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On Location of Blow-Up of Ground States of Semilinear Elliptic Equations in $\mathbb{R}^n$ Involving Critical Sobolev Exponents

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1. INTRODUCTION AND STATEMENT OF MAIN RESULT

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$$\Delta u - k(x)u + u^{p-\varepsilon} = 0, \qquad u > 0 \quad \text{in} \quad \mathbb{R}^n, \tag{1.1}$$

as $\varepsilon \to 0$, where $n \geqslant 3$, p is the critical Sobolev exponent, i.e., $p = (n+2)/(n-2)$. In [PW], Pan and Wang obtained the precise blow-up rate of the L^∞ norm of the ground states of (1.1). They also proved that any sequence u_{ε_j} of ground states contains a subsequence which blows up and concentrates at a single point as $\varepsilon_j \to 0$, under certain conditions on $k(x)$ and the ground states. The main purpose of this paper is to show that this point of blow-up and concentration is a global minimum point of $k(x)$.

Before giving the precise statements of the results described above, we first need to state a technical condition on $k(x)$.

$$\begin{gathered} k \text{ is a nonnegative } C^1 \text{ function defined on } \mathbb{R}^n, \\ k + \tfrac{1}{2} x \cdot \nabla k \text{ is bounded in } \mathbb{R}^n, \\ k(x) \geqslant k_0 > 0 \text{ for } |x| \text{ large, and } -k \in E(\rho, \mathbb{R}^n) \text{ for some } \rho \geqslant 0. \end{gathered} \tag{K}$$

Here $E(\rho, \mathbb{R}^n)$ is the set of all continuous functions u defined on $\mathbb{R}^n$ satisfying $u(y + te_i) \leqslant u(y + (2\lambda - t)e_i)$ for all $t \geqslant \lambda \geqslant \rho$ or $t \leqslant -\lambda \leqslant -\rho$, $y \in \Sigma_i = \{x = (x_1, ..., x_n) \in \mathbb{R}^n | x_i = 0\}$ with $1 \leqslant i \leqslant n$, where e_i is the unit vector pointing in the direction of the positive x_i-axis. Note if $u \in E(\rho, \mathbb{R}^n)$, then u is ultimately nondecreasing in every direction along some coordinate axis and u assumes its maximum in the cube $C(\rho)$ with length 2ρ and center at the origin.

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Any solution of (1.1) which also minimizes energy functional J_ε is called a *ground state* of (1.1), where J_ε is defined by

$$J_\varepsilon(u) = \frac{\int_{\mathbb{R}^n} |\nabla u|^2 + k(x)\, u^2}{(\int_{\mathbb{R}^n} |u|^{p+1-\varepsilon}\, dx)^{2/(p+1-\varepsilon)}}, \qquad u \in H^1(\mathbb{R}^n), \quad u \not\equiv 0$$

In the special case when $k(x) \equiv 1$, the existence of ground states (for $0 < \varepsilon < p-1$) was studied years ago ([Ne], [B] and [S]), but only until recently it was proven that every solution of (1.1) which decays at infinity must be radially symmetric about some point and achieves its maximum at that point ([GNN]), and that such solutions of (1.1) are unique up to translation in x variable ([K]).

For more general k, it is known that under Condition (**K**), (1.1) (with $0 < \varepsilon < p-1$) has a ground state u_ε which also belongs to $E(\rho, \mathbb{R}^n)$ (the condition on $k + \frac{1}{2} x \cdot \nabla k$ is unnecessary for this purpose, see [DN] or Lemma 2.1 in [PW]). Since this ground state u_ε is in $E(\rho, \mathbb{R}^n)$, it assumes its maximum at some point x_ε in the cube $C(\rho)$ and hence $\{x_\varepsilon\}$ is bounded.

Concerning the behavior of ground states of (1.1) for general $k(x)$, the following theorem is proved in [PW] (see Theorem 2 and the proof of Lemma 3.7 in [PW]).

THEOREM A. *Suppose Condition* (**K**) *holds. Let u_ε be a ground state of* (1.1) *which has a maximum point x_ε that remains bounded as $\varepsilon \to 0$. If some sequence x_{ε_j} converges to some point x_0, then each of the following holds.*

(i) *When $n = 3$,*

$$\varepsilon_j \, \|u_{\varepsilon_j}\|^2_{L^\infty} \to \frac{768\pi^3}{\sqrt{3}} \int_{\mathbb{R}^n} (k + \tfrac{1}{2} x \cdot \nabla k)\, \Gamma_k^2(x, x_0)\, dx$$

as $\varepsilon_j \to 0$, where Γ_k is the fundamental solution of $-\Delta + k$ in $\mathbb{R}^n$;

(ii) *When $n > 4$,*

$$\varepsilon_j \, \|u_{\varepsilon_j}\|^{4/(n-2)}_{L^\infty} \to (k(x_0) + \tfrac{1}{2} x_0 \cdot \nabla k(x_0))\, \frac{16n(n-1)}{(n-2)^3}$$

as $\varepsilon_j \to 0$.

(iii) $\|u_{\varepsilon_j}\|_{L^\infty} u_{\varepsilon_j}(x) \to (1/n)\, \omega_n [n(n-2)]^{n/2}\, \Gamma_k(x, x_0)$ *in* $C^2_{\mathrm{loc}}(\mathbb{R}^n \backslash \{x_0\})$ *as $\varepsilon_j \to 0$. Furthermore, for ε_j small,*

$$u_{\varepsilon_j}(x) \leqslant \begin{cases} Ce^{-a|x - x_{\varepsilon_j}|} / \|u_{\varepsilon_j}\|_{L^\infty}, & |x - x_{\varepsilon_j}| \geqslant 1, \\ C\,|x - x_{\varepsilon_j}|^{2-n} / \|u_{\varepsilon_j}\|_{L^\infty}, & |x - x_{\varepsilon_j}| \leqslant 1, \end{cases} \tag{1.2}$$

where C and a are positive constants independent of ε.

Remark 1.1. In [PW], Condition (**K**) contains one more condition: $k+\frac{1}{2}x\cdot\nabla k\geqslant 0,\ \not\equiv 0$. This is used only in the proof of Lemma 3.1 in [PW] to show the blow-up of u_ε (including the case when $n=4$). It turns out that this is still the case without this extra condition—actually, we do not even need the boundedness of $k+\frac{1}{2}x\cdot\nabla k$. See Lemma 3.1 in this paper. In [PW], u_ε is assumed to be in $E(\rho, \mathbb{R}^n)$, and x_ε in $C(\rho)$. By the proof in [PW], only the boundedness of x_ε is necessary. The condition $-k\in E(\rho, \mathbb{R}^n)$ is useful only to assure the existence of u_ε and x_ε in the statement of Theorem A. The boundedness of $k+\frac{1}{2}x\cdot\nabla k$ is used to obtain the blow-up rates ((i) and (ii) of Theorem A).

Remark 1.2. Part (ii) does not cover the case when $n=4$ (Part (iii) does). However, when $k(x)$ is identically equal to 1, it is covered in [PW, Theorem 1], where the value of the integral in (i) is also given. We conjectured in [PW] that

$$\frac{\varepsilon_j\|u_{\varepsilon_j}\|^2_{L^\infty}}{\ell n\|u_{\varepsilon_j}\|_{L^\infty}}\to 48(k(x_0)+\tfrac{1}{2}x_0\cdot\nabla k(x_0)),$$

and we were informed by Zhenchao Han that he obtained a proof of this.

From this theorem, we see that u_{ε_j} blows up and concentrates at x_0. The main purpose of this paper is to show that x_0 is a minimum point of k. More precisely, we shall prove the following.

THEOREM 1.1. *Suppose that* $n>6$ *and* (**K**) *holds. Let* u_ε *and* x_ε *be defined as in the statement of Theorem* A. *Then* $k(x_\varepsilon)\to\inf_{x\in\mathbb{R}^n}k(x)$ *as* $\varepsilon\to 0$.

Remark 1.3. In Section 3, we shall show that when $n>6$, Theorem A and Theorem 1.1 hold for an *arbitrary* ground state u_ε of (1.1) and an *arbitrary* maximum point x_ε of u_ε (i.e., the boundedness of x_ε is not needed), under an additional condition (3.4) (see Theorem 3.3). In that same section, we shall also show that this is still the case if "$-k\in E(\rho, \mathbb{R}^n)$" in Condition (**K**) is replaced by (3.6) (see Theorem 3.4). Under (3.6), the existence of a ground state is proved by Rabinowitz [R]. The main concern here is that x_ε might go off to infinity as $\varepsilon\to 0$. Indeed, this may happen if K is independent of at least one component of x.

Before describing the main arguments in the proof of Theorem 1.1, we need some preparation. Define μ_ε by $\mu_\varepsilon^{-2/(p-1-\varepsilon)}=\|u_\varepsilon\|_{L^\infty}$. Let $v_\varepsilon(x)=\mu_\varepsilon^{2/(p-1-\varepsilon)}u(x_\varepsilon+\mu_\varepsilon x)$. Then $0<v_\varepsilon\leqslant 1$, $v_\varepsilon(0)=1$ and

$$\Delta v_\varepsilon-\mu_\varepsilon^2 k(x_\varepsilon+\mu_\varepsilon x)v_\varepsilon+v_\varepsilon^{p-\varepsilon}=0 \qquad \text{in} \quad \mathbb{R}^n. \tag{1.3}$$

Then by the elliptic interior estimates and the uniqueness result of [CGS] or [CL], we have

$$v_\varepsilon \to U \qquad \text{in} \quad C^2_{\text{loc}}(\mathbb{R}^n), \tag{1.4}$$

where $U(x) = (1 + |x|^2/(n(n-2)))^{(2-n)/2}$ is the unique positive solution of

$$\Delta u + u^p = 0, \qquad x \in \mathbb{R}^n, \quad u(0) = 1. \tag{1.5}$$

Actually, more is known from Lemma 3.6 in [PW]

$$v_\varepsilon \leqslant cU \text{ and hence } v_\varepsilon \to U \text{ in } L^\infty \qquad \text{as} \quad \varepsilon \to 0, \tag{1.6}$$

where c stands for a generic constant independent of ε (we shall use this convention throughout this paper).

To prove Theorem 1.1, we adapt the method developed by Ni and Takagi in [NT] where they proved that as the diffusion coefficient shrinks to zero, least energy solutions to the Neumann problem of an elliptic equation on a bounded domain concentrate at the "most curved" part of the boundary. The basic idea is to get an asymptotic expansion (in ε or μ_ε) of the "ground energy"

$$S_\varepsilon = \inf\{J_\varepsilon(u) \mid u \in H^1(\mathbb{R}^n), u \not\equiv 0\},$$

then compare it with an upper bound of S_ε obtained by using a good trial function. To have this asymptotic expansion, we expand v_ε in μ_ε. By (1.4) and (1.6), the first approximation of v_ε should be U. Let $v_\varepsilon = U + \mu_\varepsilon^2 w_\varepsilon$. In order to get an a-priori bound for w_ε, we have to deal with the linearlized operator $L = \Delta + pU^{p-1}$. Unlike in [NT], one of the main difficulties stems from the slow decay of U and the fundamental solution of Δ. We get around this by using Lemma 2.4. Unfortunately, the case $3 \leqslant n \leqslant 6$ is left out in this approach, though we certainly believe that Theorem 1.1 holds in this case.

Finally, we mention that the blow-up behavior of "ground states" of the Dirichlet problem of Equation (1.1) with $k(x)$ identically equal to zero has been studied at least by Atkinson and Peletier [AP], Brezis and Peletier [BP], Han [H] and Rey [Re]. By using Pohozaev identity, Han and Rey proved that as $\varepsilon \to 0$ the ground states blow up at critical points of the regular part of the Green function. The approach in the present paper is entirely different from theirs.

2. Proof of Theorem 1.1

Throughout this section, we assume Condition (**K**) holds.

To prove Theorem 1.1, we just need to show x_0 in the statement of Theorem A is a minimum point of k. We begin with a result which offers a good upper bound for

$$S_j \equiv S_{\varepsilon_j} = \inf\{I_{\varepsilon_j}(u) \mid u \in H^1(\mathbb{R}^n),\ u \not\equiv 0\}. \tag{2.1}$$

Let S be the best Sobolev constant, i.e.,

$$S = \inf_{u \in H^1(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n} |\nabla u|^2\, dx}{(\int_{\mathbb{R}^n} |u|^{p+1}\, dx)^{2/(p+1)}}, \qquad u \not\equiv 0.$$

It is well-known that S is achieved by U and hence from (1.5),

$$S = \left(\int_{\mathbb{R}^n} |\nabla U|^2\, dx\right)^{2/n} = \left(\int_{\mathbb{R}^n} U^{p+1}\, dx\right)^{2/n}. \tag{2.2}$$

Recall

$$\mu_j \equiv \mu_{\varepsilon_j} = (\|u_{\varepsilon_j}\|_{L^\infty})^{-(p-\varepsilon_j-1)/2} \to 0 \qquad \text{as} \quad \varepsilon_j \to 0.$$

Lemma 2.1. *If $n > 4$, then*

$$S_j \leqslant S + \mu_j^2 \left[\inf k S^{(2-n)/2} \int_{\mathbb{R}^n} U^2\, dx + \frac{n-2}{n} C(n,k)\, S^{(2-n)/2} \right.$$
$$\left. \times \int_{\mathbb{R}^n} U^{p+1} \ell n\, U\, dx - \frac{n}{(p+1)^2} C(n,k)\, S\, \ell n\, S \right] + o(\mu_j^2),$$

where $C(n,k) = (k(x_0) + \frac{1}{2} x_0 \cdot \nabla k(x_0))\, 16n(n-1)/(n-2)^3$.

Proof. Since $-k \in E(\rho, \mathbb{R}^n)$, infimum of k is assumed at some point x_1. Let $\varphi_j(x) = U((x - x_1)/\mu_j)$. Then we have

$$\int_{\mathbb{R}^n} |\nabla \varphi_j|^2\, dx = \mu_j^{n-2} \int_{\mathbb{R}^n} |\nabla U|^2\, dx = \mu_j^{n-2} S^{n/2}, \tag{2.3}$$

and

$$\int_{\mathbb{R}^n} k(x)\, \varphi_j^2(x)\, dx = \mu_j^n \int_{\mathbb{R}^n} k(x_1 + \mu_j y)\, U^2(y)\, dy.$$

Since $k+\frac{1}{2}x\cdot\nabla k$ is bounded, by considering $f(t)=k(tx)$, it is easy to see that k is also bounded. So by the Dominated Convergence Theorem we have

$$\int_{\mathbb{R}^n} k(x_1+\mu_j y)\, U^2(y)\, dy = k(x_1)\int_{\mathbb{R}^n} U^2\, dy + o(1).$$

Thus,

$$\int_{\mathbb{R}^n} k(x)\,\varphi_j^2(x)\, dx = \mu_j^n k(x_1)\int_{\mathbb{R}^n} U^2\, dy + o(\mu_j^n). \tag{2.4}$$

From Theorem A, we obtain

$$\varepsilon_j = C(n,k)\,\mu_j^2 + o(\mu_j^2). \tag{2.5}$$

By Taylor's theorem, (2.2) and (2.5), we have

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n} \varphi^{p-\varepsilon_j+1}\, dx\right)^{2/(p-\varepsilon_j+1)}\\
&\quad=\mu_j^{2n/(p-\varepsilon_j+1)}\left(\int_{\mathbb{R}^n} U^{p+1-\varepsilon_j}\, dy\right)^{2/(p+1-\varepsilon_j)}\\
&\quad\geqslant \mu^{2n/(p+1)}\left(\int_{\mathbb{R}^n} \left(U^{p+1}-\varepsilon_j\, U^{p+1}\,\ell n\, U\right) dy + o(\varepsilon_j)\right)^{2/(p+1-\varepsilon_j)}\\
&\quad=\mu_j^{n-2}\left[\left(S^{n/2}-\varepsilon_j\int_{\mathbb{R}^n} U^{p+1}\,\ell n\, U\, dy\right)^{2/(p+1-\varepsilon_j)} + o(\varepsilon_j)\right]\\
&\quad=\mu_j{}^{n-2}\left\{(S^{n/2})^{2/(p+1)} + \varepsilon_j (S^{n/2})^{2/(p+1)}\right.\\
&\qquad\times\left.\left[\frac{2}{(p+1)^2}\,\ell n\, S^{n/2} + \frac{2}{p+1}\,\frac{1}{S^{n/2}}\left(-\int_{\mathbb{R}^n} U^{p+1}\,\ell n\, U\, dy\right)\right] + o(\varepsilon_j)\right\}\\
&\quad=\mu_j^{n-2} S^{(n-2)/2}\left[1 + C(n,k)\,\mu_j^2\left(\frac{n}{(p+1)^2}\,\ell n\, S\right.\right.\\
&\qquad\left.\left. - \frac{(n-2)}{n}\, S^{-n/2}\int_{\mathbb{R}^n} U^{p+1}\,\ell n\, U\, dy\right) + o(\mu_j^2)\right].
\end{aligned}$$

Combining this with (2.3) and (2.4), we obtain

$$S_j \leqslant \frac{\int_{\mathbb{R}^n} |\nabla \varphi_j|^2 + k\varphi_j^2 \, dx}{(\int_{\mathbb{R}^n} \varphi_j^{p+1-\varepsilon_j})^{2/(p+1-\varepsilon_j)}}$$

$$\leqslant \frac{\mu_j^{n-2} S^{n/2} + \mu_j^n k(x_1) \int_{\mathbb{R}^n} U^2 \, dy + o(\mu_j^n)}{\left[\begin{array}{l} \mu_j^{n-2} S^{(n-2)/2} [1 + C(n,k)\, \mu_j^2 (n/(p+1)^2 \, \ell n \, S \\ \qquad - (n-2)/n \, S^{-n/2} \int_{\mathbb{R}^n} U^{p+1} \, \ell n \, U \, dy) + o(\mu_j^2)] \end{array}\right]}$$

$$= \left(S + \mu_j^2 k(x_1) S^{(2-n)/2} \int_{\mathbb{R}^n} U^2 \, dx + o(\mu_j^2) \right)$$

$$\cdot \left[1 - C(n,k) \mu_j^2 \left(\frac{n}{(p+1)^2} \, \ell n \, S \right.\right.$$

$$\left.\left. - \frac{(n-2)}{n} S^{-n/2} \int_{\mathbb{R}^n} U^{p+1} \, \ell n \, U \, dy \right) + o(\mu_j^2) \right].$$

From this, Lemma 2.1 follows. ▌

Define w_j by $v_j = U + \mu_j^2 w_j$, where $v_j \equiv v_{\varepsilon_j} = \mu_j^{2/(p-1-\varepsilon_j)} u(x_{\varepsilon_j} + \mu_j x)$. Then by (1.3),

$$\Delta w_j + pU^{p-1} w_j - k_j v_j + F(w_j) = 0 \qquad \text{in } \mathbb{R}^n, \tag{2.6}$$

where $F(w_j) = [(U + \mu_j^2 w_j)^{p-\varepsilon_j} - U^p - p\mu_j^2 U^{p-1} w_j]/\mu_j^2$, $k_j(x) = k(x_{\varepsilon_j} + \mu_j x)$.

PROPOSITION 2.2. *Assume $n > 6$. Then $w_j \to w$ in L^∞ as $j \to \infty$, where w is a bounded solution of*

$$\Delta w + pU^{p-1} w - k(x_0)\, U - C(n,k)\, U^p \, \ell n \, U = 0 \qquad \textit{in } \mathbb{R}^n, \tag{2.7}$$

$w \in W^{2,s}(\mathbb{R}^n)$ *for* $s > n/(n-4)$.

More properties of w will be seen later. We delay the proof of this result, but use it to show Theorem 1.1 now.

Proof of Theorem 1.1. First, we derive an asymptotic formula for S_j. By the definitions of u_{ε_j}, v_j, S_j, and by (1.3), we have

$$\begin{aligned} S_j &= J_{\varepsilon_j}(u_{\varepsilon_j}) \\ &= \frac{\int_{\mathbb{R}^n} (|\nabla v_j|^2 + \mu_j^2 k_j v_j^2)\, dx}{(\int_{\mathbb{R}^n} v_j^{p+1-\varepsilon_j} \, dx)^{2/(p+1-\varepsilon_j)}} \\ &= \left(\int_{\mathbb{R}^n} v_j^{p+1-\varepsilon_j} \, dx \right)^{1-2/(p+1-\varepsilon_j)} \\ &= \left(\int_{\mathbb{R}^n} (U + \mu_j^2 w_j)^{p+1-\varepsilon_j} \, dx \right)^{1-2/(p+1-\varepsilon_j)}. \end{aligned} \tag{2.8}$$

From Taylor's Theorem, we obtain

$$\int_{\mathbb{R}^n} (U+\mu_j^2 w_j)^{p+1-\varepsilon_j}\,dx$$

$$=\int_{\mathbb{R}^n} \Big[U^{p+1-\varepsilon_j} + (p+1-\varepsilon_j)\, U^{p-\varepsilon_j}\mu_j^2 w_j$$

$$+\tfrac{1}{2}(p+1-\varepsilon_j)(p-\varepsilon_j)(U+t\mu_j^2 w_j)^{p-1-\varepsilon_j}\mu_j^4 w_j^2 \Big]\,dx \qquad (2.9)$$

for some $0<t<1$ which depends on x and j. By (1.6) and Proposition 2.2,

$$\int_{\mathbb{R}^n} (U+t\mu_j^2 w_j)^{p-1-\varepsilon_j}\,\mu_j^4 w_j^2\,dx \leqslant c\int_{\mathbb{R}^n} U^{p-\varepsilon_j-1}\,|v_j-U|\,\mu_j^2\,\|w_j\|_{L^\infty}\,dx$$

$$=o(\mu_j^2).$$

Now returning to (2.9) and using Proposition 2.2 and Taylor's Theorem again, we have

$$\int_{\mathbb{R}^n} (U+\mu_j^2 w_j)^{p+1-\varepsilon_j}\,dx$$

$$=\int_{\mathbb{R}^n} (U^{p+1-\varepsilon_j} + (p+1-\varepsilon_j)\, U^{p-\varepsilon_j}\mu_j^2 w)\,dx + o(\mu_j^2)$$

$$=\int_{\mathbb{R}^n} (U^{p+1} - \varepsilon_j\, U^{p+1}\,\ell n\, U + (p+1)\, U^p \mu_j^2 w)\,dx + o(\varepsilon_j) + o(\mu_j^2)$$

$$=S^{n/2} + \mu_j^2 \int_{\mathbb{R}^n} (-C(n,k)\, U^{p+1}\,\ell n\, U + (p+1)\, U^p w)\,dx + o(\mu_j^2). \qquad (2.10)$$

(At the last step, we have used (2.2) and (2.5).) Multiplying (2.7) by U and integrating by parts yield

$$\int_{\mathbb{R}^n} U^p w\,dx = \frac{1}{p-1}\int_{\mathbb{R}^n} (k(x_0)\,U^2 + C(n,k))\; U^{p+1}\,\ell n\, U)\,dx.$$

(Here (1.5), the fact that $w\in L^s(\mathbb{R}^n)$ for $s>n/(n-4)$ and $n>6$ have been used.) Plugging this identity into (2.10) and then returning to (2.8), by Taylor's Theorem, we have

$$S_j = \left[S^{n/2} + \mu_j^2 \int_{\mathbb{R}^n} \left(\frac{2}{p-1} C(n,k)\, U^{p+1} \ell n\, U \right.\right.$$

$$\left.\left. + \frac{p+1}{p-1} k(x_0)\, U^2 \right) dx \right]^{1-2/(p+1-\varepsilon_j)} + o(\mu_j{}^2)$$

$$= I^{1-2/(p+1-\varepsilon_j)} + o(\mu_j^2)$$

$$= I^{1-2/(p+1)} - \varepsilon_j I^{1-2/(p+1)} (\ell n\, I)\, \frac{2}{(p+1)^2} + o(\mu_j^2)$$

$$= (S^{n/2})^{(p-1)/(p+1)} + \frac{p-1}{p+1} (S^{n/2})^{-2/(p+1)} \mu_j^2$$

$$\times \int_{\mathbb{R}^n} \left(\frac{2}{p-1} C(n,k)\, U^{p+1} \ell n\, U + \frac{p+1}{p-1} k(x_0)\, U^2 \right) dx$$

$$- \frac{2\varepsilon_j}{(p+1)^2} (S^{n/2})^{(p-1)/(p+1)} \ell n\, S^{n/2} + o(\mu_j^2).$$

Now by (2.5), we have

$$S_j = S + \mu_j^2 S^{(2-n)/2} \left(k(x_0) \int_{\mathbb{R}^n} U^2\, dx + \frac{n-2}{n} C(n,k) \int_{\mathbb{R}^n} U^{p+1} \ell n\, U\, dx \right.$$

$$\left. - \frac{nC(n,k)}{(p+1)^2} S^{n/2} \ell n\, S \right) + o(\mu 2j). \tag{2.11}$$

Comparing this with the upper bound of S_j given in Lemma 2.1, we have $k(x_0) = \inf k$. The proof of Theoem 1.1 is complete. ∎

The remaining part of this section is devoted to the proof of Proposition 2.2. First, we need to analyze the linear operator associated with (2.6).

LEMMA 2.3. *Regard* $L = \Delta + pU^{p-1}$ *as an operator defined on* $\mathrm{Dom}(L) = W^{2,r}(\mathbb{R}^n)$, *where* $n/(n-2) < r < +\infty$. *Then*

$$\mathrm{Ker}\, L = \mathrm{span} \left\{ \frac{\partial U}{\partial x_1}, ..., \frac{\partial U}{\partial x_n}, x \cdot \nabla U + \frac{n-2}{2} U \right\}$$

Proof. We use the method in the proof of Lemma 4.2 in [NT], that is, we first show that the dimension of Ker L is less than or equal to $n+1$ by using the eigenfunctions of the Laplace–Beltrami operator Δ_θ on S^{n-1}.

Suppose $\varphi \in \mathrm{Ker}\, L$, i.e., $\varphi \in W^{2,r}(\mathbb{R}^n)$ and φ satisfies

$$\Delta\varphi + pU^{p-1}\varphi = 0 \qquad \text{in } \mathbb{R}^n. \tag{2.12}$$

By the elliptic regularity theory, $\varphi \in C^\infty(\mathbb{R}^n)$. Furthermore, from the one-sided Harnack inequality (see Theorem 8.17 in [GT]), we have

$$|\varphi(x)| \leqslant C(n, r)\|\varphi\|_{L^r(B_1(x))} \to 0 \qquad \text{as} \quad x \to \infty, \tag{2.13}$$

where $B_1(x)$ is the unit ball centered at x. Now using the interior L^p estimates and the imbedding theorem, we obtain

$$|D\varphi(x)| \to 0 \qquad \text{as} \quad x \to \infty. \tag{2.14}$$

Let λ_i and ψ_i be the eigenvalues and eigenfunctions of $-\Delta_\theta$,

$$-\Delta_\theta \psi_i = \lambda_i \psi_i, \qquad 0 = \lambda_0 < \lambda_1 = \cdots = \lambda_n = (n-1) < \lambda_{n+1} < \cdots.$$

$\{\psi_i\}$ forms an orthonormal basis of $L^2(S^{n-1})$. Define

$$\varphi_i(t) = \int_{S^{n-1}} \varphi(t, \theta)\, \psi_i(\theta)\, d\theta, \qquad t = |x|.$$

Then

$$\varphi_i'' + \frac{n-1}{t}\varphi_i' + \left(pU^{p-1} - \frac{\lambda_i}{t^2}\right)\varphi_i = 0, \qquad \varphi_i'(0) = 0. \tag{2.15}$$

If $\varphi_i \not\equiv 0$, then by uniqueness, $\varphi_i(0) \neq 0$. Without loss of generality, assume $\varphi_i(0) > 0$. Then there exists $t_i \in (0, \infty]$ such that φ_i is positive on $[0, t_i)$, $\varphi_i(t_i) = 0$. Multiplying (2.15) by $U' t^{n-1}$ and integrating by parts on $[0, t_i)$, we obtain

$$t_i^{n-1}\varphi_i'(t_i)\, U'(t_i) + \int_0^{t_i}\left(U''' + \frac{n-1}{t}U'' + (U^p)'\right)\varphi_i t^{n-1}\, dt$$
$$- \lambda_i \int_0^{t_i} U'\varphi_i t^{n-3}\, dt = 0,$$

and hence,

$$t_i^{n-1}\varphi_i'(t_i)\; U'(t_i) + (n-1-\lambda_i)\int_0^{t_i} U'\varphi_i t^{n-3}\, dt = 0.$$

(When $t_i = \infty$, we use (2.13) and (2.14); in this case, the first term vanishes.) Thus $\lambda_i \leqslant n-1$ and consequently $i \leqslant n$. We have shown $\varphi_i \equiv 0$ if $i \geqslant n+1$. Therefore,

$$\varphi(t, \theta) = \varphi_0(t) + \sum_{i=1}^{n} \varphi_i(t)\, \psi_i(\theta),$$

which implies $\dim(\operatorname{Ker} L) \leqslant n+1$.

On the other hand, by (1.5), $\partial U/\partial x_i \in \operatorname{Ker} L$ (note $r > n/(n-2)$). Furthermore, since $U_\lambda(x) = \lambda^{(n-2)/2} U(\lambda x)$ is a solution of (1.4) for any $\lambda > 0$,

$$\left.\frac{\partial U_\lambda}{\partial \lambda}\right|_{\lambda=1} = x \cdot \nabla U + \frac{n-2}{2} U \text{ also belongs to Ker } L.$$

This completes the proof of Lemma 2.3. ▮

Let $X = \operatorname{span}\{\partial U/\partial x_1, ..., \partial U/\partial x_n, x \cdot \nabla U + U(n-2)/2\}$. Then $X \subset L^s(\mathbb{R}^n)$ for any $s > n/(n-2)$. So when $1 < t < n/2$. $\varphi u \in L^1(\mathbb{R}^n)$ for any $\varphi \in X$ and $u \in L^t(\mathbb{R}^n)$. Define

$$Y_t = \left\{ u \in L^t(\mathbb{R}^n) \,\middle|\, \int_{\mathbb{R}^n} u\varphi \, dx = 0 \text{ for all } \varphi \in X \right\}.$$

Then

$$L^t(\mathbb{R}^n) = X \oplus Y_t \qquad \text{for any} \quad \frac{n}{n-2} < t < \frac{n}{2}. \tag{2.16}$$

The following result plays a crucial role in the proof of Proposition 2.2.

Lemma 2.4. *Suppose $n > 4$. For any $1 < q < n/4$, there exists a constant $C = C(q, n)$ such that*

$$\|u\|_{W^{2,r}(\mathbb{R}^n)} \leqslant C(\|Lu\|_{L^q} + \|Lu\|_{L^r}), \tag{2.17}$$

for $u \in Y_r \cap W^{2,r}(\mathbb{R}^n)$ with $Lu \in L^q(\mathbb{R}^n)$ where $1/q - 2/n = 1/r$.

Proof. We claim that

$$\|u\|_{L^r} \leqslant C(q, n)(\|Lu\|_{L^q} + \|Lu\|_{L^r}) \tag{2.18}$$

for all $u \in Y_r \cap W^{2,r}(\mathbb{R}^n)$ with $Lu \in L^q(\mathbb{R}^n)$. Once this claim is shown, (2.17) follows from Corollary 9.10 of [GT]. To show (2.18), we argue by contradiction. So assume there exists a sequence $\{u_i\} \subset Y_r \cap W^{2,r}(\mathbb{R}^n)$ such that

$$\|u_i\|_{L^r} = 1, \qquad \|f_i\|_{L^q} + \|f_i\|_{L^r} \to 0 \qquad \text{as} \quad i \to \infty, \tag{2.19}$$

where $f_i = \Delta u_i + pU^{p-1} u_i$. This and Corollary 9.10 of [GT] imply that $\{u_i\}$ is bounded in $W^{2,r}(\mathbb{R}^n)$. Consequently, there exists $u_\infty \in W^{2,r}$ such that, after passing to a subsequence, $u_i \to u_\infty$ weakly in $W^{2,r}(\mathbb{R}^n)$ and strongly in $L^r_{\text{loc}}(\mathbb{R}^n)$. Let Γ be the fundamental solution of Δ in $\mathbb{R}^n$. Then

$$u_i + T(u_i) = \Gamma * f_i, \tag{2.20}$$

where $T(u_i) = \Gamma * (pU^{p-1}u_i)$. By virtue of the Hardy–Littlewood–Sobolev inequality ([HL] and [So]), we have

$$\|\Gamma * f\|_{L^r} \leqslant C(n, q) \|f\|_{L^q} \qquad \text{for} \quad f \in L^q(\mathbb{R}^n). \tag{2.21}$$

Therefore, by (2.19), we obtain

$$\Gamma * f_i \to 0 \text{ in } L^r(\mathbb{R}^n) \text{ as } i \to \infty . \tag{2.22}$$

We claim $\{T(u_i)\}$ is Cauchy in $L^r(\mathbb{R}^n)$. Let χ_R be the characteristic function of the ball $B_R(0)$ centered at the origin with radius R. Define $v_i^R = \chi_R u_i$, $w_i^R = (1 - \chi_R)\, u_i$. Then for fixed $R > 0$, by (2.21),

$$\begin{aligned} \|T(v_i^R - v_\ell^R)\|_{L^r(\mathbb{R}^n)} &\leqslant C(n, q)\, \|U^{p-1}(v_i^R - v_\ell^R)\|_{L^q(\mathbb{R}^n)} \\ &\leqslant C(n, q, R)\, \|v_i^R - v_\ell^R\|_{L^r(B_R(0))}. \end{aligned}$$

This and the fact that $\{u_i\}$ is Cauchy in $L^r_{\text{loc}}(\mathbb{R}^n)$ imply that $\{T(v_i^R)\}$ is Cauchy in $L^r(\mathbb{R}^n)$. By virtue of (2.21) and Hölder's inequality, we have

$$\begin{aligned} \|T(w_i^R - w_\ell^R)\|_{L^r} &\leqslant C(n, q)\, \|U^{p-1}(1 - \chi_R)(u_i - u_\ell)\|_{L^q} \\ &\leqslant C(n, q, R)\, \|u_i - u_\ell\|_{L^r} \left(\int_{|x| \geqslant R} U^{2n/(n-2)}\, dx \right)^{2/n} \\ &\to 0 \qquad \text{as} \quad R \to \infty \end{aligned}$$

uniformly with respect to i, ℓ, where we have used the facts that $rq/(r - q) = n/2$ and $\{u_i\}$ is bounded in $L^r(\mathbb{R}^n)$. This, (2.20) and (2.22) imply that $\{u_k\}$ is Cauchy in $L^r(\mathbb{R}^n)$. Consequently, $\|u_\infty\|_{L^r} = 1$, $u_\infty \in Y_r$ (note since $q < n/4$, $r < n/2$), and $u_\infty + T(u_\infty) = 0$, i.e.,

$$\Delta u_\infty + pU^{p-1}u_\infty = 0 \qquad \text{in } \mathbb{R}^n.$$

Since $u_\infty \in W^{2,r}(\mathbb{R}^n)$ and $r > n/(n-2)$, then by Lemma 2.3, $u_\infty \in X$. But u_∞ also belongs to Y_r. So $u_\infty \equiv 0$ which contradicts the fact that $\|u_\infty\|_{L^r} = 1$. Now (2.18) and hence Lemma 2.4 are proved. ∎

Since u_ε decays exponentially in x for each fixed $\varepsilon > 0$ (see, e.g. Lemma 3.5 in [PW]), by the L^p estimate we have that $u_\varepsilon \in W^{2,s}(\mathbb{R}^n)$ for $s > 1$. Thus $w_j \in W^{2,s}$ for $s > n/(n-2)$ and hence by (2.16) we can write

$$w_j = \sum_{i=1}^{n+1} a_{ij} e_i + z_j \qquad j = 1, 2, ..., \tag{2.23}$$

where a_{ij}'s are constants, $e_i = \partial U/\partial x_i$, $i=1, ..., n$, $e_{n+1} = x \cdot \nabla U + U(n-2)/2$, and $z_j \in Y_r \cap W^{2,r}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ for $n/(n-2) < r < n/2$. Furthermore, from (2.6) we have

$$\Delta z_j + pU^{p-1} z_j - k_j v_j + F(w_j) = 0 \qquad \text{in } \mathbb{R}^n. \tag{2.24}$$

To finish the proof of Proposition 2.2, following the main lines in [NT], first we show that a_{ij} and $\|z_j\|_{W^{2,s}(\mathbb{R}^n)}$ $(s > n/(n-4))$ are bounded as $j \to \infty$ (Lemma 2.5); then we prove that $z_j \to z$ in $C^1_{\text{loc}}(\mathbb{R}^n)$, where z satisfies (2.7) and $z \in W^{2,s}(\mathbb{R}^n)$ for $s > n/(n-4)$ (Lemma 2.6); finally, after showing that $a_{ij} \to 0$ for $1 \leqslant i \leqslant n$ and $a_{(n+1)j} \to -2z(0)/(n-2)$ in Lemma 2.7, we prove $w_j \to w$ in $L^\infty(\mathbb{R}^n)$ as $j \to \infty$, where $w = z - 2z(0)(x \cdot \nabla U + U(n-2)/2)/(n-2)$ (Lemma 2.8).

LEMMA 2.5. *Suppose $n > 6$. Let $M_j = \max\{|a_{1j}|, |a_{2j}|, ..., |a_{(n+1)j}|\}$. Then M_j and $\|z_j\|_{W^{2,s}(\mathbb{R}^n)}$ are bounded as $j \to \infty$ for every fixed $s > n/(n-4)$.*

Proof. As in [NT], we argue by contradiction. Assume, without loss of generality, that $M_j \to \infty$ and

$$\frac{1}{M_j} (a_{1j}, a_{2j}, ..., a_{(n+1)j}) \to (b_1, b_2, ..., b_{n+1}) \neq 0$$

as $j \to \infty$. From (2.24) it follows that

$$L\left(\frac{z_j}{M_j}\right) - \frac{1}{M_j} k_j v_j + \frac{1}{M_j} F(w_j) = 0 \qquad \text{in } \mathbb{R}^n. \tag{2.25}$$

Observe that

$$\begin{aligned} |\mu_j^2 F(w_j)| &\leqslant |U^{p-\varepsilon_j} - U^p| \\ &\quad + |(U + \mu_j^2 w_j)^{p-\varepsilon_j} - U^{p-\varepsilon_j} - (p - \varepsilon_j)\, \mu_j^2\, U^{p-1-\varepsilon_j} w_j| \\ &\quad + |p\mu_j^2\, U^{p-1} w_j - (p-\varepsilon_j)\, \mu_j^2 U^{p-1-\varepsilon_j} w_j| \\ &= I_1 + I_2 + I_3. \end{aligned}$$

It is easy to see that

$$I_3 \leqslant c\varepsilon_j \mu_j^2\, |w_j|\, U^{p-1-\varepsilon_j}(|\ell n U| + 1), \qquad I_1 \leqslant c\varepsilon_j\, U^{p-\varepsilon_j}(|\ell n U| + 1). \tag{2.26}$$

Define $f(t) = (U + t\mu_j^2 w_j)^{p-\varepsilon_j}$. Since $f(1) = f(0) + f'(0) + \int_0^1 tf''(1-t)\,dt$, we have

$$\begin{aligned} I_2 &= |f(1) - f(0) - f'(0)| \\ &\leqslant \int_0^1 t\,|f''(1-t)|\,dt \\ &\leqslant c \int_0^1 t(U + (1-t)\,\mu_j^2 w_j)^{p-2-\varepsilon_j}\,\mu_j^4 w_j^2\,dt \\ &= c\mu_j^2\,|w_j|\,|v_j - U| \int_0^1 t(tU + (1-t)\,v_j)^{p-2-\varepsilon_j}\,dt \\ &\leqslant c\mu_j^2\,|w_j|\,|v_j - U| \int_0^1 t(tU)^{p-2-\varepsilon_j}\,dt. \end{aligned}$$

Thus

$$I_2 \leqslant c\mu_j^2\,|w_j|\,|v_j - U|\,U^{p-2-\varepsilon_j}. \tag{2.27}$$

From this, (2.26) and (2.5), we obtain

$$|F(w_j)| \leqslant c[\,U^{p-\varepsilon_j}(|\ell n\,U| + 1) + |w_j|\,|v_j - U|\,U^{p-2-\varepsilon_j}]. \tag{2.28}$$

Choose an arbitrary $q \in (n/(n-2), n/4)$. (Since $n > 6$, such q exists—this is the place we need $n > 6$.) Let $1/r = 1/q - 2/n$. Then $n/(n-4) < r < n/2$ and hence $z_j \in Y_r \cap W^{2,r}(\mathbb{R}^n)$. Thus we can apply Lemma 2.4 to obtain that

$$\begin{aligned} &\|z_j/M_j\|_{W^{2,r}(\mathbb{R}^n)} \\ &\quad\leqslant \frac{c}{M_j}\,(\|k_j v_j\|_{L^q} + \|k_j v_j\|_{L^r} + \|F(w_j)\|_{L^q} + \|F(w_j)\|_{L^r}) \\ &\quad\leqslant \frac{c}{M_j}\,(1 + \|F(w_j)\|_{L^q} + \|F(w_j)\|_{L^r}), \end{aligned} \tag{2.29}$$

since $v_j \leqslant cU$ and k is bounded. By virtue of (2.28) and Hölder's inequality, we have

$$\begin{aligned} \frac{1}{M_j}\,\|F(w_j)\|_{L^q} &\leqslant \frac{c}{M_j}\,(\|\,U^{p-\varepsilon_j}(|\ell n\,U| + 1))\|_{L^q} + \|w_j U^{p-2-\varepsilon_j}(v_j - U)\|_{L^q}) \\ &\leqslant \frac{c}{M_j}\,(1 + \|w_j\|_{L^r}\,\|U^{p-2-\varepsilon_j}(v_j - U)\|_{L^{n/2}}) \\ &= \frac{c}{M_j}\,(1 + o(1)\,\|w_j\|_{L^r}) \\ &= o(1) + o(1)\,\|z_j/M_j\|_{L^r}. \end{aligned} \tag{2.30}$$

In the third inequality, we have used the fact that

$$\| U^{p-2-\varepsilon_j}(v_j - U)\|_{L^{n/2}} = o(1)$$

which follows from the Dominated Convergence Theorem; in the last step, we have used (2.23). By (2.28) and (2.23) again, we have

$$\begin{aligned}\frac{1}{M_j}\|F(w_j)\|_{L^r} &\leqslant \frac{c}{M_j}\left(\| U^{p-\varepsilon_j}(|\ell n\, U|+1)\|_{L^r} + \|w_j\, U^{p-2-\varepsilon_j}(v_j - U)\|_{L^r}\right)\\ &\leqslant \frac{c}{M_j}\left(1 + o(1)\,\|w_j\|_{L^r}\right)\\ &= o(1) + o(1)\,\|z_j/M_j\|_{L^r}. \qquad (2.31)\end{aligned}$$

Now (2.29)–(2.31) imply that

$$\|z_j/M_j\|_{W^{2,r}(\mathbb{R}^n)} = o(1) \qquad (2.32)$$

for every fixed $r \in (n/(n-4), n/2)$. By the imbedding theorem,

$$\|z_j/M_j\|_{L^{r_1}} = o(1) \qquad (2.33)$$

where $1/r_1 = 1/r - 2/n$. By choosing r close to $n/2$, r_1 can be arbitrarily large. From (2.25), (2.28) and (2.23), it follows that

$$\begin{aligned}\left|L\left(\frac{z_j}{M_j}\right)\right| &\leqslant \frac{c}{M_j}\left(U + U^{p-\varepsilon_j}(|\ell n\, U|+1) + |w_j|\, U^{p-2-\varepsilon_j}\,|v_j - U|\right)\\ &\leqslant o(1)\left[U + U^{p-\varepsilon_j}(|\ell n\, U|+1) + U^{p-1-\varepsilon_j}\sum_{i=1}^{n+1}|e_i|\right] + o(1)\left|\frac{z_j}{M_j}\right|.\end{aligned}$$

(At the last step, we have also used (1.6).) In view of this, (2.33) and the L^p estimate (Corollary 9.10 in [GT]), we have

$$\begin{aligned}\|z_j/M_j\|_{W^{2,r_1}(\mathbb{R}^n)} &\leqslant c\left(\|L(z_j/M_j)\|_{L^{r_1}} + \|z_j/M_j\|_{L^{r_1}}\right)\\ &= o(1) + o(1)\,\|z_j/M_j\|_{L^{r_1}}.\end{aligned}$$

Hence

$$\|z_j/M_j\|_{W^{2,r_1}(\mathbb{R}^n)} = o(1)$$

which, by the imbedding theorem, implies that

$$\frac{z_j}{M_j} \to 0 \quad \text{in} \quad L^\infty(\mathbb{R}^n) \quad \text{and} \quad C^1_{\text{loc}}(\mathbb{R}^n). \tag{2.34}$$

Recall that $v_j(0) = U(0)$ and that both v_j and U achieve their maximum at the origin. By the definitions of w_j and M_j and in view of (2.34), we have

$$0 = w_j(0) = M_j \left(\sum_{i=1}^{n+1} b_i e_i(0) + o(1) \right),$$

$$0 = \nabla w_j(0) = M_j \left(\sum_{i=1}^{n+1} b_i \nabla e_i(0) + o(1) \right). \tag{2.35}$$

By direct calculation, one finds that $e_i(0) = \partial U/\partial x_i = 0$, $i = 1, ..., n$, $e_{n+1}(0) = (n-2)/2$, $\nabla e_{n+1}(0) = 0$, and that $\nabla e_1(0), ..., \nabla e_n(0)$ are linearly independent. These observations and (2.35) imply that $(b_1, ..., b_{n+1}) = 0$, which is impossible.

We have proved the boundedness of M_j as $j \to \infty$. The remaining part of Lemma 2.5 can be proved similarly. ▌

LEMMA 2.6. *Suppose $n > 6$. Then there exists a function z so that $z_j \to z$ in $C^1_{\text{loc}}(\mathbb{R}^n)$, z satisfies (2.7), $z \in W^{2,s}(\mathbb{R}^n)$ for $s > n/(n-4)$, and that z is radial.*

Remark 2.1. From the following proof, we shall see that $z \in Y_s$ for $n/(n-4) < s < n/2$. Thus z is the unique solution of (2.7) in the class $Y_s \cap W^{2,s}$ for $n/(n-4) < s < n/2$.

Proof. By Lemma 2.5 and the imbedding theorem, every subsequence of $\{z_j\}$ has a subsequence $\{z_{j_k}\}$ so that

$$z_{j_k} \text{ converges to some } z \text{ weakly in } W^{2,s}(\mathbb{R}^n) \left(s > \frac{n}{n-4} \right)$$
$$\text{and strongly in } C^1_{\text{loc}}(\mathbb{R}^n). \tag{2.36}$$

Observe that

$$\begin{aligned} &|F(w_j) + C(n,k)\, U^p \ell n\, U| \\ &\quad = |(U^{p-\varepsilon_j} - U^p)/\mu_j^2 + C(n,k)\, U^p \ell n\, U| + (I_2 + I_3)/\mu_j^2 \\ &\quad = I_1' + (I_2 + I_3)/\mu_j^2, \end{aligned}$$

where I_2 and I_3 are defined in the proof of Lemma 2.5. Using Lemma 2.5 and the imbedding theorem again, we have that $\|w_j\|_{L^\infty}$ is bounded. Thus by (2.26) and (2.27), we see that

$$(I_2+I_3)/\mu_j^2 \leqslant C(\varepsilon_j\,|w_j|\,U^{p-1-\varepsilon_j}(|\ell n\,U|+1)+|w_j|\,|v_j-U|\,U^{p-2-\varepsilon_j}) \\ =o(1). \tag{2.37}$$

On the other hand, by Taylor's Theorem and (2.5), it is easily seen that

$$I_1' \leqslant o(1)\,U^{p-\varepsilon_j}\,\ell n^2\,U. \tag{2.38}$$

Thus

$$F(w_j) \to -C(n,k)\,U^p\,\ell n\,U \quad \text{in} \quad L^\infty(\mathbb{R}^n) \qquad \text{as} \quad j\to\infty.$$

Now we see that z is a weak $W^{2,s}(\mathbb{R}^n)$ $(s>n/(n-4))$ and hence a classical solution of (2.7).

$\forall\,\varphi\in X$, it is easily seen that $\langle\varphi, f\rangle=\int_{\mathbb{R}^n}\varphi f\,dx$, $f\in L^r(\mathbb{R}^n)$, is a bounded linear functional on $L^r(\mathbb{R}^n)$, and hence it is also one on $W^{2,r}(\mathbb{R}^n)$ for every $1<r<n/2$. Since $z_{j_k}\in Y_r$ for $n/(n-2)<r<n/2$, we have $\langle\varphi, z_{j_k}\rangle=0$ for $\varphi\in X$. So by (2.36), $\langle\varphi, z\rangle=0$ for $\varphi\in X$. Thus z belongs not only to $W^{2,s}(\mathbb{R}^n)$ but also to Y_s for $n/(n-4)<s<n/2$. Since (2.7) has at most one such solution, the whole sequence $z_j\to z$ weakly in $W^{2,s}(\mathbb{R}^n)$ and strongly in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$, where z satisfies (2.7).

To show z is radial, let A be a rotation in $\mathbb{R}^n$. Define $z_A(x)=z(Ax)$. It is easy to see that z_A still belongs to $Y_s\cap W^{2,s}(\mathbb{R}^n)$ for $n/(n-4)<s<n/2$. On the other hand, since (2,7) is invariant under rotation, z_A-z belongs to X. Consequently, $z_A-z\equiv 0$. ∎

Lemma 2.7. *When* $n>6$, $M_j'=\{|a_{1j}|, ..., |a_{nj}|\}\to 0$ *and* $a_{(n+1)j}\to -2z(0)/(n-2)$ *as* $j\to\infty$.

Proof. Observe that the following analogue of (2.35) holds:

$$\begin{aligned} 0&=\frac{n-2}{2}\,a_{(n+1)j}+z_j(0) \\ 0&=\sum_{i=0}^{n} a_{ij}\nabla e_i(0)+\nabla z_j(0) \end{aligned} \tag{2.39}$$

On the other hand, since z is radial and C^1 smooth, $\nabla z(0)=0$. Combining this with (2.39) and the fact that $z_j\to z$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$ (Lemma 2.6), we have the conclusion of Lemma 2.7. ∎

Remark 2.2. Since z is a radial solution of (2.7), by uniqueness of solutions to IVP for ODE's, $z(0)\neq 0$.

Finally we are at the point of finishing the proof of Proposition 2.2.

LEMMA 2.8. *When* $n>6$, $w_j \to w$ *in* $L^\infty(\mathbb{R}^n)$ *as* $j\to\infty$, *where*

$$w = z - \frac{2z(0)}{n-2}\left(x\cdot\nabla U + \frac{n-2}{2}\,U\right),$$

and hence w *satisfies* (2.7), $w\in W^{2,s}(\mathbb{R}^n)$ *for* $s>n/(n-4)$.

Proof. In view of Lemma 2.7, we just need to show that $z_j\to z$ in L^∞ as $j\to\infty$.

Since z satisfies (2.7), by (2.24), (2.37) and (2.38), we have

$$\begin{aligned} |L(z_j - z)| &\leqslant |k_j v_j - k(x_0)\,U| + |F(w_j) + C(n,k)\,U^p\,\ell n\,U| \\ &\leqslant |k_j v_j - k(x_0)\,U| + C(\varepsilon_j\,|w_j|\,U^{p-1-\varepsilon_j}\,(|\ell n\,U|+1) \\ &\quad + |w_j|\,|v_j - U|\,U^{p-2-\varepsilon_j} + o(1)\,U^{p-\varepsilon_j}\,\ell n^2\,U). \end{aligned} \tag{2.40}$$

Applying Lemma 2.4 with $n/(n-2)<q<n/4$ and $1/r=1/q-2/n$, we have

$$\begin{aligned} \|z_j - z\|_{W^{2,r}(\mathbb{R}^n)} \leqslant C(&\|k_j v_j - k(x_0)\,U\|_{L^q} + \|k_j v_j - k(x_0)\,U\|_{L^r} \\ &+ \|F(w_j) + C(n,k)\,U^p\,\ell n\,U\|_{L^q} \\ &+ \|F(w_j) + C(n,k)\,U^p\,\ell n\,U\|_{L^r}). \end{aligned}$$

The first two terms on the right hand side are $o(1)$ as $j\to\infty$, which follows from the Dominated Convergence Theorem. Arguing as in (2.30) and (2.31) and using (2.40), we obtain

$$\begin{aligned} &\|F(w_j) + C(n,k)\,U^p\,\ell n\,U\|_{L^q} + \|F(w_j) + C(n,k)\,U^p\,\ell n\,U\|_{L^r} \\ &\quad\leqslant C(\varepsilon_j\,\|w_j\|_{L^r}\,\|U^{p-1-\varepsilon_j}\,\ell n\,U\|_{L^{n/2}} + \|w_j\|_{L^r}\,\|U^{p-2-\varepsilon_j}(v_j - U)\|_{L^{n/2}} \\ &\qquad + \varepsilon_j\,\|w_j\|_{L^r} + \|w_j\|_{L^r}\,\|(v_j - U)\,U^{p-2-\varepsilon_j}\|_{L^\infty} + o(1)) \\ &\quad= o(1). \end{aligned}$$

Here at the last step, we have used Lemma 2.5. Thus we see

$$\|z_j - z\|_{W^{2,r}(\mathbb{R}^n)} = o(1)$$

for every fixed $n/(n-4)<r<n/2$. From this, by using the imbedding theorem and the L^p estimates as in the proof of Lemma 2.5 (the part from (2.33) to (2.34)), we obtain

$$\|z_j - z\|_{W^{2,r_1}(\mathbb{R}^n)} = o(1)$$

for every fixed large r_1. Now the imbedding theorem implies that $z_j\to z$ in $L^\infty(\mathbb{R}^n)$ as $j\to\infty$.

3. Concluding Remarks

The main purpose of this section is to discuss the possibility of removing the conditions on x_ε and "$-k \in E(\rho, \mathbb{R}^n)$" from Theorem 1.1 and Theorem A. However, we start with what is claimed in Remark 1.1.

Lemma 3.1. *Suppose Condition* (**K**) *holds, but with the condition on* $k + \frac{1}{2} x \cdot \nabla k$ *replaced by the weaker assumption that* k *be bounded. Let* u_ε *be an arbitrary positive ground state of* (1.1). *Then the* L^∞ *norm of* u_ε *blows up as* $\varepsilon \to 0$.

Proof. We argue by contradiction. Assume that the L^∞ norm of $u_j \equiv u_{\varepsilon_j}$ is bounded by M for $j \geqslant 1$. Let x_j be a maximum point of u_j. Since we are not assuming $u_j \in E(\rho, \mathbb{R}^n)$ and $x_j \in C(\rho)$, we do not know if x_j is bounded. By Lemma 2.3 of [PW], we have $u_j(x_j) \geqslant \alpha_0 > 0$ for $j \geqslant 1$. Define $w_j(x) = u_j(x_j + x)$. Then

$$\Delta w_j - k(x_j + x) w_j + w_j^{p - \varepsilon_j} = 0 \quad \text{in } \mathbb{R}^n, \qquad \alpha_0 \leqslant w_j(0) = \max w_j \leqslant M$$

Since

$$\int_{\mathbb{R}^n} (|\nabla u_j|^2 + k(x) u_j^2)\, dx = \int_{\mathbb{R}^n} u_j^{p+1-\varepsilon_j}\, dx \to S^{n/2} \tag{3.1}$$

as $\varepsilon_j \to 0$ (see Corollary 2.6 of [PW]), u_j and hence w_j are bounded in $H^1(\mathbb{R}^n)$. Consequently, we have that after passing to a subsequence,

$$w_j \to w_0 \text{ weakly in } H^1(\mathbb{R}^n).$$

On the other hand, since $w_j \leqslant M$ and k is bounded, by the L^p interior estimates and the imbedding theorem, we have $w_j \to w_0$ in $C^{1+\alpha}_{\text{loc}}(\mathbb{R}^n)$. Thus $w_0(0) \geqslant \alpha_0 > 0$, $w_0 \not\equiv 0$.

Case 1. $\{x_j\}$ is bounded. W.L.O.G., assume $x_j \to x_0$. Then w_0 is a nontrivial and nonnegative classical solution of

$$\Delta w_0 - k(x_0 + x) w_0 + w_0^p = 0 \qquad \text{in } \mathbb{R}^n. \tag{3.2}$$

By the strong maximum principle, w_0 is positive on $\mathbb{R}^n$. Thus by (3.2), we have

$$\int_{\mathbb{R}^n} |\nabla w_0|^2\, dx < \int_{\mathbb{R}^n} w_0^{p+1}\, dx. \tag{3.3}$$

Now by the definition of the Sobolev constant S and by (3.1), we deduce

$$S \leqslant \frac{\int_{\mathbb{R}^n} |\nabla w_0|^2}{(\int_{\mathbb{R}^n} w_0^{p+1}\, dx)^{2/(p+1)}} < \left(\int_{\mathbb{R}^n} w_0^{p+1}\, dx\right)^{2/n}$$
$$\leqslant \liminf_{j\to\infty} \left(\int_{\mathbb{R}^n} w_j^{p+1-\varepsilon_j}\, dx\right)^{2/n}$$
$$= S.$$

We reach a contradiction.

Case 2. $\{x_j\}$ is unbounded. W.L.O.G., assume $x_j \to \infty$. Then w_0 satisfies that

$$\Delta w_0 - k_0 w_0 + w_0^p \geqslant 0 \text{ in the sense of } H^{-1}(\mathbb{R}^n),$$

since $k(x) \geqslant k_0 > 0$ near infinity. This will lead to (3.3) and hence to a contradiction again. ∎

Next, we discuss the possibility of removing the condition on x_ε in Theorem 1.1 and Theorem A. For an arbitrary global maximum point x_ε of an arbitrary ground state u_ε of (1.1), our worry is that x_ε may go off to infinity as ε shrinks to zero. Indeed, this may happen when k is independent of one component of x. We shall assume that

$$k \text{ is not independent of any component } x_i \text{ of } x = (x_1, ..., x_n). \quad (3.4)$$

This condition, together with Condition (**K**), implies that any maximum point of any solution of (1.1) that decays at infinity must be contained in the cube $C(\rho)$ (centered at the origin with length 2ρ). More precisely, the following is true.

LEMMA 3.2. *Let k be a nonnegative function defined on $\mathbb{R}^n$ with $k(x) \geqslant k_0 > 0$ at $x = \infty$. Suppose $-k \in E(\rho, \mathbb{R}^n)$ for some $\rho \geqslant 0$, and that (3.4) hold. Then any solution u of*

$$\Delta u - k(x) u + u^q = 0, \qquad u > 0 \quad \text{in} \quad \mathbb{R}^n, \qquad u(\infty) = 0, \qquad (3.5)$$

($q > 1$) satisfies that

$$\frac{\partial u}{\partial x_i} < 0 \qquad \text{for} \quad x_i > \rho; \qquad \frac{\partial u}{\partial x_i} > 0 \qquad \text{for} \quad x_i < -\rho.$$

In particular, all maximum points of u are contained in the cube $C(\rho)$.

The proof of this result is a slight modification of the one in Li-Ni [LN]. It will be given at the end of this section.

From this lemma, we immediately have

THEOREM 3.3. *Suppose that Condition* (**K**) *and* (3.4) *hold. Let* u_ε *be an arbitrary positive ground state of* (1.1), *and* x_ε *be an arbitrary maximum point of* u_ε. *Then* $x_\varepsilon \in C(\rho)$ *and the conclusions of Theorem* A *and Theorem* 1.1 *hold.* (*For Theorem* 1.1 *to hold, we need* $n > 6$.)

Now, we discuss the possibility of removing "$-k \in E(\rho, \mathbb{R}^n)$" in Condition (**K**). This "geometric condition" is not directly used in the previous part of this paper. It is only used in [PW] to show the existence of a ground state u_ε which also belongs to $E(\rho, \mathbb{R}^n)$ (so it has a maximum point in $C(\rho)$). Recently, Rabinowitz proved, among other things, the existence of a positive ground u_ε of (1.1) for each $0 < \varepsilon < p-1$, under the condition

$$k \text{ is a nonnegatrive } C^1 \text{ function defined in } \mathbb{R}^n \text{ satisfying}$$
$$\lim_{x \to \infty} k(x) = \sup_{x \in \mathbb{R}^n} k(x) > \inf_{x \in \mathbb{R}^n} k(x) \tag{3.6}$$

(see Theorem 4.27 of [R]). Actually "$\inf k > 0$" is assumed in [R]. But as can be checked, his arguments go through without this condition.

THEOREM 3.4. *Suppose that* (3.6) *holds, and that* $k + \frac{1}{2} x \cdot \nabla k$ *is bounded. Let* u_ε *and* x_ε *be as given in Theorem* 3.3. *Then* x_ε *remains bounded as* $\varepsilon \to 0$ *and the conclusions of Theorem* A *and Theorem* 1.1 *hold.* ($n > 6$ *is needed for Theorem* 1.1.)

Proof. We just need to show the boundedness of x_ε as $\varepsilon \to 0$. We argue by contradiction. So, W.L.O.G., assume $x_\varepsilon \to \infty$ $\varepsilon \to 0$. Define μ_ε and v_ε as before.

Claim. There exists a constant C independent of small ε such that

$$v_\varepsilon \leqslant CU \qquad \text{in } \mathbb{R}^n. \tag{3.7}$$

(In the case that x_ε is bounded, this is Lemma 3.6 in [PW].)

We put off the proof of this claim and use it to reach the desired conclusion now. By this claim and by (3.16) in [PW], we have

$$\varepsilon = O(\mu_\varepsilon^2). \tag{3.8}$$

(Note in the argument leading to (3.16) in [PW], we just need the boundedness of $k + \frac{1}{2} x \cdot \nabla k$ and the exponential decay of u_ε and $|\nabla u_\varepsilon|$ for each fixed ε.) From (3.6) and (3.8), there exists a sequence $\varepsilon_j \to 0$ and constants $\bar{c} \geqslant 0$ and $\bar{k}$ so that

$$\varepsilon_j = \bar{c}\mu_j^2 + o(\mu_j^2), \quad \lim_{j \to \infty} k(x_{\varepsilon_j}) = \bar{k} > \inf k, \tag{3.9}$$

where $\mu_j = \mu_{\varepsilon_j}$. The first part of (3.9) is an analogue of (2.5). In the present case, Lemma 2.1 with $C(n, k)$ replaced by $\bar{c}$ holds (by modifying the proof in the obvious way); Proposition 2.2 with $k(x_0)$ in (2.7) replaced by $\bar{k}$ also holds by almost the same proof. (Note when proving that z satisfies the modified version of (2.7) in the proof of Lemma 2.6, we can use the uniform continuity of k on $\mathbb{R}^n$.) Now as in the proof of Theorem 1.1, we are led to $\bar{k} \leqslant \inf k$, which contradicts (3.9). The proof of Theorem 3.4 is complete except we now have to show (3.7). To this end, first we observe that Lemma 3.2 in [PW] still remains true. Then the proof of Lemma 3.4 in [PW] implies that for any $\delta > 0$, there exists a small $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then

$$\int_{|x - x_\varepsilon| \geqslant 1/2} u_\varepsilon^{p+1}\, dx \leqslant \delta.$$

Using this and the one-sided Harnack inequality (Lemma 2.7 in [PW]), we have

$$u_\varepsilon(x) \leqslant \delta \qquad \text{for} \quad |x - x_\varepsilon| \geqslant 1 \text{ and small } \varepsilon. \tag{3.10}$$

Recall $k(x) \geqslant k_0 > 0$ for $|x|$ large, say, $|x| \geqslant R$. Choose $k_1 \in (0, k_0)$. Suppose δ in (3.10) is chosen so small that

$$g_\varepsilon(x) \equiv (k_1 - k(x))\, u_\varepsilon(x) + u_\varepsilon^{p-\varepsilon}(x) \leqslant 0 \tag{3.11}$$

for x satisfying both $|x - x_\varepsilon| \geqslant 1$ and $|x| \geqslant R$, and for small ε. Since for each fixed ε, u_ε decays exponentially and satisfies

$$\Delta u_\varepsilon - k_1 u_\varepsilon + g_\varepsilon(x) = 0 \qquad \text{in } \mathbb{R}^n,$$

we have

$$u_\varepsilon(x) = \int_{\mathbb{R}^n} \Gamma_{k_1}(x - y)\, g_\varepsilon(y)\, dy,$$

where Γ_{k_1} is the fundamental solution of $-\Delta + k_1$. By (3.11),

$$u_\varepsilon(x) \leqslant \int_\Omega \Gamma_{k_1}(x - y)\, g_\varepsilon(y)\, dy + \int_{|y| \leqslant R} \Gamma_{k_1}(x - y)\, g_\varepsilon(y)\, dy = I_1 + I_2,$$

where $\Omega = \{ y \in \mathbb{R}^n \mid |y - x_\varepsilon| \leqslant 1, |y| \geqslant R \}$. By (4.2) of [GNN],

$$\Gamma_{k_1}(x) \leqslant C(n, k_1)\, \frac{\exp(-\sqrt{k_1}\, |x|)}{|x|^{n-2}}\, (1 + |x|)^{(n-3)/2}.$$

From this and (3.10), it is easy to see that for small ε,

$$I_2 \leqslant C \exp(-\sqrt{k_1}\,|x|), \qquad x \in \mathbb{R}^n.$$

On the other hand, if $|x - x_\varepsilon| \geqslant 2$,

$$\begin{aligned} I_1 &\leqslant \int_\Omega \Gamma_{k_1}(x-y)\, u_\varepsilon^{p-\varepsilon}(y)\, dy \\ &\leqslant \|u_\varepsilon\|_{L^{p+1-\varepsilon}}^{p-\varepsilon} \left(\int_\Omega (\Gamma_{k_1}(x-y))^{p+1-\varepsilon}\, dy \right)^{1/(p+1-\varepsilon)} && \text{(Hölder's inequality)} \\ &\leqslant C \left(\int_{|y-x_\varepsilon| \leqslant 1} (\Gamma_{k_1}(x-y))^{p+1-\varepsilon}\, dy \right)^{1/(p+1-\varepsilon)} && ((3.1)) \\ &\leqslant Ce^{-a\,|x-x_\varepsilon|} \end{aligned}$$

for some constant $a > 0$. Thus we have shown that for small ε,

$$u_\varepsilon(x) \leqslant I_1 + I_2 \leqslant Ce^{-a\,|x-x_\varepsilon|}, \qquad |x - x_\varepsilon| \geqslant 2 \tag{3.12}$$

which is an analogue of Lemma 3.5 in [PW]. Now (3.7) follows from almost the same proof of Lemma 3.6 in [PW] (whenever Lemma 3.5 is used there, we apply (3.12) above instead). ▮

Finally, we give

Proof of Lemma 3.2. We shall only prove $\partial u/\partial x_1 < 0$, $x_1 > \rho$, in detail. The proof for the other cases is similar and hence is omitted.

We use the "moving plane" method.

For any real number λ, set

$$\Sigma_\lambda = \{x = (x_1, ..., x_n) \mid x_1 < \lambda\}, \qquad T_\lambda = \{x = (x_1, ..., x_n) \mid x_1 = \lambda\}.$$

For any $x \in \mathbb{R}^n$, let x^λ be the reflection point of x about the hyperplane T_λ, i.e., $x^\lambda = (2\lambda - x_1, x_2, ..., x_n)$. Define $v_\lambda(x) = u(x) - u(x^\lambda)$ and

$$\Lambda = \left\{ \lambda' \geqslant \rho \mid v_\lambda > 0 \text{ in } \Sigma_\lambda,\ \frac{\partial v_\lambda}{\partial x_1} < 0 \text{ on } T_\lambda,\ \lambda \geqslant \lambda' \right\}.$$

Claim 1. Λ is nonempty. Since $k(x) \geqslant k_0 > 0$ at $x = \infty$ and $u(\infty) = 0$, there exists a large $\rho_1 > \rho$ such that

$$k(x) \geqslant k_0 \quad \text{on} \quad (C(\rho_1))^c \quad \text{and} \quad \max_{(C(\rho_1))^c} u < \left(\frac{1}{q} k_0 \right)^{1/(q-1)}.$$

We can also choose a large $\rho_2 > \rho_1$ such that

$$\min_{C(\rho_1)} u > \max_{(C(\rho_2))^c} u.$$

By (3.5), we have

$$\Delta v_\lambda(x) - k(x)\, u(x) + k(x^\lambda)\, u(x^\lambda) + u^q(x) - u^q(x^\lambda) = 0 \qquad \text{in } \mathbb{R}^n. \quad (3.13)$$

Since $-k \in E(\rho, \mathbb{R}^n)$, $k(x^\lambda) \geqslant k(x)$ for $\lambda \geqslant \rho$, $x \in \Sigma_\lambda$. So if $\lambda \geqslant \rho$, we have

$$\Delta v_\lambda(x) + (c(x) - k(x^\lambda))\, v_\lambda(x) \leqslant 0, \qquad x \in \Sigma_\lambda, \quad (3.14)$$

where $c(x) = (u^q(x) - u^q(x^\lambda))/(u(x) - u(x^\lambda))$, which is between $qu^{q-1}(x)$ and $qu^{q-1}(x^\lambda)$.

From our choices for ρ_1 and ρ_2, we see that for $\lambda \geqslant \rho_2$,

$$v_\lambda > 0 \quad \text{on } C(\rho_1), \qquad c(x) - k(x^\lambda) < 0,\ x \in \Sigma_\lambda \backslash C(\rho_1). \quad (3.15)$$

Note also that $v_\lambda \equiv 0$ on T_λ and $\lim_{x \to \infty} v_\lambda(x) = 0$. This and (3.15) enable us to apply the strong maximum principle to (3.14) on $\Sigma_\lambda \backslash C(\rho_1)$, to conclude that for $\lambda \geqslant \rho_2$, $v_\lambda > 0$ on $\Sigma_\lambda \backslash C(\rho_1)$ and hence on Σ_λ. Furthermore, by Hopf boundary point lemma (see [GT]), $\partial v_\lambda / \partial x_1 < 0$ on T_λ. Thus $\rho_2 \in \Lambda$ and Claim 1 is proved.

Let $\lambda_0 = \inf \Lambda$. We shall prove $\lambda_0 = \rho$. Once this is shown, the proof of Lemma 3.2 is complete.

Claim 2. $\lambda_0 \in \Lambda$ if $\lambda_0 > \rho$. By the definition of λ_0 and the continuity of u, $v_{\lambda_0} \geqslant 0$ on Σ_{λ_0}. Applying the strong maximum principle and the Hopf boundary point lemma, we have that either $v_{\lambda_0} \equiv 0$ in Σ_{λ_0}, or $v_{\lambda_0} > 0$ on Σ_{λ_0} and $\partial v_{\lambda_0} / \partial x_1 < 0$ on T_{λ_0}. If the latter occurs, then by the definition of Λ, Claim 2 is true; if the former occurs, by (3.13) we have

$$k(x^{\lambda_0}) \equiv k(x), \qquad x \in \Sigma_{\lambda_0}. \quad (3.16)$$

This implies that k is independent of x_1.

This is shown as follows. Since $-k \in E(\rho, \mathbb{R}^n)$, k is nondecreasing in $x_1 \geqslant \rho$. So if $\lambda_0 > \rho$ and (3.16) occurs, then k is independent of $x_1 \in [\rho, 2\lambda_0 - \rho]$. For $x = (x_1, ..., x_n)$ with $2\lambda_0 - \rho < x_1 \leqslant 3\lambda_0 - 2\rho$, we have

$$k(x) = k(x^{\lambda_0}) \leqslant k((x^{\lambda_0})^\rho) \leqslant k(x),$$

where $(x^{\lambda_0})^\rho$ stands for the reflection point of x^{λ_0} about T_ρ. Thus k is independent in $x_1 \in [\rho, 3\lambda_0 - 2\rho]$ (recall k nondecreasing in $x_1 \geqslant \rho$). Continuing this process, we have k is constant in $x_1 \in [\rho, \infty)$ and hence in $x_1 \in (-\infty, +\infty)$.

We have reached a contradiction to the assumption (3.4). Claim 2 is proved.

Now we show $\rho = \lambda_0$. We argue by contradiction, so assume $\lambda_0 > \rho$. By Claim 1 and Claim 2, $\lambda_0 \leqslant \rho_2$ and $\lambda_0 \in \Lambda$. In particular $\partial v_{\lambda_0}/\partial x_1 < 0$ on T_{λ_0}, i.e., $\partial u/\partial x_1 < 0$ on T_{λ_0}. So there exists a small $\varepsilon > 0$ such that

$$\frac{\partial u}{\partial x_1} < 0 \qquad \text{on} \quad C(\rho_2) \cap \{\lambda_0 - 2\varepsilon \leqslant x_1 \leqslant \lambda_0 + 2\varepsilon\}.$$

Thus for $\lambda \in [\lambda_0 - \varepsilon, \lambda_0]$,

$$u(x) > u(x^\lambda), \text{ i.e., } v_\lambda(x) > 0, \qquad x \in C(\rho_2) \cap \{\lambda_0 - 2\varepsilon \leqslant x_1 < \lambda\}.$$

On the other hand, since $v_\lambda(x) > 0$ in Σ_{λ_0}, by the continuity of u, there exists a small $0 < \delta < \varepsilon$ such that for $\lambda \in [\lambda_0 - \delta, \lambda_0]$,

$$v_\lambda(x) > 0 \qquad \text{on} \quad C(\rho_2) \cap \{x_1 \leqslant \lambda_0 - 2\varepsilon\}.$$

So now we have

$$v_\lambda(x) > 0 \qquad \text{on} \quad C(\rho_2) \cap \Sigma_\lambda, \qquad \lambda \in [\lambda_0 - \delta, \lambda_0]. \tag{3.17}$$

For $x \in \Sigma_\lambda \backslash C(\rho_2)$ and $\lambda \in [\lambda_0 - \delta, \lambda_0]$, both x and x^λ fall off $C(\rho_2)$ (recall $\rho_2 > \rho_1$). So by our choice for ρ_1 and the definition of $c(x)$, we have

$$c(x) - k(x^\lambda) < 0, \qquad x \in \Sigma_\lambda \backslash C(\rho_2), \quad \lambda \in [\lambda_0 - \delta, \lambda_0]. \tag{3.18}$$

Observe that $v_\lambda \geqslant 0$, $v_\lambda \not\equiv 0$ on the boundary of $\Sigma_\lambda \backslash C(\rho_2)$ and that $\lim_{x \to \infty} v_\lambda(x) = 0$. By using this and (3.18), we can apply the strong maximum principle to (3.14) on $\Sigma_\lambda \backslash C(\rho_2)$ to conclude that

$$v_\lambda > 0 \qquad \text{on} \quad \Sigma_\lambda \backslash C(\rho_2), \qquad \lambda \in [\lambda_0 - \delta, \lambda_0].$$

Combining this with (3.17), we see that v_λ is positive on whole Σ_λ, $\lambda \in [\lambda_0 - \delta, \lambda_0]$. Now once again, the Hopf boundary point lemma implies that

$$\frac{\partial v_\lambda}{\partial x_1} < 0 \qquad \text{on } T_\lambda, \qquad \lambda \in [\lambda_0 - \delta, \lambda_0].$$

We have thus shown $[\lambda_0 - \delta, \lambda_0] \subset \Lambda$, which contradicts the definition of λ_0. ▮

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References

[AP] F. Atkinson and L. Peletier, Elliptic equations with nearly critical growth, *J. Differential Equations* **70** (1987), 349–365.

[B] M. S. Berger, On the existence and structure of stationary states for a nonlinear Klein–Gordon equation, *J. Funct. Anal.* **9** (1972), 249–261.

[BP] H. Brezis and L. Peletier, Asymptotics for elliptic equations involving critical growth, *in* "Partial Differential Equations and Calculus of Variations," (F. Colombini, Eds.), Vol. 1, pp. 149–192, Birkhäuser, Basel, 1989.

[CGS] L. Caffarelli, B. Gidas, and J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth, *Comm. Pure Appl. Math.* **XLII** (1989), 271–297.

[CL] W. Chen and C. Li, Classification of solutions of some semilinear elliptic equations, *Duke Math. J.* **63** (1991), 615–622.

[DN] W.-Y. Ding and W.-M. Ni, On the existence of positive entire solutions of a semilinear elliptic equation, *Arch. Rational Mech. Anal.* **91** (1986), 283–308.

[GNN] B. Gidas, W.-M. Ni, and L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$, *Adv. Math. Suppl. Stud. (Math. Anal. Appl. Part A)* **7** (1981), 369–402.

[GT] D. Gilbarg and N. Trudinger, "Elliptic Partial Differential Equations of Second Order," 2nd ed., Springer, New York/Berlin, 1983.

[H] Z. C. Han, Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent, *Anal. Nonlinière* **8** (1991), 159–174.

[HL] G. Hardy and J. Littlewood, Some properties of fractional integrals, *Math. Z.* **27** (1928), 565–606.

[K] M. K. Kwong, Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^n$, *Arch. Rational Mech. Anal.* **105** (1989), 243–266.

[LN] Yi Li and W.-M. Ni, Radial symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$, *Comm. Partial Differential Equations* **18** (1993), 1043–1054.

[Ne] Z. Nehari, On a nonlinear differential equation arising in nuclear physics, *Proc. Roy. Irish Acad. Sect. A* **62** (1963), 117–135.

[NT] W.-M. Ni and I. Takagi, Locating the peaks of least–energy solutions to a semilinear Neumann problem, *Duke Math. J.* **70** (1993), 247–281.

[PW] X.-B. Pan and X. Wang, Blow-up behavior of ground states of semilinear elliptic equations in $\mathbb{R}^n$ involving critical Sobolev exponents, *J. Differential Equations* **99** (1992), 78–107.

[R] P. H. Rabinowitz, On a class of nonlinear Schrödinger equations, *Z. Angew. Math. Phys.* **43** (1992), 270–291.

[Re] O. Rey, Proof of two conjectures of H. Brezis and L. A. Peletier, *Manuscr. Math.* **65** (1989), 19–37.

[So] S. Sobolev, On a theorem of functional analysis, *AMS Transl. Ser.* **2**, No. 34 (1963), 39–68.

[S] W. Strauss, Existence of solitary waves in higher dimensions, *Comm. Math. Phys.* **55** (1977), 149–162.