM 1 (Bollobás [2]). *If G is a graph of order n,* $\text{Hom}(G, K^k) \neq \emptyset$ *and* $\delta(G) > (3k-5)/(3k-2)$, *then G is uniquely K^k-colorable.*

It is shown in [3] that if $k = \chi(G)$ and if G is uniquely K^k-colorable, then for any $\theta \in \text{Hom}(G, K^k)$, the following condition holds:

$$G[\theta^{-1}(i) \cup \theta^{-1}(j)] \text{ is connected,} \qquad \text{for any } ij \in E(K^k). \qquad (1)$$

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For graphs satisfying (1) for some $\theta \in \mathrm{Hom}(G, K^k)$, the bound in Theorem 1 can be improved.

THEOREM 2 (Bollobás [2]). *Let G be a graph of order n such that* $\exists \theta \in \mathrm{Hom}(G, K^k)$ *satisfying* (1). *If*

$$\delta(G) > n\frac{k-2}{k-1},$$

then G is uniquely K^k*-colorable.*

Bollobás gave two classes of graphs to show that both bounds are best possible.

In a recent paper [6], we proved

THEOREM 3. *Let G be a graph of order n. If, for* $k > 1$,

(i) $\delta(G) \geqslant n/(k+1)$, *and*

(ii) $\exists \theta \in \mathrm{Hom}(G, C^{2k+1})$ *such that* $\theta(G) = C^{2k+1}$,

then G is uniquely C^{2k+1}*-colorable.*

This result is also best possible.

It is clear that Theorem 3 is analogous to Theorem 1. In this note we obtain the corresponding analogue to Theorem 2 for C^{2k+1}-colorings with $k \geqslant 2$ such that a condition analogous to (1) holds.

We note first that if G is a bipartite graph and $G \notin \{K^1, K^2\}$, then G is not uniquely $\mathbb{Z}_{2k+1}$-colorable.

Let G be a graph with $G \neq K^2$. For any vertex $v \in V(G)$, let $\Gamma(v)$ denote the neighborhood of v in G. Suppose $\theta \in \mathrm{Hom}(G, \mathbb{Z}_{2k+1})$, where $\theta(G) = \mathbb{Z}_{2k+1}$. If $|\theta(\Gamma(x))| = 1$, for some $x \in V(G)$, then x can be recolored, and so G is not uniquely C^{2k+1}-colorable.

From the above observations, we conclude that if $G \neq K^1$, $G \neq K^2$, and G is uniquely $\mathbb{Z}^{2k+1}$-colorable, then for any $\theta \in \mathrm{Hom}(G, \mathbb{Z}_{2k+1})$, the following holds:

$$\theta(G) = \mathbb{Z}_{2k+1} \qquad \text{and} \qquad |\theta(\Gamma(x))| = 2, \qquad \text{for all } x \in V(G). \qquad (2)$$

THEOREM 4. *Let G be a graph of order n and let* $k \geqslant 2$ *be an integer. If*

(i) $\exists \theta \in \mathrm{Hom}(G, \mathbb{Z}_{2k+1})$ *such that* (2) *is satisfied, and*

(ii) $\delta(G) > 3n/(4k+2)$,

then G is uniquely $\mathbb{Z}_{2k+1} -$ *colorable.*

We start with some lemmas.

LEMMA 1. *Let G be a graph of order n. If* $\exists\theta \in \mathrm{Hom}(G, \mathbb{Z}_{2k+1})$ *such that* $\theta(G) = \mathbb{Z}_{2k+1}$, *then*

$$\theta_M \leqslant n - k(\delta(G)), \tag{3}$$

where $\theta_M = \max\{|\theta^{-1}(i)| : i \in \mathbb{Z}_{2k+1}\}$.

Proof. Let $n_i = |\theta^{-1}(i)|$, $i \in \mathbb{Z}^{2k+1}$. We may assume that $|\theta^{-1}(0)| = \theta_M$. Suppose that k is even. Consider the following k inequalities:

$$\begin{aligned} n_1 + n_3 &\geqslant \delta(G), \\ n_2 + n_4 &\geqslant \delta(G), \\ n_5 + n_7 &\geqslant \delta(G), \\ n_6 + n_8 &\geqslant \delta(G), \\ &\dots \\ n_{2k-3} + n_{2k-1} &\geqslant \delta(G), \\ n_{2k-2} + n_{2k} &\geqslant \delta(G). \end{aligned}$$

Adding them all together with n_0, we get (3).

The proof when k is odd uses $n_1 + n_{2k} \geqslant \delta(G)$ as one of the k inequalities, and it is similar. ∎

LEMMA 2. *Suppose that G satisfies the hypotheses of Theorem* 4. *For any* $x \in V(G)$, *there exists an* $m(2k+1)$*-cycle of G containing x, where m equals* 1 *or* 2.

Proof. Denote, for subsets V, V' of $V(G)$.

$$(V, V') = \{v \in V' : \exists w \in V \text{ such that } vw \in E(G)\}.$$

Pick $x \in \theta^{-1}(0)$. Let

$$\begin{aligned} &V_1 = (\{x\}, \theta^{-1}(1)), && W_{2k} = (\{x\}, \theta^{-1}(2k)), \\ &V_2 = (V_1, \theta^{-1}(2)), && W_{2k-1} = (W_{2k}, \theta^{-1}(2k-1)), \\ &\dots \\ &V_i = (V_{i-1}, \theta^{-1}(i)), && W_i = (W_{i+1}, \theta^{-1}(i)), \\ &\dots \\ &V_{2k} = (V_{2k-1}, \theta^{-1}(2k)), && W_1 = (W_2, \theta^{-1}(1)), \\ &V_0 = (V_{2k}, \theta^{-1}(0)), && W_0 = (W_1, \theta^{-1}(0)). \end{aligned}$$

Then we have, for all $i \in \mathbb{Z}_{2k+1}$,

$$|\theta^{-1}(i)| + |V_{i+2}| \geq \delta(G),$$
$$|\theta^{-1}(i+4)| + |W_{i+2}| \geq \delta(G).$$

Notice that condition (2) implies that $V_i \neq \varnothing$ and $W_i \neq \varnothing$ for all $i \in \mathbb{Z}_{2k+1}$. If $V_i \cap W_i = \varnothing$, for all $i \in \mathbb{Z}_{2k+1}$, then we would have

$$\begin{aligned} 3n &\geq \sum_{i \in Z_{2k+1}} \{|\theta^{-1}(i)| + |\theta^{-1}(i+4)| + (|V_{i+2}| + |W_{i+2}|)\} \\ &\geq 2(2k+1)(\delta(G)) \\ &> 3n, \end{aligned}$$

a contradiction.

Hence for some $i \in \mathbb{Z}_{2k+1}$, $V_i \cap W_i \neq \varnothing$. If $i=0$, then x lies on a $2(2k+1)$-cycle of G. In other cases, x is in a $(2k+1)$-cycle of G. ∎

LEMMA 3. *If G satisfies the hypotheses of Theorem 4, then G does not have an odd cycle of length less than $2k+1$.*

Proof. This follows from the fact that $\mathrm{Hom}(G, \mathbb{Z}_{2k+1}) \neq \varnothing$. ∎

LEMMA 4. *Suppose that G satisfies the hypotheses of Theorem 4. Then G has a $(2k+1)$-cycle.*

Proof. By Lemma 2, G has a cycle whose length is $2k+1$ or $2(2k+1)$. Let C be a cycle of G of length $m(2k+1)$ such that m is minimized. We shall show $m=1$.

By contradiction, suppose $m>1$. By Lemma 2, $m=2$. Let

$$C = v_1 v_2 \cdots v_{m(2k+1)-1} v_{m(2k+1)} v_1.$$

By (ii) of Theorem 4, we have

$$\sum_{j=1}^{m(2k+1)} d(v_j) > 3nm \frac{2k+1}{4k+2} = \frac{3}{2} nm > nm. \tag{4}$$

Denote $\partial C = \{e \in E(G) - E(C) \colon e$ is incident with $V(C)\}$. We claim that there is no edge in ∂C that is incident with two vertices of $V(C)$. Suppose not, we can find an edge $e = v_i v_j \in \partial C$. Without loss of generality, we assume $1 \leq i \leq 2k+1$. Hence $j = (2k+1+i) \pm 1$.

If $j = 2k+i$, then $v_1 v_2 \cdots v_i v_{2k+i+1} \cdots v_{2(2k+1)} v_1$ is a $(2k+1)$-cycle, a contradiction.

If $j=2k+i+2$, then the cycle $v_j v_{j+1} \cdots v_{2(2k+1)} v_1 \cdots v_{i-1} v_i$ has length $2k-1$, contrary to Lemma 3. Hence the claim.

Thus no edge $e \in \partial C$ can be incident with two vertices of $V(C)$ and so

$$|\partial C| = \sum_{j=1}^{m(2k+1)} (d(v_j)-2), \tag{5}$$

since exactly two edges incident with $v_j \in V(C)$ are in $E(C)$, and the $d(v_j)-2$ other edges are in ∂C. Denote

$$Y = \{ y \in V(G) - V(C) \colon \Gamma(y) \cap V(C) \neq \varnothing \}.$$

We claim that for all $y \in Y$,

$$|\Gamma(y) \cap V(C)| \leqslant 2. \tag{6}$$

Since $\theta \in \operatorname{Hom}(G, \mathbb{Z}_{2k+1})$,

$$\theta[\Gamma(y) \cap V(C)] \subseteq \{\theta(y)-1, \theta(y)+1\}. \tag{7}$$

Suppose that (7) holds with equality. Then by the minimality of m, $|\Gamma(y) \cap V(C)| = 2$. Suppose that (7) is strict. Then $|\Gamma(y) \cap V(C)| \leqslant m$. By Lemma 2, $m \leqslant 2$. Hence in either case, (6) holds, as claimed.

By (6) and the definitions of Y and C,

$$|\partial C| = \sum_{y \in Y} |\Gamma(y) \cap V(C)| \leqslant 2\,|Y|, \tag{8}$$

and

$$n \geqslant |Y| + |V(C)| = |Y| + m(2k+1). \tag{9}$$

We combine (9), (8), (5), and (4) to get

$$\begin{aligned} 2n &\geqslant 2|Y| + 2m(2k+1) \geqslant |\partial C| + 2m(2k+1) \\ &= \sum_{j=1}^{m(2k+1)} (d(v_j)-2) + 2m(2k+1) \\ &\geqslant \sum_{j=1}^{m(2k+1)} d(v_j) \geqslant m(2k+1)(\delta(G)) > \frac{3}{2}mn > mn. \end{aligned}$$

It follows that $m<2$ and we are done. ∎

For a graph H that has a $(2k+1)$-cycle, we define a new graph $C^{2k+1}(H)$ whose vertex set is the set of all $(2k+1)$-cycles of H, where two vertices of $C^{2k+1}(H)$ are adjacent if and only if the corresponding $(2k+1)$-cycles have at least one edge in common.

Let X be the subset of $V(G)$ that consists of all vertices lying on $(2k+1)$-cycles of G. By Lemma 4, $X \neq \emptyset$. Let $W = V(G) - X$.

Let C_1 and C_2 be two $(2k+1)$-cycles of G. For any $\theta_1, \theta_2 \in \operatorname{Hom}(G, \mathbb{Z}_{2k+1})$, consider the following condition on the restrictions of θ_1 and θ_2:

$$\theta_1|_{C_1} = \theta_2|_{C_1} \Leftrightarrow \theta_1|_{C_2} = \theta_2|_{C_2}. \tag{10}$$

We say that C_1 is *related to* C_2 if and only if (10) holds for any $\theta_1, \theta_2 \in \operatorname{Hom}(G, \mathbb{Z}_{2k+1})$. It is clear that this defines an equivalence relation on $V(C^{2k+1}(G))$. We shall first show that $V(C^{2k+1}(G))$ has only one equivalence class. This will mean for $\theta_1 \in \operatorname{Hom}(G, \mathbb{Z}_{2k+1})$, if $\theta_1(v) = \theta(v)$ for all vertices v in a $(2k+1)$-cycle of G, then $\theta_1(x) = \theta(x)$ for all $x \in X$. Then we shall show that $\theta_1(w) = \theta(w)$ for all $w \in W$ also.

Let C_1 and C_2 be two $(2k+1)$-cycles of G that have an edge in common. If θ_1 is a homomorphism from C_1 onto $\mathbb{Z}_{2k+1}$, then there is a unique way to extend θ_1 to a homomorphism from $C_1 \cup C_2$ onto $\mathbb{Z}_{2k+1}$. Hence C_1 is related to C_2. As a consequence, if there is a (C_1, C_2)-path in $C^{2k+1}(G)$ for two $(2k+1)$-cycles C_1 and C_2 in G, then C_1 is related to C_2.

LEMMA 5. *If the hypotheses of Theorem 4 hold, then $V(C^{2k+1}(G))$ has only one equivalence class.*

Proof. Let C_1 and C_2 be two $(2k+1)$-cycles of G. We shall also use C_1 and C_2 to denote the corresponding vertices in $C^{2k+1}(G)$. It suffices to show that C_1 is related to C_2.

Case 1. $V(C_1) \cap V(C_2) \neq \emptyset$. We assume that $C_1 = x_0 x_1 \cdots x_{2k} x_0$ and $C_2 = y_0 y_1 \cdots y_{2k} y_0$ and that $x_0 = y_0$.

Suppose that C_1 is not related to C_2. Thus there is no (C_1, C_2)-path in $C^{2k+1}(G)$.

First we claim that $|V(C_1) \cap V(C_2)| = 1$. Suppose not. We have $x_i = y_j$ for some $0 < i, j < 2k+1$.

If $i = j$, then $x_0 x_1 \cdots x_i y_{i+1} \cdots y_{2k} x_0$ is a $(2k+1)$-cycle in G that is adjacent to both C_1 and C_2 in $C^{2k+1}(G)$, a contradiction.

If $i = 2k+1-j$, then $x_0 x_1 \cdots x_i y_{2k+2-j} \cdots y_{2k} x_0$ is a $(2k+1)$-cycle in G that is adjacent to both C_1 and C_2 in $C^{2k+1}(G)$, a conradiction.

If $i \not\equiv \pm j \pmod{2k+1}$, then one of the following cycles,

$$x_0 x_1 \cdots x_i y_{i+1} \cdots y_{2k-1} y_{2k} x_0,$$

$$x_0 x_1 \cdots x_i y_{i-1} \cdots y_2 y_1 x_0,$$

$$x_0 x_{2k} \cdots x_i y_{i+1} \cdots y_{2k-1} y_{2k} x_0,$$

$$x_0 x_{2k} \cdots x_i y_{i-1} \cdots y_2 y_1 x_0,$$

is an odd cycle of length less than $2k+1$, contrary to Lemma 3. Hence the claim holds and so $|V(C_1 \cup C_2)| = 4k+1$.

Subcase I.1. Suppose $k \geqslant 3$. Define $S_i = \{x_{i-1}, y_{i-1}, x_{i+1}, y_{i+1}\}$ for all $i \in \mathbb{Z}_{2k+1}$, where S_i has fewer than four elements if $i \in \{1, -1\}$.

Let m_0 be the number of incidences of edges of G with $V(C_1 \cup C_2)$. We shall estimate m_0 in two ways.

We claim that $\forall i \in \mathbb{Z}^{2k+1} - \{0\}$, $|\Gamma(z) \cap S_i| \leqslant 2$, for all $z \in \theta^{-1}(i)$. Suppose not. Since $x_0 = y_0$, if $i \in \{-1, 1\}$, then $\{y_0, x_{2i}, y_{2i}\} \subseteq \Gamma(z)$ and so $y_0 z$ lies on a $(2k+1)$-cycle C_3 adjacent to C_1 and on a $(2k+1)$-cycle C_4 adjacent to C_2. Thus $C_1 C_3 C_4 C_2$ is a (C_1, C_2)-path in $C^{2k+1}(G)$, a contradiction. If $i \notin \{-1, 0, 1\}$, then without loss of generality, we may assume $\{y_{i-1}, x_{i+1}\} \subseteq \Gamma(z)$ with $z \in \theta^{-1}(i)$. Then $y_0 \cdots y_{i-1} z x_{i+1} \cdots x_{2k} y_0$ is a $(2k+1)$-cycle adjacent to both C_1 and C_2 in $C^{2k+1}(G)$, a contradiction. Hence the claim.

Note that $|\Gamma(z) \cap S_0| \leqslant 4$, $\forall z \in \theta^{-1}(0)$. Therefore, by the above claim, we sum over all $z \in V(G)$ to get

$$m_0 = \sum_{z \in V(G)} |\Gamma(z) \cap V(C_1 \cup C_2)| = \sum_{i \in Z_{2k+1}} \sum_{z \in \theta^{-1}(i)} |\Gamma(z) \cap S_i|$$
$$\leqslant 2n + 2|\theta^{-1}(0)|.$$

By (ii) of Theorem 4, and since $|V(C_1 \cup C_2)| = 4k+1$,

$$m_0 \geqslant |V(C_1 \cup C_2)| \delta(G) > (4k+1) \frac{3n}{4k+2} = 3n - \frac{3n}{4k+2}.$$

Combine these bounds to get

$$2\,|\theta^{-1}(0)| > n - \frac{3n}{4k+2}.$$

But (3) and (ii) of Theorem 4 give

$$|\theta^{-1}(0)| < n - \frac{3kn}{4k+2}.$$

Hence we must have

$$2\left(n - \frac{3kn}{4k+2}\right) > n - \frac{3n}{4k+2}.$$

It follows that $k < \frac{5}{2}$, contrary to $k \geqslant 3$.

Subcase I.2. Suppose $k=2$. Then $\delta(G)>3n/10$. Now we have

$C_1=x_0x_1x_2x_3x_4x_0$ and $C_2=y_0y_1y_2y_3y_4y_0$ with $x_0=y_0$.

CLAIM. $\Gamma(x_1)$, $\Gamma(y_2)$, $\Gamma(x_2)$, $\Gamma(y_3)$ are pairwise disjoint.

Proof of the Claim. By Lemma 3, we have $\Gamma(x_1)\cap\Gamma(x_2)=\varnothing$ and $\Gamma(y_2)\cap\Gamma(y_3)=\varnothing$.

If $\Gamma(x_1)\cap\Gamma(y_2)\neq\varnothing$, then picking $w\in\Gamma(x_1)\cap\Gamma(y_2)$, we can see that $x_0x_1wy_2y_1x_0$ is a 5-cycle in G that is adjacent to both C_1 and C_2 in $C^5(G)$, a contradiction.

If $\Gamma(x_1)\cap\Gamma(y_3)\neq\varnothing$, then picking $w\in\Gamma(x_1)\cap\Gamma(y_3)$, we can see that $x_0x_1wy_3y_4x_0$ is a 5-cycle in G that is adjacent to both C_1 and C_2 in $C^5(G)$, a contradiction.

Suppose $\Gamma(x_2)\cap\Gamma(y_2)\neq\varnothing$. Let $\theta_1,\theta_2\in\mathrm{Hom}(G,\mathbb{Z}_{2k+1})$ and let $w\in\Gamma(x_2)\cap\Gamma(y_2)$. Without loss of generality, we assume that $\theta_1(x_i)=\theta_2(x_i)=i$, $\forall i\in\mathbb{Z}_{2k+1}$. Since $w\in\Gamma(x_2)$, $\theta_1(w)\in\{1,3\}$. Since $\theta_1(x_0)=\theta_1(y_0)$ and since $C_2=y_0y_1y_2y_3y_4y_0$ is a 5-cycle, $\theta_1(y_2)\in\{2,3\}$ and so we must have $\theta_1(y_2)=2$, by $w\in\Gamma(y_2)$, $\theta_1(w)\in\{1,3\}$, and $\theta_1(y_2)\in\{2,3\}$. It follows that $\theta_1(y_i)=i$, $\forall i\in\mathbb{Z}_{2k+1}$. Similarly, we can see that $\theta_2(y_i)=i$, $\forall i\in\mathbb{Z}_{k+1}$. Hence C_1 is related to C_2, a contradiction.

Similarly we can see that $\Gamma(x_2)\cap\Gamma(y_3)=\varnothing$. Hence the claim. ∎

By the claim, we have

$$n\geqslant|\Gamma(x_1)\cup\Gamma(x_2)\cup\Gamma(y_2)\cup\Gamma(y_3)|\geqslant 4(\delta(G))>\frac{12}{10}n>n,$$

a contradiction.

Case 2. $V(C_1)\cap V(C_2)=\varnothing$. Fix a $(2k+1)$-cycle C of G. Define $C=v_0v_1\cdots v_{2k}v_0$ such that $\theta(v_i)=i$, $\forall i\in\mathbb{Z}_{2k+1}$, Let

$$N(C)=\bigcup_{i\in Z_{2k+1}}[\Gamma(v_{i-1})\cap\Gamma(v_{i+1})].$$

Note that

$$\begin{aligned}|N(C)|&=\sum_{i\in Z_{2k+1}}|\Gamma(v_{i-1})\cap\Gamma(v_{i+1})|\\&\geqslant\sum_{i\in Z_{2k+1}}\{|\Gamma(v_{i-1})\cap\theta^{-1}(i)|+|\Gamma(v_{i+1})\cap\theta^{-1}(i)|-|\theta^{-1}(i)|\}\\&=\sum_{i\in Z_{2k+1}}\deg(v_i)-n\\&>(2k+1)\frac{3n}{4k+2}-n\\&=\frac{n}{2}.\end{aligned}$$

Hence

$$|N(C_1)\cap N(C_2)|>\frac{n}{2}+\frac{n}{2}-n=0. \tag{11}$$

By (11), there is a vertex $u\in N(C_1)\cup N(C_2)$. Thus there are i and j, $0\leqslant i, j\leqslant 2k$, such that

$$u\in\Gamma(x_{i-1})\cap\Gamma(x_{i+1})\cap\Gamma(y_{j-1})\cap\Gamma(y_{j+1}).$$

Let

$$C_3=x_0x_1\cdots x_{i-1}ux_{i+1}\cdots x_{2k-1}x_{2k},$$

$$C_4=y_0y_1\cdots y_{j-1}uy_{j+1}\cdots y_{2k-1}y_{2k}.$$

Then C_3 has an edge in common with C_1 and C_4 has an edge in common with C_2 and C_3 and C_4 have a vertex, namely u, in common. Thus C_1 is related to C_3 and C_2 is related to C_4. By the result of Case 1, C_3 is related to C_4 and so by transitivity of the equivalence relation, C_1 is related to C_2.

Thus C_1 is related to C_2 and we are done. ▌

Proof of Theorem 4. Suppose that $\exists\theta, \theta_1\in\mathrm{Hom}(G, \mathbb{Z}_{2k+1})$ such that θ and θ_1 agree in X, the set of vertices lying in some $(2k+1)$-cycle of G. Let $w\in W$, where $W=V(G)-X$. By Lemma 2, there is a $2(2k+1)$-cycle C_5 that contains w. We may assume

$$C_5=w_0w_1\cdots w_{2k+1}w_{2k+2}\cdots w_{4k+1}w_{4k+2}$$

such that $w=w_0=w_{4k+2}$ and $\theta(w_i)\equiv i \pmod{2k+1}$.

For each $i\in\mathbb{Z}_{2k+1}$, define

$$T_i=\{w_{i-1}, w_{i+1}, w_{2k+i}, w_{2k+i+2}\}.$$

We claim that

$$\forall i\in\mathbb{Z}_{2k+1}-\{0\} \quad \text{and} \quad \forall z\in\theta^{-1}(i), \quad |\Gamma(z)\cap T_i|\leqslant 3. \tag{12}$$

If, instead, we have $T_i\subseteq\Gamma(z)$ for some $i\in\mathbb{Z}_{2k+1}-\{0\}$ and $z\in\theta^{-1}(i)$, then $w_0w_i\cdots w_{i-1}zw_{2k+i+2}\cdots w_{4k+1}w_0$ is a $(2k+1)$-cycle containing w, a contradiction. Hence the claim.

Let m_1 be the number of incidences of edges of G with $V(C_5)$.

Suppose $|\Gamma(z)\cap T_0|\leqslant 3$, $\forall z\in\theta^{-1}(0)$. Then by (12),

$$m_1=\sum_{i\in Z_{2k+1}}\sum_{z\in\theta^{-1}(i)}|\Gamma(z)\cap T_i|\leqslant 3n.$$

On the other hand,

$$m_1 = \sum_{i=1}^{4k+2} \deg(w_i) \geqslant (4k+2)(\delta(G)) > 3n,$$

a contradiction. Hence we must have

$$|\Gamma(z) \cap T_0| = 4, \qquad \text{for some } z \in \theta^{-1}(0). \tag{13}$$

By (13), $zw_1w_2 \cdots w_{2k}z$ and $zw_{2k+2}w_{2k+3} \cdots w_{4k+1}z$ are $(2k+1)$-cycles of G. Hence $w_1, w_{4k+1} \in X$. Thus $\theta(w_1) = \theta_1(w_1) = 1$ and $\theta(w_{4k+1}) = \theta_1(w_{4k+1}) = 2k$. Since $w_0w_1, w_0w_{4k+1} \in E(G)$, we must have $\theta(w_0) = \theta_1(w_0) = 0$.

Thus G is uniquely $\mathbb{Z}_{2k+1}$-colorable. ▮

Theorem 4 is best possible in some sense. Let $k \geqslant 2$ and $m \geqslant 1$ be integers. Let $C_1 = v_0v_1 \cdots v_{2k}v_0$ and $C_2 = u_0u_1 \cdots u_{2k}u_0$ be two $(2k+1)$-cycles with $V(C_1) \cap V(C_2) = \varnothing$. Let H be the graph such that $V(H) = V(C_1) \cup V(C_2)$ and $E(H) = E(C_1) \cup E(C_2) \cup \{u_iv_i : i = 0, 1, 2 \cdots 2k\}$. Let H_m be the graph obtained from H by replacing each vertex v_i by a set V_i of m vertices and replacing each vertex u_i by a set U_i of m vertices, where two vertices of H_m are adjacent in H_m if and only if they correspond to different vertices that are adjacent in H.

Hence $|V(H_m)| = (4k+2)m$ and $\delta(H_m) = 3m$. There are two homomorphisms $\theta_i \in \mathrm{Hom}(H_m, \mathbb{Z}_{2k+1})$, $i = 1, 2$, where $\theta_1^{-1}(j) = U_j \cup V_{j-1}$ and $\theta_2^{-1}(j) = U_j \cup V_{j+1}, j \in \mathbb{Z}_{2k+1}$. Clearly $\theta_i(H_m) = \mathbb{Z}_{2k+1}, i = 1, 2$. Also, θ_1, θ_2 satisfy (2). But there is no automorphism φ of $\mathbb{Z}_{2k+1}$ satisfying $\varphi\theta_1 = \theta_2$.

Examples of Bollobás [2] showing that Theorem 1 is best possible also show that Theorem 4 fails for $k = 1$.

Let $\theta \in \mathrm{Hom}(G, \mathbb{Z}_{2k+1})$. One may consider the following analogue to (1):

$$G[\theta^{-1}(i) \cup \theta^{-1}(i+1)] \text{ is connected, } \forall i \in \mathbb{Z}_{2k+1}. \tag{14}$$

One may ask if (14) is a necessary condition for unique $\mathbb{Z}_{2k+1}$-colorings, and when (14) holds, if it is not such a necessary condition.

THEOREM 5. *Let G be a graph of order n and let $k \geqslant 1$ be an integer. Suppose that $\exists \theta \in \mathrm{Hom}(G, \mathbb{Z}_{2k+1})$ such that $\theta(G) = \mathbb{Z}_{k+1}$. If one of the following holds*:

(i) *k is even and $\delta(G) > n/(k+1)$,*

(*ii*) *k is odd and $\delta(G) > 4n/(4k+3)$,*

then (14) *holds.*

Proof. By contradiction, we may assume that $G[\theta^{-1}(0)\cup\theta^{-1}(1)]$ is the disjoint union of two nontrivial subgraphs G_1 and G_2 such that there are no edges of G joining a vertex of G_1 and a vertex of G_2. Let

$$V_0=V(G_1)\cap\theta^{-1}(0),$$
$$W_0=V(G_2)\cap\theta^{-1}(0),$$
$$V_1=V(G_1)\cap\theta^{-1}(1),$$
$$W_1=V(G_2)\cap\theta^{-1}(1).$$

Assume (i) of Theorem 5 first. We claim that V_0, V_1, W_0, and W_1 are nonempty. Suppose, to the contrary, that $V_0=\varnothing$. Since $\theta(G)=\mathbb{Z}_{2k+1}$ and $G[\theta^{-1}(0)\cup\theta^{-1}(1)]$ is disconnected, we may assume that $V_1\neq\varnothing$ and there is no edge in G joining one vertex in V_1 and one vertex in W_0. Let $x\in V_1$. Then $\Gamma(x)\subseteq\theta^{-1}(2)$. Thus

$$|\theta^{-1}(2)|\geqslant\delta(G)>\frac{n}{k+1}. \tag{15}$$

By (3), $|\theta^{-1}(2)|\leqslant n-k\delta(G)\leqslant n/(k+1)$, a contradiction. Hence V_0, V_1, W_0 and W_1 are nonempty. Without loss of generality, suppose

$$|V_0|+|W_0|\leqslant|V_1|+|W_1|. \tag{16}$$

Considering the degrees, we have

$$|W_0|+|\theta^{-1}(2)|\geqslant\delta(G), \tag{17}$$
$$|V_1|+|\theta^{-1}(2k)|\geqslant\delta(G), \tag{18}$$
$$|\theta^{-1}(2k-1)|+|\theta^{-1}(0)|\geqslant\delta(G).$$

If $k>2$, we have the following besides the above three inequalities,

$$|\theta^{-1}(4j+3)|+|\theta^{-1}(4j+5)|\geqslant\delta(G),$$
$$|\theta^{-1}(4j+4)|+|\theta^{-1}(4j+6)|\geqslant\delta(G),$$

where $j=0, 1, ..., (k/2-2)$.

Adding all these inequalities, we get

$$\begin{aligned} n+(|W_0|-|W_1|)&\geqslant|W_0|+|V_1|+\sum_{j\in\mathbb{Z}_{2k+1}-\{1\}}|\theta^{-1}(j)|\\ &\geqslant\left[2\left(\frac{k}{2}-1\right)+3\right]\delta(G)\\ &=(k+1)\,\delta(G)>n.\end{aligned}$$

This implies $|W_0| > |W_1|$. Replacing W_0 by V_0 in (17) and V_1 by W_1 in (18), we get similarly $|V_0| > |V_1|$, contrary to (16).

Now we assume that (ii) of Theorem 5 holds. A similar argument shows that V_0, V_1, W_0, and W_1 are nonempty. Without loss of generality, suppose

$$|V_0| \leqslant |V_1|. \tag{19}$$

Considering the degrees, we get

$$|W_0| + |\theta^{-1}(2)| \geqslant \delta(G),$$
$$|\theta^{-1}(2k-1)| + |\theta^{-1}(0)| \geqslant \delta(G),$$

and

$$|\theta^{-1}(2k)| + |W_1| \geqslant \delta(G),$$
$$|\theta^{-1}(2i-1)| + |\theta^{-1}(2i+1)| \geqslant \delta(G),$$
$$|\theta^{-1}(2i)| + |\theta^{-1}(2i+2)| \geqslant \delta(G),$$

where $i = 1, 2, ..., k-1$.

Adding all these inequalities, we get

$$\begin{aligned} 2n - (|V_0| + |V_1|) &\geqslant \sum_{i \in Z_{2k+1} - \{0\}} |\theta^{-1}(i)| + |W_0| \\ &\quad + \sum_{i \in Z_{2k+1} - \{1\}} |\theta^{-1}(j)| + |W_1| \\ &\geqslant (2k+1)\,\delta(G) \\ &> (2k+1)\frac{4n}{4k+3} \\ &= 2n - \frac{2n}{4k+3}. \end{aligned}$$

Thus $|V_0| + |V_1| < 2n/(4k+3)$, and so (19) implies that $|V_0| < n/(4k+3)$. Considering the degrees of vertices in V_1, we have

$$|V_0| + |\theta^{-1}(2)| \geqslant \delta(G) > 4n/(4k+3).$$

It follows that $|\theta^{-1}(2)| > 3n/(4k+3) > n - k\delta(G)$, contrary to (3).

This completes the proof of Theorem 5. ∎

Theorem 5 is best possible in some sense also. Let H be a theta graph obtained from $\mathbb{Z}_{2k+1}$ by adding two vertices α, β to $\mathbb{Z}_{2k+1}$ in the following way: α is adjacent to β and 0, and β is adjacent to α and β.

Let $k>1$ and $s>1$ be integers. Let $G_j(k, s)$, $j=1, 2$, be the graphs obtained from H by replacing each vertex $j \in \{\alpha, \beta\} \cup \mathbb{Z}_{2k+1}$ by a set $V_{i,j}$ of $n_{i,j}$ vertices, where the $n_{i,j}$'s are defined as follows:

When $j=1$, we let

$$n_{\alpha,1}=n_{1,1}=n_{\beta,1}=n_{2,1}=s,$$
$$n_{0,1}=n_{3,1}=3s,$$
$$n_{i,1}=2s, \qquad \text{for all other } n_{i,1}\text{'s};$$

and when $j=2$ and k is odd, we let

$$n_{\alpha,2}=n_{1,2}=n_{\beta,2}=n_{2,2}=s,$$
$$n_{3,2}=3s,$$

and for $p=1, 2, ..., (k-1)/2$,

$$n_{4p,2}=n_{4p+2,2}=2s,$$
$$n_{4p+1,2}=s,$$
$$n_{4p+3,2}=3s.$$

Two vertices in $H_j(k, s)$ are adjacent if and only if they correspond to different vertices that are adjacent in H.

Let H_j denote $H_j(k, s)$. It is easy to see that $\mathrm{Hom}(H_j, \mathbb{Z}_{2k+1}) \neq \varnothing$, $j=1, 2$. Note that

$$\delta(H_1)=4s=|V(H_1)|/(k+1) \qquad \text{and} \qquad \delta(H_2)=4s=|V(H_2)|/(4k+3).$$

By Theorem 3, both H_1 and H_2 are uniquely $\mathbb{Z}_{2k+1}$-colorable. Hence for $j \in \{1, 2\}$, $\forall \theta_j \in \mathrm{Hom}(H_j, \mathbb{Z}_{2k+1})$, we have

$$\theta_j(V_{\alpha,j})=\theta_j(V_{i,j}) \qquad \text{and} \qquad \theta_j(V_{\beta,j})=\theta_j(V_{2,j}).$$

But $H_j[V_{\alpha,j} \cup V_{i,j} \cup V_{\beta,j} \cup V_{2,j}]$ is disconnected. These extremal graphs also show that (13) is not a necessary condition for unique $\mathbb{Z}_{2k+1}$-colorings.

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