Superconvergence of discontinuous Galerkin methods for hyperbolic systems

Tie Zhang\(^{a}\), Jiandong Li\(^{a}\), Shuhua Zhang\(^{b,c,\ast}\)

\(^{a}\) Department of Mathematics, School of Information Science and Engineering, Northeastern University, Shenyang 110004, China
\(^{b}\) Department of Mathematics, Tianjin University of Finance and Economics, Tianjin 300222, China
\(^{c}\) Liu Hui Center for Applied Mathematics, Nankai University 300071, Tianjin University 300072, China

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In this paper, the discontinuous Galerkin method for the positive and symmetric, linear hyperbolic systems is constructed and analyzed by using bilinear finite elements on a rectangular domain, and an \(O(h^2)\)-order superconvergence error estimate is established under the conditions of almost uniform partition and the \(H^3\)-regularity for the exact solutions. The convergence analysis is based on some superclose estimates derived in this paper. Finally, as an application, the numerical treatment of Maxwell equation is discussed and computational results are presented.

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1. Introduction

Recently finite element methods for hyperbolic problems have attracted more and more attention; see, e.g., \([8,17,21,29,33]\) for the Galerkin finite element method, \([7,10,14,19,20,23,25,26,30]\) for the discontinuous Galerkin finite element method, \([1,12,16,28]\) for the Petrov–Galerkin method, and \([9,15,22,27]\) for the streamline diffusion method.

It is well known that for the \(k\)th order finite element approximations to elliptic or parabolic problems with an exact solution \(u\) in \(H^{k+1}(\Omega)\), the optimal error estimate in \(L^2\)-norm is of \(O(h^{k+1})\) order. However, for linear hyperbolic problems, it is still a completely unsolved problem that whether or not the finite element solutions admit this optimal error estimate. Generally speaking, the convergence rate of the Galerkin finite element method for hyperbolic problems is of \(O(h^k)\)-order, that is one order lower than the approximation order of finite element space; cf. \([8,17]\). In addition, in \([8]\) Dupont gave a counterexample by using the third-order Hermitian element to indicate that this convergence rate is sharp. However, this does not exclude the possibility that under a certain condition of partition the optimal or superconvergence error estimate of \(O(h^{k+1})\) can be obtained \([31]\).
In order to obtain high accuracy and to cope with the challenges of hyperbolic problems, Reed and Hill [24] proposed the discontinuous Galerkin finite element method for the neutron transport equation in 1973. Later on, Lesaint and Raviart gave the detailed theoretical analysis of this method for the neutron transport equation [19]. In [18] Lesaint investigated it for the linear system of first-order hyperbolic equations, and gained the error estimate of “gap 1”. Johnson and Huang [13] considered it for the Friedrichs system of equations, and gained the error estimate of “gap $\frac{1}{2}$”, which was extended to the linear system of first-order hyperbolic equations in [30,32]. In [23] Peterson proved that under the condition of quasi-uniform meshes the estimate $O(h^{k+\frac{1}{2}})$ is sharp. With the strict restriction of the characteristic direction and the triangular partition, Richter [25] obtained the superconvergence estimate for the first-order hyperbolic equation. Moreover, Huang [10,11] discussed discontinuous Galerkin finite element methods for mixed Tricomi equations and nonlinear vorticity transport equations. Also, Johnson and Pitkaranta considered this method for a class of nonlinear conservation equations [14].

Since 1989, Cockburn and Shu has systematically studied the discontinuous Galerkin finite element method for nonlinear conservation equations. By using numerical flux of finite difference with higher resolution, TVB (total variation bounded), and gradient limiters, a new type of discontinuous Galerkin finite element methods for aerodynamic problems was designed. See, for instance, [2–5]. Furthermore, discontinuous Galerkin methods were also investigated in [6] for the Maxwell equations with periodic boundary conditions.

In this paper, we discuss the discontinuous bilinear finite element approximation to positive and symmetric hyperbolic systems. Under the conditions of almost uniform rectangular partition and $H^3$-regularity for the exact solutions, an $O(h^2)$-order superconvergence is established. To the authors’ knowledge, very few superconvergence results have been obtained for hyperbolic problems, even in the one-dimensional case.

In the present paper, we use the standard notation $W^m_p(\Omega)$ for the Sobolev spaces with the corresponding norms and seminorms. Especially, when $p = 2$, $W^m_2(\Omega) = H^m(\Omega)$, $\| \cdot \|_{m,2} = \| \cdot \|_m$. Denote by $(\cdot, \cdot)$ and $\| \cdot \|_0$ the standard inner product and norm in $L^2(\Omega)$ space. For a Banach space $X$ and a constant $T > 0$, we use the space

$$L_p(0, T; X) = \left\{ v(t) : (0, T) \rightarrow X : \|v\|_{L_p(X)} = \left( \int_0^T \|v(t)\|^p_X dt \right)^{\frac{1}{p}} < \infty \right\}.$$ 

Throughout the paper, C represents a generic positive constant independent of the mesh size $h$.

This paper is organized as follows. In Section 2, some superclose estimates for interpolation functions are established. In Section 3, the discontinuous bilinear finite element approximations are analyzed for steady and nonsteady positive and symmetric hyperbolic systems, respectively, and the superconvergence error estimates are derived. Finally, as an application, the numerical treatment of Maxwell equation is discussed and the computational results are presented in Section 4.

2. Superclose estimates

Let $\Omega \subset \mathbb{R}^2$ be a rectangular domain with boundary $\partial \Omega = \cup_{i=1}^4 I_i$ (see Fig. 1), and $J_h = \{e\}$ a rectangular partition of domain $\Omega$ parameterized by mesh size $h$ so that $\mathcal{T} = \cup_{e \in J_h} \{\bar{e}\}$.

We introduce the discontinuous bilinear finite element space $S_h$ as follows:

$$S_h = \{ v \in L_2(\Omega) : v|_e \text{ is bilinear, } \forall e \in J_h\}.$$
Definition 1. Let rectangular element \( e = (x_e - h_e/2, x_e + h_e/2) \times (y_e - \tau_e/2, y_e + \tau_e/2) \). The partition \( J_0 \) is said to be uniform if
\[
h_e = h_1, \quad \tau_e = h_2, \quad \forall e \in J_0.
\] (1)
In addition, \( J_0 \) is said to be almost uniform if, for any two adjacent elements \( e \) and \( e' \),
\[
|h_e - h_{e'}| + |\tau_e - \tau_{e'}| = O(h^2), \quad \forall e, e' \in J_0, \quad \partial e \cap \partial e' \neq \emptyset.
\] (2)
Obviously, an almost uniform partition is also quasi-regular, which implies that the finite element inverse inequalities hold in \( S_0 \).

Set
\[
(u, v)_h = \sum_{e \in J_0} \int_e u v, \quad (u, v)_e = \int_e u v.
\]
For an arbitrary piecewise smooth function \( \phi \), we denote its jump along \( \partial e \) by
\[
[\phi] = \phi^+ - \phi^-, \quad \phi^+(p_0) = \lim_{p \to p_0^+, p \in e} \phi(p), \quad \phi^-(p_0) = \lim_{p \to p_0^-, p \in e} \phi(p),
\]
and define \( \phi_\partial = 0 \).

Let \( w_i \in C(\bar{\Omega}) \cap S_0 \) stand for the bilinear interpolation function of \( w \). By means of integral identities, we will establish the superclose properties for \( w_i \) (see, for example, [22]). To this end, we define two error functions by
\[
E(x) = \frac{1}{2} \left[ (x - x_e)^2 - \left( \frac{h_e}{2} \right)^2 \right], \quad F(y) = \frac{1}{2} \left[ (y - y_e)^2 - \left( \frac{\tau_e}{2} \right)^2 \right].
\] (3)
Obviously, we have at the vertices of each element \( e \) that
\[
(w - w_i) \left( x_e \pm \frac{h_e}{2}, y_e \pm \frac{\tau_e}{2} \right) = 0, \quad E \left( x_e \pm \frac{h_e}{2} \right) = 0, \quad F \left( y_e \pm \frac{\tau_e}{2} \right) = 0.
\] (4)

Lemma 1. Let partition \( J_0 \) be uniform, \( \beta = (\beta_1, \beta_2) \) a constant vector, \( w \in H^3(\Omega) \), and \( \phi \in S_0 \). Then, we have the following superclose estimate:
\[
\| (\beta \cdot \nabla (w - w_i), \phi)_{\partial e} \| = O(h^2) \| w \|_3 \| \phi \|_0 + \frac{1}{12} h_n^2 \beta_1 n \phi, w_{x \xi} \right|_{\Gamma_1^0} + \frac{1}{12} h_n^2 \beta_2 n \phi, w_{y \eta} \right|_{\Gamma_1^0} + \frac{1}{12} \sum_1 \left( h_n^2 \left( [\phi] \beta_1 n, w_{x \xi} \right|_{\Gamma_i^0} + h_n^2 \left( [\phi] \beta_2 n, w_{y \eta} \right|_{\Gamma_i^0} \right),
\]
where \( n = (n_x, n_y) \) is the outward unit normal vector along \( \partial e \), and \( \sum_1 \) stand for summing up the interior boundaries \( l \in \partial e, \quad l \notin \partial \Omega \) of all the elements.

Proof. For an arbitrary \( \phi \in S_0 \), since \( \phi \) is bilinear, it follows from the Taylor expansion of \( \phi 
\[
\phi(x, y) = \phi(x_e, y_e) + (x - x_e) \phi_x(x_e, y_e) + (y - y_e) \phi_y(x_e, y_e) + (x - x_e)(y - y_e) \phi_{xy}(x_e, y_e),
\]
that
\[
\int_e (w - w_i) \phi = \int_e (w - w_i) \phi(x_e, y_e) + \int_e (w - w_i) (x - x_e) \phi_x(x_e, y_e) + \int_e (w - w_i) (y - y_e) \phi_y(x_e, y_e) + \int_e (w - w_i) (x - x_e)(y - y_e) \phi_{xy}(x_e, y_e)
\]
\[
= I_1 + I_2 + I_3 + I_4.
\]
Next we deal with \( I_1, I_2, I_3, \) and \( I_4 \), respectively.

From (3) and (4), and integration by parts we have (see Fig. 1)
\[
I_1 = \int_e (w - w_i) \phi(x_e, y_e) = \phi(x_e, y_e) \int_e F_{x} (w - w_i)_x
\]
\[
= \phi(x_e, y_e) \left( \int_{I_1} (w - w_i) F_x \right) - \phi(x_e, y_e) \int_e (w - w_i) F_y
\]
\[
= -\phi(x_e, y_e) \int_e (w - w_i) F_y \phi_x = \phi(x_e, y_e) \int_e F w_{xy}
\]
\[
= \int_e F w_{xy} \phi(x, y) - E_\delta \phi_y(x, y) - F_\delta \phi_x(x, y) - E_\delta F \phi_{xy}
\]
\[
= O(h^2) \| w \|_{3, e} \| \phi \|_{0, e},
\]
where we have used the finite element inverse inequality.
\[ h^2 \| \phi \|_{k,e} \leq C \| \phi \|_{0,e}, \quad k = 1, 2, \forall \phi \in S_h. \]

Similarly, from \((y - y_e) = \frac{1}{6} (F^2)_{yy} \) we know by integration by parts that

\[
I_3 = \int_e \left( w - w_i \right) (y - y_e) \phi_\gamma (x_e, y_e) = \int_e \left( w - w_i \right) \frac{1}{6} (F^2)_{yy} \phi_\gamma (x_e, y)
\]

\[
= \frac{1}{6} \left( \int_{l_3} \left( (F^2)_{yy} (w - w_i) \phi_\gamma (x_e, y) \right) + \int_e \left( (F^2)_{yy} (w - w_i) \phi_\gamma (x_e, y) \right) \right)
\]

\[
= \frac{1}{6} \int_e \left( (F^2)_{yy} (w - w_i) \phi_\gamma (x_e, y) \right)
\]

\[
= \frac{1}{6} \int_e \left( (F^2)_{yy} (w - w_i) \phi_\gamma (x_e, y) \right) + \frac{1}{6} \int_e \left( (F^2)_{yy} (w - w_i) \phi_\gamma (x_e, y) \right)
\]

\[
= \frac{1}{6} \int_e \left( (F^2)_{yy} (w - w_i) \phi_\gamma (x_e, y) \right) = O(h^2) \| w \|_{1, e} \| \phi \|_{0, e}.
\]

where we have also used the finite element inverse estimates.

In addition, we have

\[
I_4 = \int_e \left( w - w_i \right) E_\gamma F_\gamma \phi_\gamma (x_e, y_e) = - \int_e w_{\gamma e} E_\gamma F_\gamma \phi_\gamma (x_e, y_e)
\]

\[
= \int_e w_{\gamma e} E_\gamma F_\gamma \phi_\gamma = O(h^2) \| w \|_{1, e} \| \phi \|_{0, e}.
\]

We note that

\[
E(x) = \frac{1}{6} (E^2)_{xx} - \frac{1}{12} h_e^2, \quad \phi_\gamma (x, y) = \phi_\gamma (x, y) + (y - y_e) \phi_\gamma (x_e, y_e),
\]

and \(\phi_\gamma (x_e, y_e) = \phi_\gamma (x_e, y_e)\). Integrating by parts and from (4) we obtain

\[
I_2 = \int_e \left( w - w_i \right) \phi_\gamma (x_e, y_e) = - \int_e w_{\gamma e} E_\gamma \phi_\gamma (x_e, y_e)
\]

\[
= - \int_e w_{\gamma e} \left[ \phi_\gamma (x, y) - (y - y_e) \phi_\gamma (x_e, y_e) \right]
\]

\[
= - \int_e w_{\gamma e} E_\gamma \phi_\gamma + \int_e w_{\gamma e} E_\gamma \phi_\gamma
\]

\[
= - \int_e w_{\gamma e} (E^2)_{xx} \phi_\gamma + \frac{1}{12} h_e^2 \int_e w_{\gamma e} \phi_\gamma - \int_e w_{\gamma e} E_\gamma \phi_\gamma
\]

\[
= \frac{1}{6} \int_e w_{\gamma e} (E^2)_{xx} \phi_\gamma + \frac{1}{12} h_e^2 \int_e w_{\gamma e} \phi_\gamma - \int_e w_{\gamma e} E_\gamma \phi_\gamma
\]

\[
= O(h^2) \| w \|_{1, e} \| \phi \|_{0, e} + \frac{1}{12} h_e^2 \left( \int_{l_3} \int_{l_4} \right) w_{\gamma e} \phi,
\]

which, together with the estimates for \(I_1, I_3, \) and \(I_4\), leads to

\[
\int_e \left( w - w_i \right) \phi = O(h^2) \| w \|_{1, e} \| \phi \|_{0, e} + \frac{1}{12} h_e^2 \left( \int_{l_3} \int_{l_4} \right) w_{\gamma e} \phi.
\]  \hspace{1cm} (5)

Analogously, we have

\[
\int_e \left( w - w_i \right) \phi = O(h^2) \| w \|_{1, e} \| \phi \|_{0, e} + \frac{1}{12} \tau_{yy}^2 \left( \int_{l_3} \int_{l_4} \right) \phi_{yy} \phi.
\]  \hspace{1cm} (6)

It follows from summing up (5) and (6) that

\[
\sum_{e} \int_{e} \beta \cdot \varphi (w - w_i) \phi = O(h^2) \| w \|_{1, e} \| \phi \|_{0, e} + \frac{1}{12} h_e^2 \left( \langle \beta_1 \phi, w_{\gamma e} \rangle_{l_2} - \langle \beta_1 \phi, w_{\gamma e} \rangle_{l_4} \right) + \frac{1}{12} h_e^2 \sum_{l_2} \langle \beta_1 \phi, w_{\gamma e} \rangle_{l_2}
\]

\[
+ \frac{1}{12} \tau_{yy}^2 \left( \langle \beta_2 \phi, w_{yy} \rangle_{l_3} - \langle \beta_2 \phi, w_{yy} \rangle_{l_4} \right) + \frac{1}{12} \tau_{yy}^2 \sum_{l_3} \langle \beta_2 \phi, w_{yy} \rangle_{l_3}
\]

\[
= O(h^2) \| w \|_{1, e} \| \phi \|_{0, e} + \frac{1}{12} h_e^2 \left( \langle \beta_1 n_e \phi, w_{\gamma e} \rangle_{l_2} + \langle \beta_2 n_e \phi, w_{yy} \rangle_{l_3} \right)
\]

\[
+ \frac{1}{12} \tau_{yy}^2 \left( \langle \beta_1 n_e \phi, w_{\gamma e} \rangle_{l_3} + \tau_{yy}^2 \langle \beta_2 n_e \phi, w_{yy} \rangle_{l_4} \right)
\],
where we have used the fact that for the outward unit normal vector \( n = (n_x, n_y) \) we have \( n_x |_l = 0 \) (or \( n_y |_l = 0 \)) when \( l \) is parallel to \( x \)-axis (or \( l \) is parallel to \( y \)-axis).

For the sake of simplicity, we assume in Lemma 1 that \( \beta \) is a constant vector and partition \( J_h \) is uniform. We now relax the limitations.

**Lemma 2.** Let partition \( J_h \) be almost uniform, \( \beta = (\beta_1(x, y), \beta_2(x, y)) \in W^1_\infty(\Omega) \) a vector, and \( w \in H^3(\Omega) \). Then, for any \( \phi \in S_h \) we have the following superclose estimate:

\[
| (\beta \cdot \nabla (w - w_l), \phi)_h | = O(h^2) \| \nabla \phi \|_0 + \frac{1}{12} \sum_{\partial e \cap J \in \mathcal{J}_4} h_e^2 (\beta_1 n_x \phi, w_{xx})_{ae} + \frac{1}{12} \sum_{\partial e \cap J \in \mathcal{J}_3} r_e^2 (\beta_2 n_y \phi, w_{yy})_{ae} \\
+ \frac{1}{12} \sum_{\partial e \cap J \in \mathcal{J}_3} (h_e^2 |(\beta_1 n_x \phi, w_{xx})| + r_e^2 |(\beta_2 n_y \phi, w_{yy})|).
\]

(7)

**Proof.** Let \( \beta^e \) be the constant approximation to \( \beta \) over \( J_h \) defined by

\[
\beta^e |_\ell = \frac{1}{|\ell|} \int_\ell \beta, \quad \forall \ell \in J_h.
\]

Then, we have

\[
| \beta - \beta^e | = O(h) \| \beta \|_{1, \infty}.
\]

From (5) and (6) we derive

\[
(\beta \cdot \nabla (w - w_l), \phi)_h = ((\beta - \beta^e) \cdot \nabla (w - w_l), \phi)_h + (\beta^e \cdot \nabla (w - w_l), \phi)_h \\
= O(h^2) \| \nabla \phi \|_0 + O(h^2) \| \phi \|_0 + \frac{1}{12} \sum_{\partial e \cap J \in \mathcal{J}_4} h_e^2 (\beta_1 n_x \phi, w_{xx})_{ae} \\
+ \frac{1}{12} \sum_{\partial e \cap J \in \mathcal{J}_3} r_e^2 (\beta_2 n_y \phi, w_{yy})_{ae} + \frac{1}{12} \sum_{\partial e \cap J \in \mathcal{J}_3} h_e^2 (\beta_1 \phi, w_{xx})_{ae} \\
+ \frac{1}{12} \sum_{\partial e \cap J \in \mathcal{J}_3} r_e^2 (\beta_2 \phi, w_{yy})_{ae}.
\]

(8)

With

\[
[\beta^e \phi] = \beta_1 [\phi] + [(\beta^e - \beta_1) \phi] = \beta_1 [\phi] + O(h) [\phi].
\]

it follows from the trace inequality and the finite element inverse inequality that

\[
\left( \int_{\partial e} w^2 \right)^{\frac{1}{2}} \leq C h^{-\frac{1}{2}} (h \| \nabla w \|_{0, e} + \| w \|_{0, e}), \quad w \in H^1(e),
\]

(9)

\[
\| \nabla \phi \|_0 \leq h^{-1} \| \phi \|_0, \quad \left( \int_{\partial e} \phi^2 \right)^{\frac{1}{2}} \leq C h^{-\frac{1}{2}} \| \phi \|_{0, e}, \quad \phi \in S_h,
\]

(10)

which, together with (8), implies (7). Now let the partition \( J_h \) be almost uniform. From (5) and (6) we obtain

\[
(\beta \cdot \nabla (w - w_l), \phi)_h = O(h^2) \| \nabla \phi \|_0 + \frac{1}{12} \sum_{\partial e \cap J \in \mathcal{J}_4} h_e^2 (\beta_1 n_x \phi, w_{xx})_{ae} + \frac{1}{12} \sum_{\partial e \cap J \in \mathcal{J}_3} r_e^2 (\beta_2 n_y \phi, w_{yy})_{ae} \\
+ \frac{1}{12} \sum_{\partial e \cap J \in \mathcal{J}_3} (h_e^2 - h_e^2)(\beta_1 \phi, w_{xx})_{ae} + \frac{1}{12} \sum_{\partial e \cap J \in \mathcal{J}_3} h_e^2 (\beta_1 \phi, w_{xx})_{ae} \\
+ \frac{1}{12} \sum_{\partial e \cap J \in \mathcal{J}_3} (r_e^2 - r_e^2)(\beta_2 \phi, w_{yy})_{ae} + \frac{1}{12} \sum_{\partial e \cap J \in \mathcal{J}_3} r_e^2 (\beta_2 \phi, w_{yy})_{ae},
\]

(11)

where \( e \) and \( e' \) are two adjacent elements with a common edge \( l = \partial e \cap \partial e' \). The proof of Lemma 2 is completed by using condition (2) and inequalities (9) and (10).

\[
\square
\]

### 3. Discontinuous Galerkin finite element methods

A discontinuous Galerkin finite element method was proposed in [30], which is modified to fit for the superconvergence analysis for our problem here.
3.1. Steady problems

Consider the following first-order hyperbolic problem:

\[ A \cdot \nabla u + B u = f(x, y), \quad (x, y) \in \Omega, \]  
\[ Nu = \frac{1}{2} (M - D) u = 0, \quad (x, y) \in \partial \Omega, \]  

where \( A = (A_1, A_2), A_k = (a_y^{(k)}(x, y)), k = 1, 2, B = (b_y(x, y)) \) and \( M = (m_y(x, y)) \) are some given \( m \times m \) order matrices, \( a_y^{(k)} \in W^1_{\infty}(\Omega), b_y, m_y \in L^\infty(\Omega), D = A \cdot n, n = (n_x, n_y) \) is the outward unit normal vector, \( u = (u_1, \ldots, u_m)^T \) and \( f = (f_1, \ldots, f_m)^T \) are \( m \)-dimensional vector functions. We assume that problem (12) and (13) is a positive and symmetric hyperbolic system, that is (see [9])

\[ A_1 = A_1^T, \quad A_2 = A_2^T, \quad (x, y) \in \Omega, \]  
\[ B + B^T \geq \text{div} A \geq \sigma_0 I, \quad \text{constant } \sigma_0 > 0, \quad (x, y) \in \Omega, \]  
\[ M + M^T \geq 0, \quad (x, y) \in \partial \Omega, \]  
\[ \text{Ker} (M - D) + \text{Ker} (M + D) = R^m, \quad (x, y) \in \partial \Omega. \]  

Introduce the bilinear form

\[ A(u, v) = (A \cdot \nabla u, v)_h + (B u, v) + \frac{1}{2} \sum_{e \in \mathcal{E}} \langle (M_e - D) [u], v \rangle_{\partial e}, \]  

where

\[ M_e = M, \quad (x, y) \in \partial e \cap \partial \Omega, \quad M_e = h^{-1} I, \quad (x, y) \in \partial e \setminus \partial \Omega, \]  

and \( I \) is the identity matrix. Now we are in a position to define the discontinuous finite element approximation to problem (12) and (13) by finding \( u_h \in [S_h]^m \) such that

\[ A(u_h, v_h) = (f, v_h), \quad \forall \ v_h \in [S_h]^m. \]  

**Lemma 3.** Let \( w \) be an arbitrary piecewise smooth vector function on \( J_h \). We have the following identity:

\[ A(w, w) = \frac{1}{2} \left( (B + B^T - \text{div} A) w, w \right)_h + \frac{1}{2} \sum_{e} \langle (M_e - D) [w], w \rangle_{\partial e} + \frac{1}{2} \sum_{I} \langle M_e [w], [w] \rangle_I. \]  

**Proof.** It follows from the Green formula and (18) that

\[ A(w, w) = \frac{1}{2} \left( (B + B^T - \text{div} A) w, w \right)_h + \frac{1}{2} \sum_{e} \langle D w, w \rangle_{\partial e} + \frac{1}{2} \sum_{e} \langle (M_e - D) [w], w \rangle_{\partial e} \]
\[ = \frac{1}{2} \left( (B + B^T - \text{div} A) w, w \right)_h + \frac{1}{2} \sum_{e} \langle M_e [w], w \rangle_{\partial e} + \frac{1}{2} \sum_{e} \langle D w', w' \rangle_{\partial e}. \]  

Note that \( D = A \cdot n, \quad D = D^T, \quad w^- |_{\partial e} = 0. \) Therefore, we have \( \sum_e \langle Dw^+, w^+ \rangle_{\partial e} = 0. \) Let \( I = \partial e \cap \partial e', e \) and \( e' \) be two adjacent elements with common edge \( l. \) Since

\[ (w^+ - w^-) w^+ |_{\partial e} + (w^+ - w^-) w^+ |_{\partial e'} = (w^+ - w^-) w^+ |_{\partial e} + (w^+ - w^+) w^- |_{\partial e} = [w] [w] |_{\partial e}, \]  

we have

\[ \sum_{e} \langle M_e [w], w \rangle_{\partial e} = \langle M [w], w \rangle_{\partial e} + \sum_{I} \langle M_e [w], [w] \rangle_{I}, \]  

which, together with (21), completes the proof of Lemma 3. \( \square \)

From Lemma 3, (15) and the Cauchy inequality, we can easily see that the solution \( u_h \) of the problem (19) uniquely exists and satisfies the following stability estimate:

\[ \sigma_0 \| u_h \|^2_0 + 2 \langle M u_h, u_h \rangle_{\partial e} + \sum_{I} \langle M_e [u_h], [u_h] \rangle_{I} \leq \frac{4}{\sigma_0} \| f \|^2_0. \]  

In order to make the error analysis here we assume a stronger condition than (16), which can be satisfied by many hyperbolic problems. That is, there exists a constant \( \sigma_1 > 0 \) such that

\[ (H) \quad \| (M + M^T) v_h, v_h \|_{\partial e} \geq \sigma_1 \| v_h \|^2_{\partial e}, \quad \forall \ v_h \in [S_h]^m. \]
Theorem 1. Let \( u \) and \( u_h \) be the solutions of the problems (12), (13) and (19), respectively, \( u \in [H^2(\Omega)]^m \), partition \( J_h \) be almost uniform, and Hypothesis \((H)\) hold. Then \( u_h \) satisfies the following superconvergence estimate:

\[
\| u - u_h \|_0 + \| u - u_h \|_{0,\partial^2} + \left( \sum_i \langle M_e[u - u_h], [u - u_h] \rangle_i \right)^{\frac{1}{2}} \leq Ch^2 \| u \|_3.
\]

Proof. When \( u \in C(\Omega) \), we have \( [u] = 0 \) on the inner element boundary \( \partial e \). Then, from (12), (13) and (19) we obtain the following error equation:

\[
A(u - u_h, v_h) = 0, \quad \forall v_h \in [S_h]^m.
\]  

(23)

Let \( u_i \in C(\Omega) \) be the bilinear interpolation approximation of function \( u \). We have from (23) that

\[
A(u_i - u_h, v_h) = A(u - u_i, v_h) = (A \cdot \nabla(u - u_i), v_h)_h + \frac{1}{2} (M - D)(u - u_i), v_h\|_{\partial^2}, \quad \forall v_h \in [S_h]^m.
\]

It follows from taking \( v_h = u_i - u_h \) in the above equation, using Lemma 3, hypothesis \((H)\), Lemma 2, and the interpolation approximation property that

\[
\frac{1}{2} \sigma_0 \| u_h - u_i \|_0^2 + \frac{1}{2} \sigma_1 \| u_h - u_i \|_{0,\partial^2}^2 + \frac{1}{2} \sum_i \langle M_e[u_h - u_i], [u_h - u_i] \rangle_i \leq Ch^2 \| u \|_3 \| u - u_i \|_0 + Ch^2 \sum_i h^{\frac{1}{2}} \| u \|_{2,\partial^2} h^{-\frac{1}{2}} \| u - u_i \|_{0,\partial^2}.
\]

(24)

\[
\leq Ch^4 \| u \|_3^2 + \sigma_0 \| u_h - u_i \|_0^2 + \sigma_1 \| u_h - u_i \|_{0,\partial^2} + \frac{1}{4} \sum_i \langle M_e[u_h - u_i], [u_h - u_i] \rangle_i,
\]

where we have used the trace inequality (9), \( M_e \|_0 = h^{-1} I \), and

\[
\| u - u_i \|_{0,\partial^2} \leq Ch^2 \| u \|_{2,\partial^2} \leq Ch^2 \| u \|_3.
\]

(25)

Using \( \| u - u_i \|_0 = 0 \), we complete the proof of Theorem 1 by the triangular inequality. \( \square \)

3.2. Nonsteady problems

Consider the time-dependent first-order hyperbolic problem:

\[
u_t + A \cdot \nabla u + Bu = f(t), \quad (t, x, y) \in [0, T) \times \Omega,
\]

(24)

\[
Nu = \frac{1}{2} (M - D)u = 0, \quad (t, x, y) \in [0, T) \times \partial \Omega,
\]

(25)

\[
u(0) = u_0, \quad (x, y) \in \Omega,
\]

(26)

where the notation representations in (24)–(26) are the same as those in (12) and (13).

Define the discontinuous Galerkin finite element approximation for problem (24)–(26) by finding \( u_h : [0, T) \rightarrow [S_h]^m \) such that

\[
(u_h, v_h) + A(u_h, v_h) = (f, v_h), \quad \forall v_h \in [S_h]^m, \quad (27)
\]

\[
u_h(0) \in [S_h]^m, \quad (28)
\]

where the bilinear form \( A(u, v) \) is given by (18).

Taking \( v_h = u_h \) in (27), from (20) we obtain

\[
\frac{1}{2} \frac{d}{dt} \| u_h(t) \|_0^2 + \frac{\sigma_0}{2} \| u_h(t) \|_{0,\partial}^2 + \frac{1}{2} \langle M_h[u_h], [u_h] \rangle_{i} \leq \| f(t) \|_0 \| u_h \|_0.
\]

(29)

or

\[
\frac{1}{2} \frac{d}{dt} \| u_h(t) \|_0^2 + \frac{\sigma_0}{2} \| u_h(t) \|_{0,\partial}^2 + \frac{1}{4} \langle (M + M^T)u_h, [u_h] \rangle_{i} + \frac{1}{4} \sum_i \langle [M_e + M^e]u_h, [u_h] \rangle_{i} \leq \| f(t) \|_0 \| u_h \|_0.
\]

Thus, from the two inequalities

\[
\langle (M + M^T)u_h, [u_h] \rangle_{\partial^2} \geq 0 \quad \text{and} \quad \sum_i \langle [M_e + M^e]u_h, [u_h] \rangle_{i} \geq 0
\]
we have
\[
\frac{1}{2} \frac{d}{dt} ||u_h(t)||_0^2 + \frac{\sigma_0}{2} ||u_h(t)||_0^2 \leq ||f(t)||_0 ||u_h||_0,
\]
or
\[
\frac{d}{dt} ||u_h(t)||_0 + \frac{\sigma_0}{2} ||u_h(t)||_0 \leq ||f(t)||_0,
\]
which yields
\[
\frac{d}{dt} (e^{\tau t} ||u_h(t)||_0) \leq e^{\tau t} ||f(t)||_0.
\]
This, together with the integration with respect to \(t\), implies the stability estimate:
\[
||u_h(t)||_0 \leq e^{-\tau t} ||u_h(0)||_0 + \int_0^t e^{-\tau (t-\tau)} ||f(\tau)||_0 d\tau, \quad t > 0.
\]

**Theorem 2.** Let \(u\) and \(u_h\) be the solutions of problems (24)–(26) and, (27)–(28), respectively, \(u(0) \in [H^2(\Omega)]^m, u_h(t) \in L_1(0, T; [H^2(\Omega)]^m), J_h\) be almost uniform, and Hypothesis (H) hold. Then, there exists a constant \(C\) independent of \(t \in [0, T)\) such that
\[
||u - u_h||_0 \leq C h^2 \left( ||u(0)||_3 + \int_0^t e^{-\tau (t-\tau)} ||f(\tau)||_3 d\tau \right), \quad t > 0.
\]

**Proof.** First introduce the projection approximation of solution \(u\) in \([S_n]^m\) by setting \(R_h u(t) : [0, T) \rightarrow [S_n]^m\) such that
\[
A(u(t) - R_h u(t), v_h) = 0, \quad \forall v_h \in [S_n]^m.
\]
From **Theorem 1** we know that
\[
||D^j_t u - R_h u(t)||_0 \leq C h^2 ||D^j_t u(t)||_3, \quad t \in [0, T), j = 0, 1.
\]

Now we write the error function as
\[
u(t) - u_h(t) = u(t) - R_h u(t) + R_h u(t) - u_h(t) = \eta + \theta.
\]
Then, from the equations satisfied by \(u(t), u_h(t)\) and \(R_h u(t)\) we see that \(\theta \in [S_n]^m\) satisfies
\[
(\theta, v_h) + A(\theta, v_h) = -(\eta, v_h), \quad \forall v_h \in [S_n]^m.
\]
Taking \(v_h = \theta\), similarly to the argument of (30), and using the triangular inequality and (31), we complete the proof. \(\square\)

**Remark.** For nonsteady problems, the condition (15) is not necessary for the error analysis. In fact, we may use the transformation: \(u = e^{\sigma t} w\) with \(\sigma\) satisfying \(\sigma - \|B + B^T - \text{div} A\|_\infty \geq \sigma_0\).

**4. Maxwell equations**

Maxwell equations are a class of very important partial differential equations in the electromagnetism field, such that various approximation methods have been proposed for them. In this section, as an application of our discontinuous finite element method discussed in Section 3, we discuss two-dimensional linear Maxwell equations in the following form (see, for example, [6]):
\[
\frac{\partial H_1}{\partial t} = -\frac{\partial H_3}{\partial y}, \quad \frac{\partial H_2}{\partial t} = \frac{\partial H_3}{\partial x}, \quad \frac{\partial H_3}{\partial t} = -\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y},
\]
with periodic boundary value condition on rectangular domain \(\Omega = [0, a] \times [0, b]\).

Setting vector function \(w = (H_1, H_2, H_3)^T\), we rewrite the problem (33) into the following first-order hyperbolic system:
\[
w_t + A_1 \partial_x w + A_2 \partial_y w = 0, \quad t > 0, (x, y) \in \Omega,
\]
\[
Nw = \frac{1}{2} (M - D_n) w = 0, \quad t > 0, (x, y) \in \partial \Omega,
\]
with the given initial value \(w(0, x, y) = w_0(x, y)\) and the matrices
\[
A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad D_n = \begin{pmatrix} 0 & 0 & n_y \\ 0 & 0 & -n_x \\ n_y & -n_x & 0 \end{pmatrix}.
\]
For the periodic boundary value problem, we can simply choose the boundary matrix \(N = O\) or \(M = D_n\).
implies that the hypothesis with periodic can be removed for the problem and and the remark in Section Theorem 2 we can find that the approximation accuracy is Let Table 1 Theorem 3 and, which validates the corresponding theoretical results computationally. Acknowledgment

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