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# Block-cutvertex trees and block-cutvertex partitions

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**Abstract**

The block-cutvertex graph of the connected graph $G$, denoted bc($G$), is the graph whose vertices are the blocks and cutvertices of $G$. The edges of bc($G$) join cutvertices with those blocks to which they belong. Gallai, Harary and Prins defined this concept and showed that a graph $G$ is the block-cutvertex graph of some connected graph $H$ if and only if $G$ is a tree in which the distance between any two leaves is even. A block-cutvertex partition of the tree $T$ is a collection $\{T_1, \ldots, T_k\}$ of block-cutvertex trees such that each $T_i$ is a subtree of $T$ and each edge of $T$ is in exactly one $T_i$. We prove that a tree has a block-cutvertex partition if and only if it does not have a perfect matching. Various concepts and algorithms related to block-cutvertex partitions will be presented.


## 1. Introduction

All graphs considered will be finite, undirected and simple. $V(\mathbf{G})$ and $E(\mathbf{G})$ denote the vertex set and the edge set of the graph $\mathbf{G}$. The order of $\mathbf{G}$ is the number of vertices and the size of $\mathbf{G}$ is the number of edges. A nonempty graph has at least one edge. The degree of the vertex $v$, denoted $d(v)$, is the number of edges incident with $v$. The following definition is due to Gallai [3] and Harary and Prins [6].

**Definition 1.** The *block-cutvertex graph* of a connected graph $\mathbf{G}$, denoted bc($\mathbf{G}$), is defined as the graph whose vertices are the blocks and cutvertices of $\mathbf{G}$. The edges of bc($\mathbf{G}$) join cut vertices with those blocks to which they belong.

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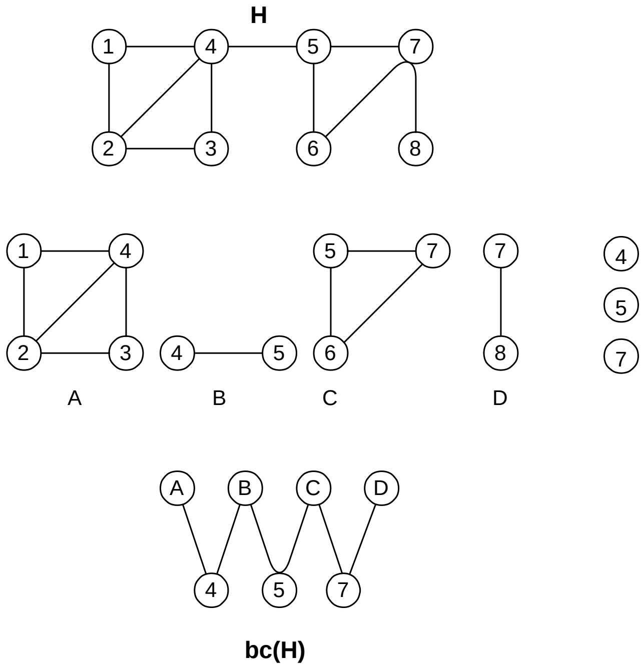



Fig. 1.

**Example 2.** A graph, **H**, of order 8 with blocks A, B, C and D, cutvertices 4, 5 and 7 and the block-cutvertex-graph, bc(**H**) (see Fig. 1).

**Theorem 3** (Harary, Gallai and Prins). *A graph* **G** *is the block-cutvertex graph of a connected graph* **H** *if and only if* **G** *is a tree in which the distance between every pair of leaves is even.*

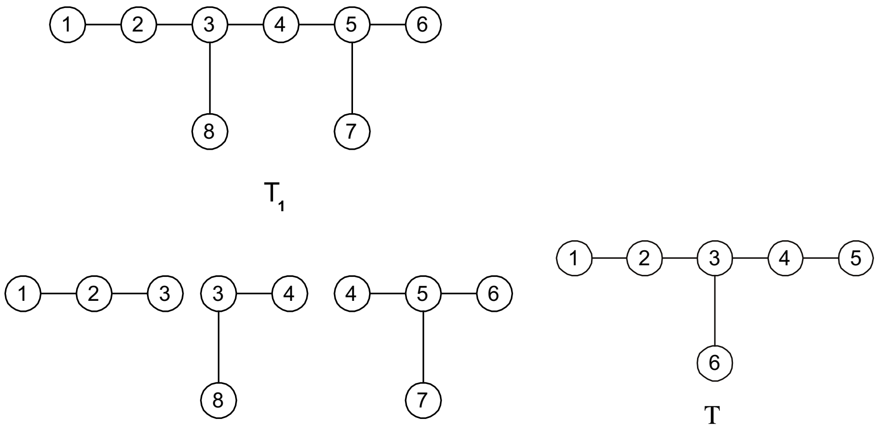
Thus, if **G** is a connected graph then bc(**G**) is a tree and we can speak of the block-cutvertex tree of a connected graph. This fact leads to the following definition.

**Definition 4.** A bc-*tree* is a tree of order 1 or a nonempty tree in which the distance between every pair of leaves is even. A bc-tree will also be called an *even tree*. A tree that is not even will be called *odd*.

Harary and Palmer enumerated bc-trees in [5].

**Definition 5.** A bc-*cover* of the nonempty tree $T$ is a collection

$$C = \{T_1, \ldots, T_k\}$$




(a) (b)

Fig. 2. (a), (b) A tree with no bc-partition.

of nonempty bc-trees such that each $T_j$ is a subtree of $T$ and each edge of $T$ is in at least one of the $T_i$. If each edge is in exactly one of the $T_i$ then $C$ is called a bc-*partition*.

**Example 6.** A bc-partition of $T_1$ (Fig. 2(a)).

The tree **T** below does not have a bc-partition : If $\mathbb{B}$ were a bc-partition then edges 45 and 34 would be together in some bc-subtree $\mathbf{S}_i \in \mathbb{B}$. Also, edges 12 and 23 would be together in some bc-subtree $\mathbf{S}_j \in \mathbb{B}$. This implies that edge 36 cannot be in a bc-subtree of $\mathbb{B}$ with edge 23 or 34. Thus, **T** does not have a bc-partition (Fig. 2(b)).

In the next section, we prove that a tree has a bc-partition if and only if it does not have a perfect matching. Section 3 will consider bc-covers in trees with a perfect matching and Section 4 describes various algorithms for finding bc-partitions and bc-covers.

## 2. bc-Partitions in trees

**Definition 7.** A *type* 1 tree has a path $u_1$, $u_2$, $u_3$, where $d(u_1) \geqslant 2$, $d(u_2) = 2$ and $d(u_3) = 1$. Any tree that is not type 1 will be called *type* 2.

**Theorem 8.** *A tree has a* bc-*partition if and only if it does not have a perfect matching.*

**Proof.** The assertion is easy to verify for trees of order at most 6. Assume, for purposes of induction, that the theorem holds for all trees of order $k \leqslant n$ and let $T$ be a tree of order $n+1$.

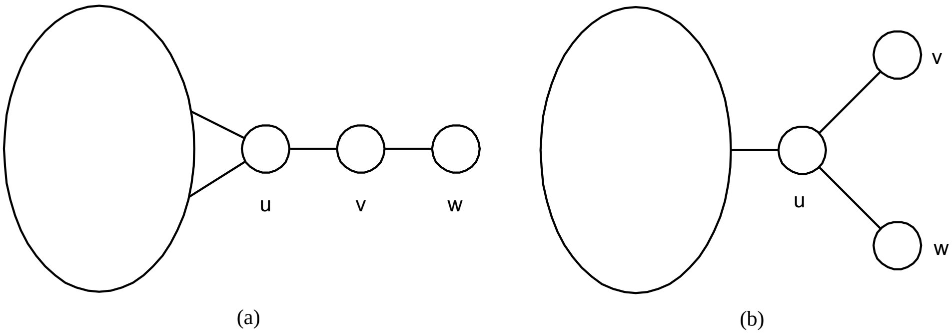



Fig. 3. (a) A type 1 configuration; (b) a type 2 configuration.

*Case* 1: Assume that $T$ is of type 1 with path $P = u, v, w$, where $w$ is a leaf, $v$ has degree 2 and $u$ has degree at least 2 (Fig. 3(a)).

Let $T^* = T - \{v, w\}$. Suppose that $T$ has no bc-partition. We must show that $T$ has a perfect matching. Note that $T^*$ cannot have a bc-partition $\mathbb{B}$, otherwise $\mathbb{B} \cup \{uv, vw\}$ would be a bc-partition of $T$. Thus, by induction, $T^*$ has a perfect matching $M^*$. Therefore, $M^* \cup \{vw\}$ is a perfect matching of $T$.

Now, assume that $T$ has the bc-partition

$$\mathbb{B} = \{T_1, T_2, \ldots, T_k\},$$

where $w \in V(T_1)$. We must show that $T$ does not have a perfect matching. If $T_1 = P$ then $\mathbb{B} - \{T_1\}$ is a bc-partition of $T^*$. If $T_1 \neq P$ let $T_1^* = T_1 - \{v, w\}$. Since the distance from $u$ to $w$ is even, $T_1^*$ is also a bc-tree so that

$$\{T_1^*, T_2, \ldots, T_k\}$$

is a bc-partition of $T^*$. Thus, by induction, $T^*$ has no perfect matching. Now, if $M$ were a perfect matching of $T$, then $M - \{vw\}$ would be a perfect matching of $T^*$. Thus $T$ has no perfect matching.

*Case* 2: Assume that $T$ is of type 2. By considering the longest path in $T$, we conclude that there is a vertex $u$ of degree at least three that is adjacent to the two leaves $v$ and $w$ (Fig. 3(b)).

Since $T$ does not have a perfect matching we must show that $T$ has a bc-partition. Let $T^* = T - \{v, w\}$. If $T^*$ has a bc-partition then so does $T$ and we are done. If $T^*$ does not have a bc-partition then, by induction, $T^*$ has a perfect matching. Let $T' = T - \{v\}$. Since $T'$ has odd order it does not have a perfect matching. By induction, let

$$\{T_1, T_2, \ldots, T_k\}$$

be a bc-partition of $T'$, where $w \in V(T_1)$. If $z$ is a leaf of $T_1$ then the distance from $z$ to $w$ is even. Therefore, the distance from $z$ to $v$ is also even in

$$T_1^* = T_1 + \{uv\}.$$

Thus, $T_1^*$ is a bc-tree and

$$\{T_1^*, T_2, \ldots, T_k\}$$

is a bc-partition of $T$. □

## 3. Semipartitions and bc-covers

If $T$ is a tree with perfect matching $M$ then $T$ does not have a bc-partition. In this section we will see that if $T \neq K_2$ then there is a bc-cover that is "almost" a bc-partition.

**Definition 9.** Let $C$ be a bc-cover of the tree $T$. The cover weight of edge $e \in E(T)$, denoted $C(e)$, is the number of bc-trees of $C$ that contain $e$. The cover weight of $T$ is

$$C(T) = \sum_e C(e).$$

**Example 10.** Let $\Psi$ be the set of all even subtrees of **T**. Note that if $S \in \Psi$ then the leaves of $S$ must have labels of the same parity. Using this observation, we find that **T** has 7 even subtrees (see Fig. 4).

In this case

$$\Psi(12) = \Psi(45) = 2, \quad \Psi(23) = \Psi(34) = 5, \quad \Psi(36) = 3 \quad \text{and} \quad \Psi(\mathbf{T}) = 17.$$

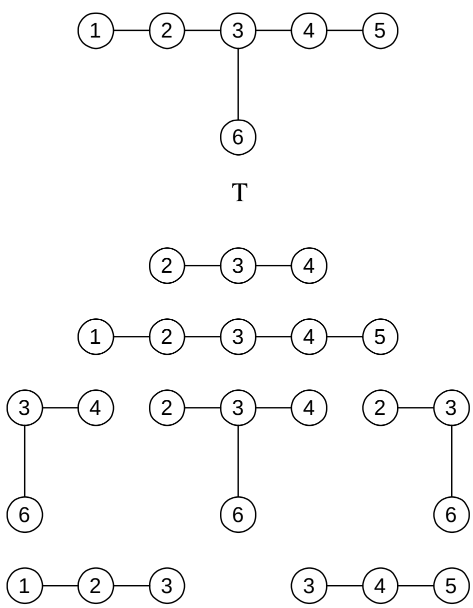



Fig. 4.

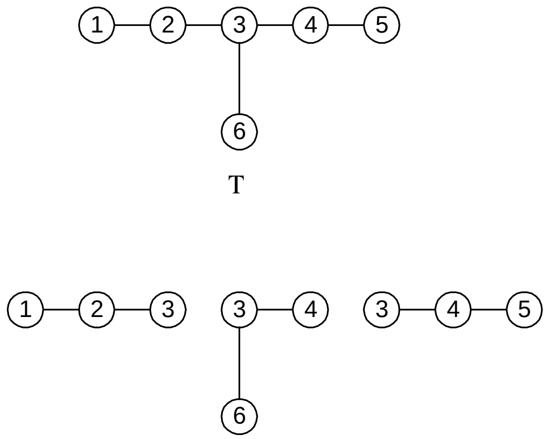



Fig. 5.

**Example 11.** **T** has a minimum cover weight for the following bc-cover (see Fig. 5). Thus, if $C$ is a bc-cover of **T** then $6 \leqslant C(\mathbf{T}) \leqslant 17$. It is not difficult to show that if $i \neq 16$ and $6 \leqslant i \leqslant 17$ then $C(\mathbf{T}) = i$ for some bc-cover $C$ of **T**.

If $T$ is a tree of order $2n \geqslant 4$ with perfect matching $M$ and bc-cover $C$ then $2n \leqslant C(T)$ because $C(T) = 2n - 1$ if and only if $C$ is a bc-partition.

**Definition 12.** Let $T$ be a tree of order $n \geqslant 3$. A *semipartition* of $T$ is a bc-cover $S$ such that $S(T) = n$.

**Definition 13.** Let $u$ be a vertex of tree $T$. A *branch* of $u$ is a maximal subtree containing $u$ as a leaf. If $ux \in E(T)$ then $\mathrm{Br}[u,x]$ is the branch of $u$ that contains $x$ and $\mathrm{Br}(u,x)$ is the subtree $\mathrm{Br}[u,x] - \{u\}$ (Fig. 6).

The following theorem shows that $2n \leqslant C(T)$ is a sharp bound.

**Theorem 14.** *Let $T$ be a tree of order $2n \geqslant 4$ with perfect matching $M$. If $e \notin M$ then there is a semipartition $S$ such that $S(e) = 2$. Furthermore, if $S$ is a semipartition of $T$ such that $S(e) = 2$ then $e \notin M$.*

**Proof.** If $e = uv \notin M$ then the required semipartition $S$ is easy to construct. Note that both components of $T - e$ have perfect matchings. Consider the subtrees $\mathrm{Br}[u,v]$ and $\mathrm{Br}[v,u]$. Each tree has a bc-partition $C_u$ and $C_v$, respectively. Therefore, $S = C_u \cup C_v$ is the required semipartition. It remains to prove that if $S$ is a semipartition of $T$ for which $S(e) = 2$ then $e \notin M$. The proof is by induction on $n$. Since the assertion is true when $n = 2$, assume that the theorem holds for every tree of order $2k$ with a perfect matching, where $2 \leqslant k \leqslant n$. Let $T$ be a tree of order $2n + 2$ with perfect matching $M$,

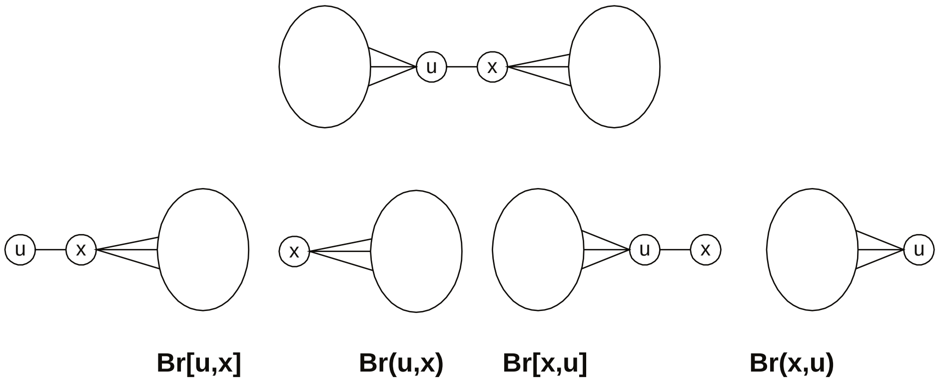



Fig. 6.

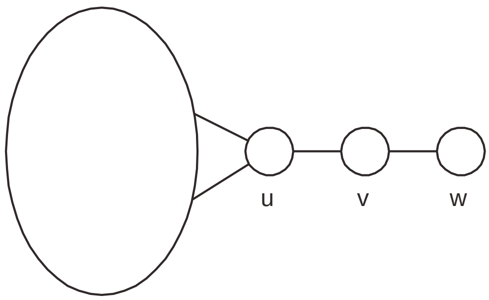



Fig. 7. A type 1 tree in the proof of Theorem 14.

semipartition

$$S = \{T_1, T_2, \ldots, T_k\}$$

and edge $e$, where $S(e) = 2$. Since a type 2 tree does not have a perfect matching, we know that $T$ is of type 1.

Let $P = u, v, w$ be a path in $T$, where $w$ is a leaf, $v$ has degree 2 and $u$ has degree at least 2 (Fig. 7).

First, note that $T^* = T - \{v, w\}$ has perfect matching $M^* = M - \{vw\}$. Assume that $e = vw$, where $e$ is in $T_1$ and $T_2$. Note that any bc-tree that contains $vw$ must also contain $uv$. Therefore, $T_1$ and $T_2$ contain $uv$ and $vw$. Since this is impossible we conclude that $e \neq vw$. If $e = uv$ then $e \notin M$ and the theorem is verified. The only other possibility is that $e$ is an edge of $T^* = T - \{v, w\}$. If $P \in S$ then $S^* = S - \{P\}$ is a semipartition of $T^*$ with $S^*(e) = 2$. If $vw \in T_i$ and $T_i \neq P$ then

$$S^* = \{T_1, \ldots, T_i - \{v, w\}, \ldots, T_k\}$$

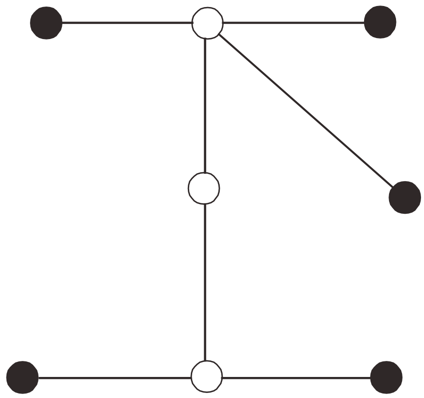


Fig. 8.

is a semipartition of $T^*$ such that $S^*(e)=2$. By induction, $e \notin M^*$. Since a tree has at most one perfect matching (see [1]) and $e \neq vw$, we conclude that $M=M^* \cup \{vw\}$ and $e \notin M$. □

If $T$ is even then $\{T\}$ is a trivial bc-partition. There are also nontrivial bc-partitions associated with each nonleaf.

**Definition 15.** Let

$$U=\{u_1,u_2,\ldots,u_k\}$$

be the neighbors of vertex $u$. The graph induced by $u$ and $U$ is denoted $\overline{N(u)}$.

**Definition 16.** Let $T$ be a nonempty even tree with bipartition $(\mathbf{X}_1,\mathbf{X}_2)$, where $\mathbf{X}_1$ contains all leaves of $T$. The vertices of $\mathbf{X}_1$ will be called the *black vertices* and the vertices of $\mathbf{X}_2$ will be called the *white vertices.*

**Example 17.** An even tree with 6 black vertices and 2 white vertices is shown in Fig. 8.

**Theorem 18.** *Every nonempty even tree except $K_{1,2}$ and $K_{1,3}$ has a nontrivial* bc-*partition.*

**Proof.** The theorem is obvious for $K_{1,n}$, where $n \geqslant 4$. Let $w$ be a white vertex of the even tree $T \neq K_{1,n}$, where

$$\{b_1,\ldots,b_k\}$$

are the neighbors of $w$ and $k \geqslant 2$. Since each component of $T-\{w\}$ is even

$$\overline{N(w)} \cup B_1 \cup \cdots \cup B_k$$

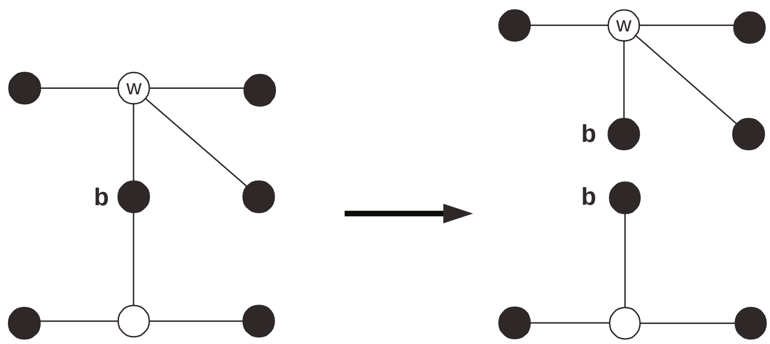



Fig. 9.

is a nontrivial bc-partition of $T$, where

$$B_j = \begin{cases} \mathrm{Br}(w, b_j), & |\mathrm{Br}(w, b_j)| > 1, \\ \emptyset, & |\mathrm{Br}(w, b_j)| = 1. \end{cases} \qquad \square$$

**Corollary 19** (Local bc-partition). *Every nonempty tree without a perfect matching has a* bc-*partition*

$$C = \{T_1, \ldots, T_h\},$$

*where $T_i$ is isomorphic to $K_{1,2}$ or $K_{1,3}$.*

If $T$ is even then there is also a natural bc-partition associated with each black vertex $b$ of degree at least two. Let

$$W = \{w_1, \ldots, w_j\}$$

be the neighbors of $b$, where $j \geqslant 2$. The bc-partition associated with $b$ is

$$\mathrm{Br}[b, w_1] \cup \cdots \cup \mathrm{Br}[b, w_j].$$

**Example 20.** The bc-partition shown below (in Fig. 9) is associated with $w$ and $b$.

## 4. Algorithms

**Problem 21.** *Let $T$ be a tree of order $n \geqslant 3$. Find a* bc-*partition of $T$ if one exists, otherwise find a semipartition of $T$.*

The following lemma is essential for the first algorithm.

**Lemma 22.** *Let $T$ be an odd type* 2 *tree* (*Fig.* 10). *Then $T^\star = T - \{v, w\}$ has a* bc-*partition.*

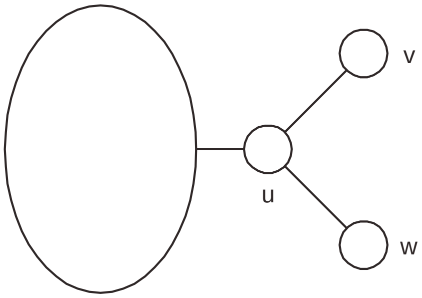



Fig. 10. A type 2 tree in the proof of Lemma 22.

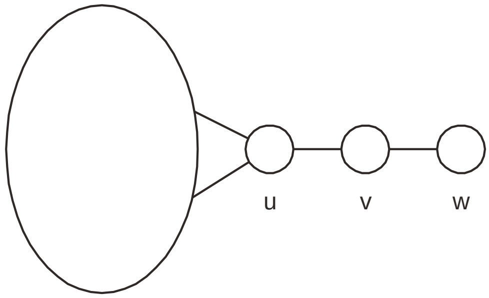



Fig. 11.

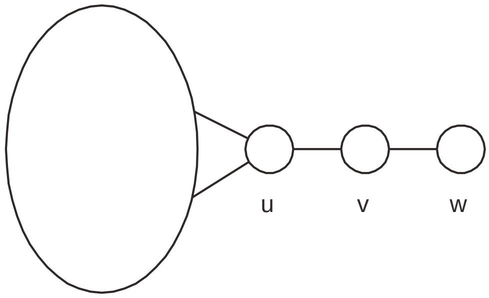



Fig. 12. An odd type 1 tree with a bc-partition in Algorithm 23.

**Proof.** Since $d(u)\geqslant 3$ there are at least 3 leaves in $T$. If all leaves of $T$ are adjacent to $u$ then $T=K_{1,p}$, where $p=d(u)$. This is a contradiction because $T$ is odd. Thus there are leaves $x$ and $y$ adjacent to $z\neq u$. Therefore, $T^\star$ does not have a perfect matching and hence $T^\star$ has a bc-partition. □

Note that if $T$ is an odd type 1 tree as in Fig. 11 with a bc-partition then

$$T^\star = T - \{v,w\}$$

also has a bc-partition. These observations lead to the following algorithm.

**Algorithm 23.** *Procedure* BCP($T$): *Find a* bc-*partition of the tree* $T$, *where* $T$ *does not have a perfect matching.*

*Input*: $T$, a tree with no perfect matching.

*Output*: $C$, a bc-partition of $T$.

(1) If $T$ is even then return the bc-partition $C=\{T\}$.

(2) (*Type* 1 *reduction*): If $T$ is odd and type 1 (Fig. 12). then return the bc-partition

$$C=\{uv,vw\}\cup \mathrm{BCP}(T-\{v,w\}).$$

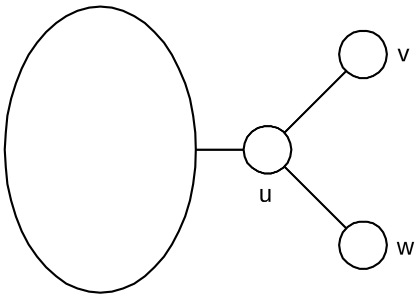



Fig. 13. An odd type 2 tree with a bc-partition in Algorithm 23.

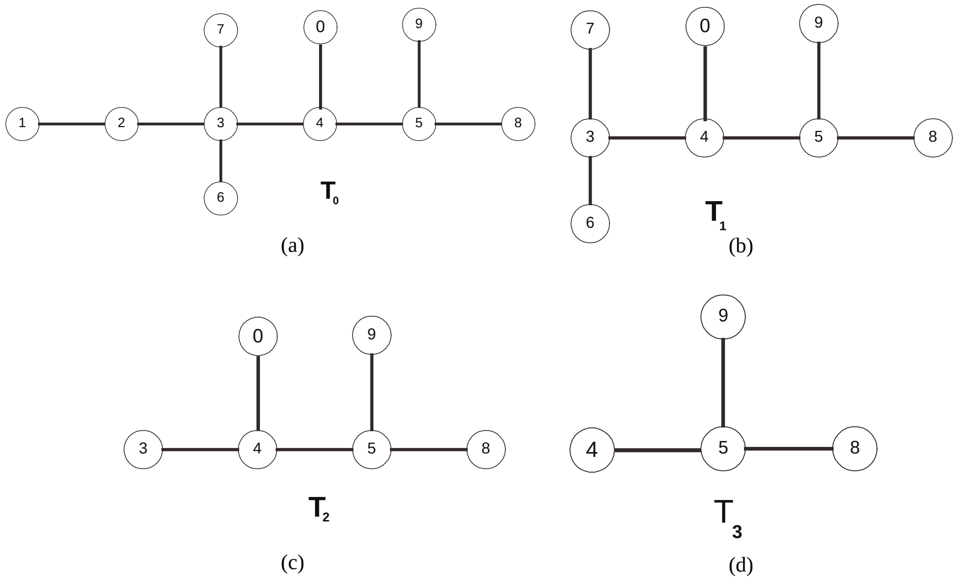



Fig. 14.

(3) (*Type* 2 *reduction*): If $T$ is odd and type 2 (Fig. 13). then return the bc-partition

$$C = \{uv, uw\} \cup \text{BCP}(T - \{v, w\}).$$

**Example 24.** Consider the tree in Fig. 14. The initial call to BCP gives

$$C = \{12, 23\} \cup \text{BCP}(T_1).$$

The first recursive call gives

$$\text{BCP}(T_1) = \{36, 37\} \cup \text{BCP}(T_2).$$

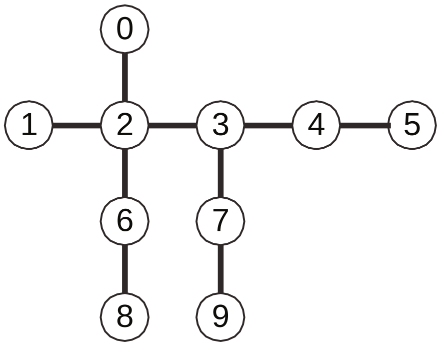



Fig. 15. A type 1 tree in which a type 2 reduction does not work in Algorithm 23.

The next call to BCP gives

$$\mathrm{BCP}(T_2) = \{04, 34\} \cup \mathrm{BCP}(T_3),$$

where the relation given in Fig. 14(d) holds and the final call to BCP gives

$$\mathrm{BCP}(T_3) = \{T_3\}.$$

A type 2 reduction may not work in a type 1 tree. In the tree below in Fig. 15 the procedure would fail if the leaves incident with vertex 2 were deleted first!

## *4.1. The unsaturated leaf algorithm*

It turns out that a maximum matching of the tree $T$ provides all of the information needed to construct a bc-partition or a semipartition.

**Lemma 25.** *Let $T$ be a tree with the maximum matching $M$. Suppose that the $M$-unsaturated vertex $u$ is a nonleaf and $x$ is a neighbor of $u$. Then* $\mathrm{Br}[u,x]$ *has maximum matching $M_x = M \cap E(\mathrm{Br}[u,x])$ and $M_x$-unsaturated leaf $u$.*

**Proof.** $T$ and $\mathrm{Br}[u,x]$ have the configurations given in Fig. 16. If $M^*$ is a matching of $\mathrm{Br}[u,x]$ with $|M^*| > |M_x|$ then

$$(M - M_x) \cup M^*$$

is a matching of $T$ with more edges than $M$. Since this is impossible we conclude that $M_x$ is a maximum matching of $\mathrm{Br}[u,x]$. □

First, note that the tree described in the previous lemma has a bc-partition and the subtree $\mathrm{Br}[u,x]$ has maximum matching $M_x$ with $M_x$-unsaturated leaf $u$. Thus, if $T$ has an $M$-unsaturated vertex $u$ of degree at least two then it can be partitioned into a collection of subtrees $\{T_i\}$, where $T_i$ has maximum matching $M_i$ and $M_i$-unsaturated

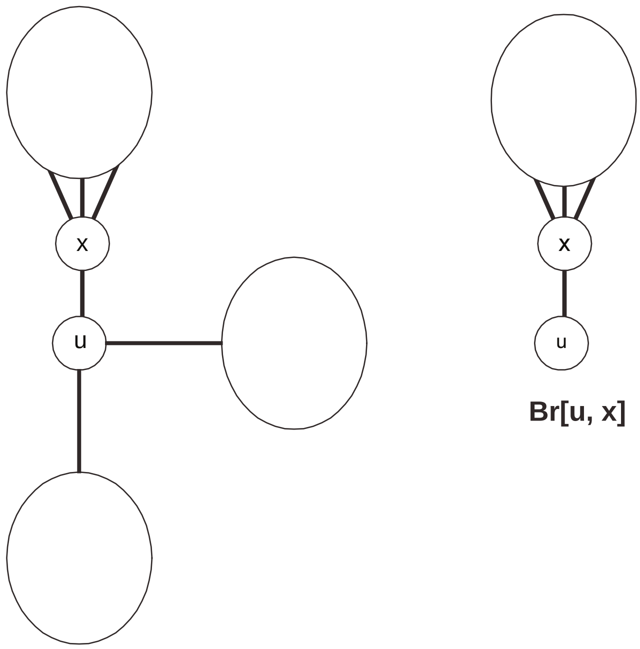



Fig. 16.

leaf $u$. If we can figure out how to partition a tree $T$ with maximum matching $M$ and $M$-unsaturated leaf $u$ then we can devise another algorithm to find a bc-partition.

Suppose that $u$ is an $M$-unsaturated leaf of $T$, where $M$ is a maximum matching. Let

$$P = u_0, u_1, \ldots, u_k$$

be an $M$-alternating path, where $u = u_0$. If $k \geqslant 3$ then let (Fig. 17)

$$M^* = M \cup \{u_0u_1\} - \{u_1u_2\}.$$

Note that $M^*$ is a maximum matching and $u_2$ is an $M^*$-unsaturated vertex of degree at least two. Thus, $T$ can be partitioned into subtrees as discussed in the previous lemma. If $k = 2$ then $T$ has the configuration given in Fig. 18.

The neighbors of $u_1$ (the black vertex in Fig. 18) are classified as follows, where $B(x) = \mathrm{Br}(u_1, x)$:

(i) $L_1, \ldots, L_a$ are leaves, where $u = L_1$ and $u_2 = L_2$.
(ii) $B(n_i)$ has an $M$-unsaturated vertex $z_i$ for every $i$, $1 \leqslant i \leqslant b$.
(iii) Each vertex of $B(p_j)$ is $M$-saturated for every $j$, $1 \leqslant j \leqslant c$.

Following the argument in the previous lemma, we conclude that $B(n_i)$ has maximum matching $M_i = M \cap E(B(n_i))$ and $M_i$-unsaturated vertex $z_i$. On the other hand, each subtree $B(p_j)$ has a perfect matching $M_j = M \cap E(B(p_j))$. Therefore, $\mathrm{Br}[u_1, p_j]$ has

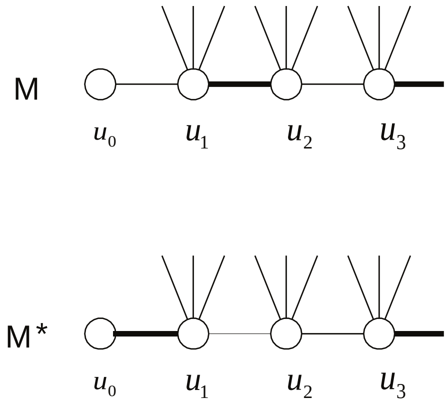



Fig. 17.

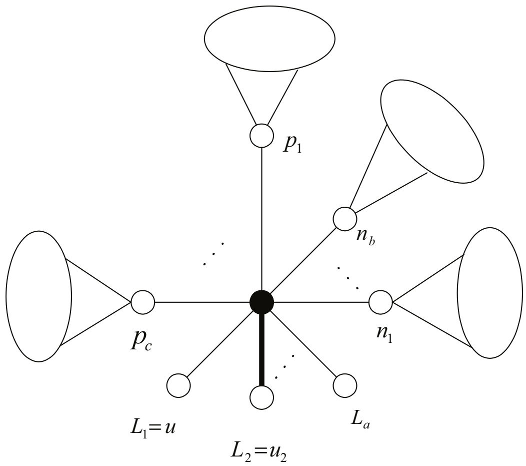



Fig. 18. An unsaturated leaf configuration.

maximum matching $M_j$ and $M_j$-unsaturated leaf $u_1$. Thus, $T$ can be partitioned as depicted in Fig. 19.

**Algorithm 26.** *Procedure* **UnsatLeaf**$(T, M, u)$: *Find a* bc-*partition of the tree* $T$, *where* $T$ *does not have a perfect matching.*

*Input*: $T$, maximum matching $M$ and $M$-unsaturated vertex $u$.

*Output*: $C$, a bc-partition of $T$.

(1) If $T$ is even then return the bc-partition $C = \{T\}$.

(2) If $T$ is odd and $u$ is not a leaf then let

$$\{B_1, B_2, \ldots, B_k\}$$

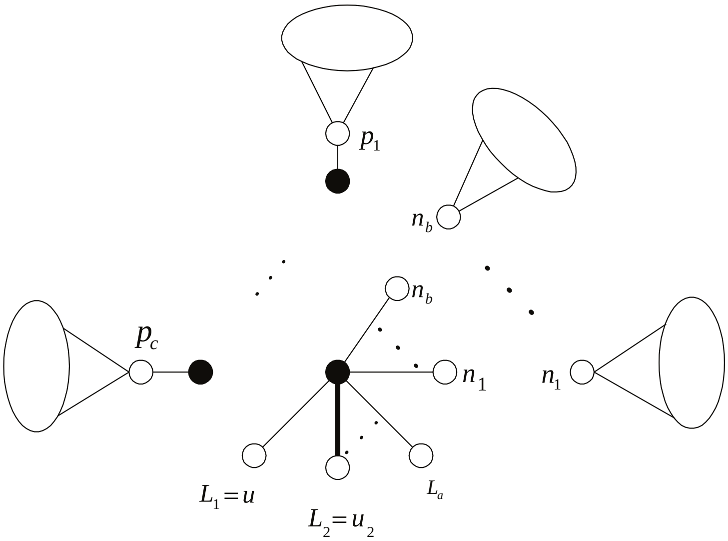



Fig. 19. The decomposition of a tree with an unsaturated leaf configuration.

be the branches at vertex $u$. $B_i$ has maximum matching $M_i = M \cap E(B_i)$ and $M_i$-unsaturated vertex $u$. Return the bc-partition

$$C = \bigcup_{i=1}^{k} \textbf{UnsatLeaf}(B_i, M_i, u).$$

(3) (*Unsaturated leaf reduction*): If $T$ is odd and $u$ is a leaf then let

$$P = u_0, u_1, \ldots, u_k$$

be an $M$-alternating path with initial vertex $u_0 = u$.

(a) If $k \geqslant 3$ let

$$M^* = M \cup \{u_0 u_1\} - \{u_1 u_2\}$$

and let

$$\{B_1, B_2, \ldots, B_h\}$$

be the branches at $u_2$, where $u_2$ has degree $h \geqslant 2$. Return the bc-partition

$$C = \bigcup_{i=1}^{k} \textbf{UnsatLeaf}(B_i, M_i, u_2),$$

where $M_i = M^* \cap E(B_i)$.

(b) If $k = 2$ then $T$ has the configuration of Fig. 18. Let

$$B_i = \mathrm{Br}[u_1, p_i], \quad \text{where } 1 \leqslant i \leqslant c.$$

$B_i$ contains the maximum matching $M_i = M \cap E(B_i)$ and $M_i$-unsaturated leaf $u_1$. Let

$$\beta_j = \mathrm{Br}(u_1, n_j), \quad \text{where } 1 \leqslant j \leqslant b.$$

$\beta_j$ contains the maximum matching $M_j = M \cap E(\beta_j)$. By definition, $\beta_j$ also contains at least one $M_j$-unsaturated vertex. Let $z_j$ be an $M_j$-unsaturated vertex of maximum degree. Finally, let $B_0$ be the subtree of $T$ induced by

$$u_1, L_1, \ldots, L_a, n_1, \ldots, n_b.$$

Return the bc-partition $C$

$$\{B_0\} \cup \left[\bigcup_{i=1}^{c} \textbf{UnsatLeaf}(B_i, M_i, u_1)\right] \cup \left[\bigcup_{j=1}^{b} \textbf{UnsatLeaf}(\beta_j, M_j, z_j)\right].$$

**Example 27.** In the following tree, $T$, the edges of maximum matching $M$ are shown and vertex 1 is $M$-unsaturated. Step 3a is executed (Fig. 20). In $M^* = M \cup \{12\} - \{23\}$

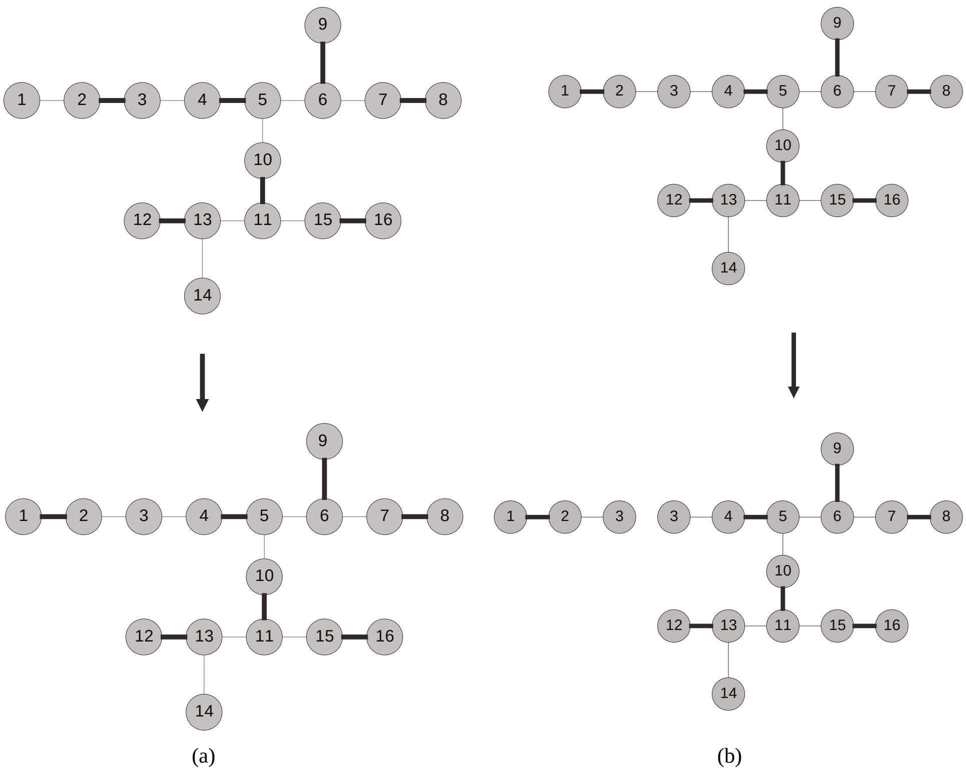
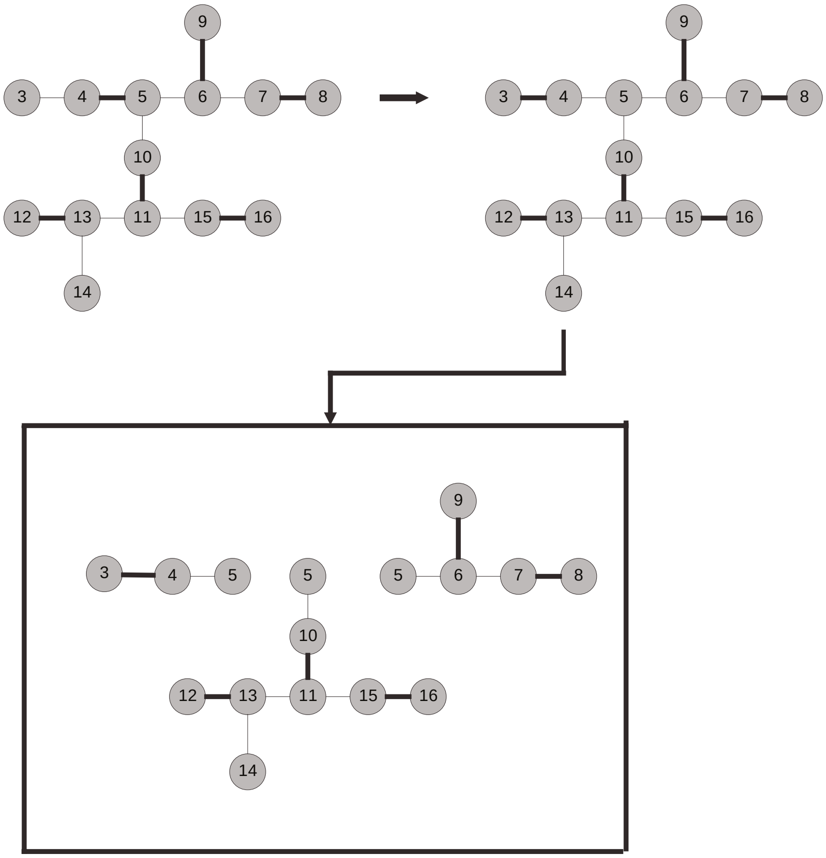
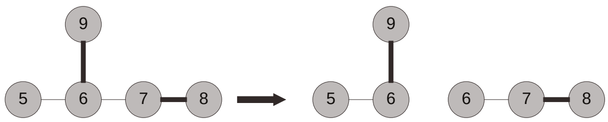


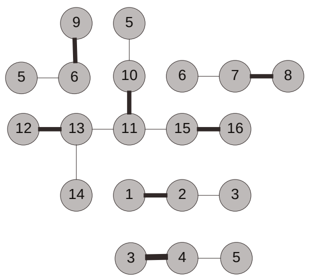

Fig. 20. (a–e). A bc-partition for the tree in Fig. 20.




(c)




(d)




(e)

Fig. 20. (*continued*)

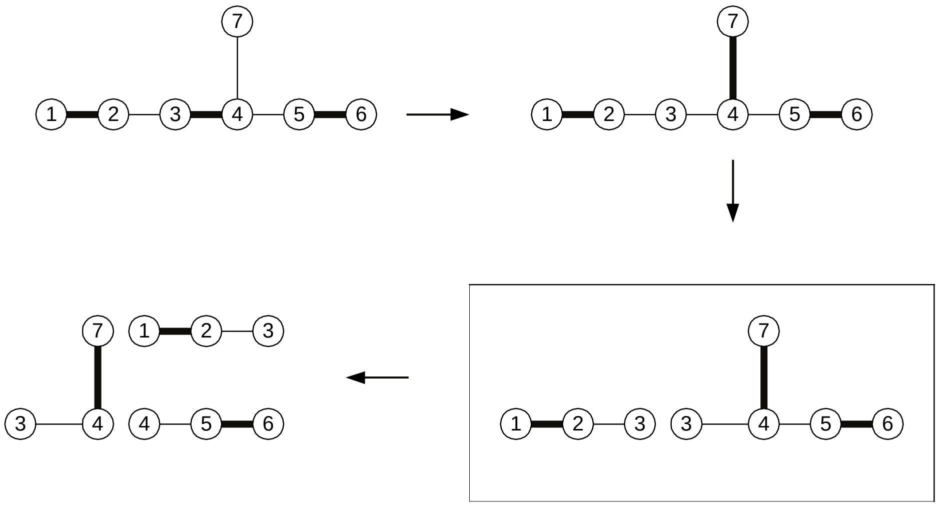



Fig. 21.

vertex 3 is unsaturated. Thus $T$ can be partitioned as in Fig. 20(a). In Br[3,4] vertex 3 is an unsaturated leaf and step 3a applies as can be seen from Fig. 20(b). Now, only Br[5,6] is odd and we have Fig. 20(c). Therefore, a bc-partition of $T$ is given in Fig. 20(e).

**Example 28.** In the tree below vertex 7 is unsaturated and the $M$-alternating path is 74321. Vertex 3 is unsaturated in the new matching and the bc-partition is generated as shown in Fig. 21.

Now, if $M$ is a perfect matching of $T$ and $e = uv \notin M$ then a semipartition can be found by finding bc-partitions $C_u$ of $\mathrm{Br}[u,v]$ and $C_v$ of $\mathrm{Br}[v,u]$. Note that

$$M_u = M \cap E(\mathrm{Br}[u,v])$$

is a maximum matching of $\mathrm{Br}[u,v]$ with $M_u$-unsaturated leaf $u$ and

$$M_v = M \cap E(\mathrm{Br}[v,u])$$

is a maximum matching of $\mathrm{Br}[v,u]$ with $M_v$-unsaturated leaf $v$. Thus, let

$$C_u = \textbf{UnsatLeaf}(\mathrm{Br}[u,v], M_u, u) \quad \text{and} \quad C_v = \textbf{UnsatLeaf}(\mathrm{Br}[v,u], M_v, v)$$

so that

$$S = C_u \cup C_v$$

is a semipartition of $T$ with $S(e) = 2$.

**Example 29.** Edge $e = 34$ is not in the perfect matching $M$. A semipartition $S$ is constructed from Br[4,3] and Br[3,4] so that $S(e) = 2$ (Fig. 22).

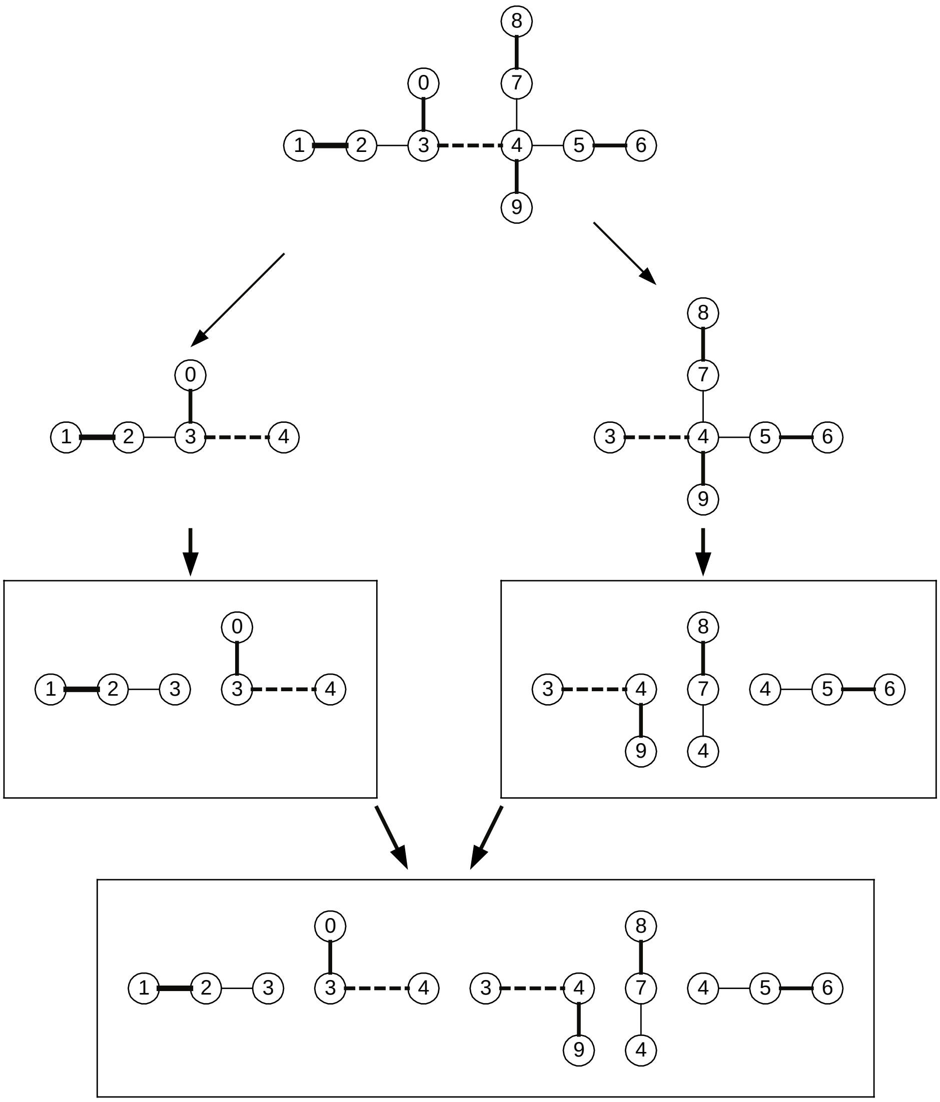



Fig. 22. Finding a semipartition in a tree with a perfect matching.

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