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The impact of eigenvalue locality on the convergence behavior of the PSD method for two-cyclic matrices

M.A. Louka, N.M. Missirlis [*,1], F.I. Tzaferis

Department of Informatics and Telecommunications, University of Athens, Panepistimiopolis, 15784 Athens, Greece

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ABSTRACT

In this paper, we analyse the convergence of the preconditioned simultaneous displacement (PSD) method applied to linear systems of the form $Au = b$ where A is a two-cyclic matrix. Convergence conditions and optimum values of the parameters of the method are determined in the cases where the eigenvalues of the associated Jacobi iteration matrix are either all real or all imaginary. It is shown that the convergence behavior of the PSD method is greatly affected by the locality of the eigenvalues of the associated Jacobi iteration matrix. In particular, it is shown that when these eigenvalues are real the PSD method degenerates into the extrapolated Gauss–Seidel method whereas when they are imaginary its convergence is increased by an order of magnitude and becomes equivalent to the extrapolated SOR method. Finally, a comparison with the SSOR method reveals that the PSD method possesses a better convergence behavior in all cases.

1. Introduction

The preconditioned simultaneous displacement (PSD) iterative method was introduced in [2,11] for the numerical solution of the linear system

$$Au = b, \tag{1}$$

where $A \in \mathbb{C}^{N,N}$ is a nonsingular, sparse matrix with nonvanishing diagonal entries and $u, b \in \mathbb{C}^N$ with b given and u to be determined.

* Corresponding author.
E-mail addresses: mlouka@di.uoa.gr (M.A. Louka), nmis@di.uoa.gr (N.M. Missirlis), ftzaf@di.uoa.gr (F.I. Tzaferis).

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The PSD method is a first order extrapolation of the symmetric successive overrelaxation (SSOR) method and as such it was shown in [2] to be asymptotically twice as fast as the SSOR method for natural ordering. In [11], various acceleration techniques were applied to the PSD method, which increased its rate of convergence by an order of magnitude. Comparisons with the successive overrelaxation (SOR) method in [11] proved that the PSD method combined with semi-iterative methods [18] gives an increased rate of convergence to that of the SOR method in certain cases. However, all these results were based on "good" (near the optimum) values of the parameters of the PSD method. Subsequently, a functional relationship was shown to exist in [19,8] for the SSOR method between the eigenvalues of its preconditioned matrix and those of the associated Jacobi matrix which has proved useful to the analysis of the convergence of the method where the associated Jacobi matrix has a p-cyclic form. Such a functional relationship can be derived and the theory concerning p-cyclic matrices (see e.g. [5,6,8,17]) can be applied for the PSD method too. It is the purpose of the present work to proceed in this direction and analyse the convergence of the PSD method using such a functional relationship in the cases where the associated block Jacobi iteration matrix is of a consistently ordered weakly two-cyclic form [18] and possesses either all real (the real case) or all imaginary (the imaginary case) eigenvalues. In such cases we find that the associated functional relationship is the same as the one derived for the extrapolated SOR method in [13] with the only difference that the SOR parameter ω is replaced by $\omega(2-\omega)$. As a consequence the theory developed in [13,14] can be applied to derive sufficient and necessary conditions for the PSD method to converge as well as determine the optimum values of the parameters of the PSD method under the aforementioned conditions on the associated Jacobi iteration matrix. Our approach is also applied to the SSOR method for comparison purposes. It is shown that the convergence behavior of the PSD method (and the SSOR method) is greatly affected by the locality of the eigenvalues of the associated Jacobi iteration matrix. More specifically, in the real case the PSD method attains a maximum rate of convergence equivalent to the extrapolated Gauss–Seidel (EGS) method [10], whereas in the imaginary case its convergence is improved by an order of magnitude. This phenomenon was unexpected as the convergence behavior of the SOR method, for example, remains unaffected under the same conditions [15].

This paper is organized as follows. In Section 2, we state the functional relationship for the PSD method between the eigenvalues of the preconditioned matrix and the associated Jacobi matrix, where this Jacobi matrix has a block p-cyclic form. In Section 3, we deduce sufficient and necessary conditions for the PSD method to converge under the assumptions that the associated Jacobi matrix has a block two-cyclic form and that the eigenvalues of this matrix are either all real or all imaginary. In Section 4, under the same assumptions we find optimum values for the parameters of the PSD method. In Section 5, we compare the PSD method with the SSOR method. Finally, in Section 6, we state our remarks and conclusions.

2. The functional relationship

Let us consider the linear system (1), with

$$A=\begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,p} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,p} \\ \vdots & \vdots & \ddots & \vdots \\ A_{p,1} & A_{p,2} & \cdots & A_{p,p} \end{bmatrix}, \quad p \geqslant 2, \tag{2}$$

where each diagonal $A_{i,i}$ $(1 \leqslant i \leqslant p)$ is square and nonsingular. Assume that the coefficient matrix A has the splitting

$$A = D - C_L - C_U,$$

where $D = \operatorname{diag}(A_{1,1}, A_{2,2}, \ldots, A_{p,p})$ and $-C_L$ and $-C_U$ are the block strictly lower and upper triangular parts of A, respectively. The associated block Jacobi matrix is defined by

$$B = L + U, \tag{3}$$

where $L = D^{-1}C_L$ and $U = D^{-1}C_U$. Accordingly, the preconditioned simultaneous displacement (PSD) method is given by the following scheme [2,4,12]:

$$u^{(n+1)} = \mathscr{D}_{\tau,\omega}u^{(n)} + \delta_{\tau,\omega}, \tag{4}$$

where

$$\mathscr{D}_{\tau,\omega} = I - \tau\mathscr{B}_{\omega}, \quad \mathscr{B}_{\omega} = (I-\omega U)^{-1}(I-\omega L)^{-1}D^{-1}A,$$
$$\delta_{\tau,\omega} = \tau(I-\omega U)^{-1}(I-\omega L)^{-1}D^{-1}b, \tag{5}$$

and $\tau \neq 0, \omega \in \mathbb{R}$. Note that if $\tau = \omega(2-\omega)$, then (4) reduces to the "well known" SSOR method.

Further, let B in (3) be a weakly cyclic matrix of index p [18] of the form

$$B = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & B_{1,p} \\ B_{2,1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & B_{3,2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & B_{p,p-1} & 0 \end{bmatrix}, \quad p \geqslant 2. \tag{6}$$

Theorem 2.1. *Assume that the block-partitioned matrix A of (2) is such that all diagonal submatrices $A_{i,i}$ are square and nonsingular, $1 \leqslant i \leqslant p$, and B of (6) is its associated weakly cyclic of index p block Jacobi matrix. If $\omega \neq 0, 2$, if $\lambda \neq 1$ is an eigenvalue of $\mathscr{B}_{\omega}$, and if μ satisfies*

$$(1-\lambda)^p = (1-\lambda\omega)^{p-2}[1-\lambda\omega(2-\omega)]\mu^p, \tag{7}$$

then μ is an eigenvalue of the block Jacobi matrix B of (6). Conversely, if μ is an eigenvalue of B and if $\hat{\lambda} \neq 1$ satisfies (7), then $\hat{\lambda}$ is an eigenvalue of $\mathscr{B}_{\omega}$.

Proof. The proof is similar to Theorem 1 of [19] and therefore is omitted. □

If ν is an eigenvalue of $\mathscr{D}_{\tau,\omega}$, then because of (5) we have

$$\nu = 1 - \tau\lambda.$$

Expressing (7) in terms of ν yields

$$(\nu+\tau-1)^p = \tau(\tau-\omega+\omega\nu)^{p-2}[\tau-\omega(2-\omega)(1-\nu)]\mu^p. \tag{8}$$

The above functional equation relates the eigenvalues of $\mathscr{D}_{\tau,\omega}$ with those of the block Jacobi matrix B. In case $\tau = \omega(2-\omega)$, (8) becomes

$$[(\nu-(1-\omega)^2]^p = \nu(\nu+1-\omega)^{p-2}(2-\omega)^2\omega^p\mu^p, \tag{9}$$

where now ν is an eigenvalue of the SSOR iteration matrix $\mathscr{D}_{\omega(2-\omega),\omega}$. Note that (9) was also obtained by Varga et al. [19]. If $p = 2$, then (9) reduces to

$$[\nu-(1-\omega)^2]^2 = \nu(2-\omega)^2\omega^2\mu^2, \tag{10}$$

which was obtained earlier by D'Sylva and Miles [1]. By letting

$$\hat{\omega} = \omega(2-\omega), \tag{11}$$

(10) becomes

$$(\nu+\hat{\omega}-1)^2 = \nu\hat{\omega}^2\mu^2, \tag{12}$$

which is Young's relation for the SOR method [21]. Using (12) Niethammer [15] and Lynn [9] produced interesting results concerning the SSOR method in the case where the Jacobi matrix B is weakly cyclic of index 2. Also, when $p = 2$ (7) reduces to

$$(1-\lambda)^2 = [1-\lambda\omega(2-\omega)]\mu^2$$

or

$$(1-\lambda)^2 = (1-\lambda\hat{\omega})\mu^2$$

obtained also in [12]. Finally, if $\tau = 1$, (8) becomes

$$\nu^p = (1-\omega+\omega\nu)^{p-2}[1-\omega(2-\omega)(1-\nu)]\mu^p, \tag{13}$$

where now ν denotes the eigenvalues of $\mathscr{D}_{1,\omega}$, the preconditioned Jacobi (PJ) iteration matrix [2,3]. Letting $p=2$ (13) yields

$$\nu^2 = [1-\omega(2-\omega)+\omega\nu(2-\omega)]\mu^2$$

or

$$\nu^2 - \hat{\omega}\mu^2\nu + \mu^2(\hat{\omega}-1) = 0 \tag{14}$$

obtained also in [12]. Our conclusion so far is that (8) is the generalization of all known functional relationships. In the sequel we will use (7) to analyse the convergence of the PSD method.

3. Convergence

In this section, we analyse the convergence of the PSD method in the case where the matrix B is weakly cyclic of index 2, i.e., where

$$B = \begin{bmatrix} 0 & B_{12} \\ B_{21} & 0 \end{bmatrix}. \tag{15}$$

In such a case A is a two-cyclic and consistently ordered matrix [19]. In particular, we derive sufficient and necessary conditions for the PSD method to converge under the assumption that the eigenvalues of the associated Jacobi iteration matrix B are either all real or all imaginary. We also derive analogous results for the SSOR and PJ methods.

3.1. The real case

In the present case, we assume that all eigenvalues $\{\mu_j\}_{j=1}^N$ of the Jacobi iteration matrix are real. This case occurs, for example, when A is a Hermitian matrix. Let $\underline{\mu} = \min_{1\leqslant j\leqslant N}|\mu_j|$ and $\bar{\mu} = \max_{1\leqslant j\leqslant N}|\mu_j|$.

Lemma 3.1. *If the eigenvalues of the matrix B of (15) are real, then the eigenvalues of $\mathscr{B}_\omega$ are also real.*

Proof. Letting $p=2$ in (7) and using (11) we have

$$\lambda^2 - (2-\hat{\omega}\mu^2)\lambda + 1 - \mu^2 = 0. \tag{16}$$

Furthermore, the roots of (16) are

$$\lambda_\pm(\mu) = \frac{2-\hat{\omega}\mu^2 \pm \sqrt{\Delta}}{2}, \tag{17}$$

where

$$\Delta \equiv \Delta(\mu) = \mu^2(\hat{\omega}^2\mu^2 - 4\hat{\omega} + 4). \tag{18}$$

But $\hat{\omega} = \omega(2-\omega) \leqslant 1$. Thus, for a real μ, the discriminant $\Delta \geqslant 0$. □

Theorem 3.1. *If the matrix B of (15) has real eigenvalues, then the PSD method converges if and only if*

$$\bar{\mu} < 1 \quad \textit{and} \quad 0 < \tau < \frac{2}{\lambda_+(\bar{\mu})}, \tag{19}$$

where $\lambda_+(\bar{\mu})$ is given by (17).

Proof. For the PSD method to converge, $\lambda_\pm(\mu)$ must either be positive or negative [16]. This implies that $1-\mu^2 > 0$ (see (16)) proving the first part of (19). Since $2-\hat{\omega}\mu^2 > 0$, $\lambda_\pm(\mu)$ are positive and the range of τ is given by [16,18,21]

$$0 < \tau < \frac{2}{\lambda_+(\mu)}. \tag{20}$$

By studying the monotonicity of $\lambda_+(\mu)$ with respect to μ^2 we find that

$$\operatorname{sign}\frac{\partial\lambda_+(\mu)}{\partial\mu^2} = +1.$$

Hence (20) yields the range of τ given by (19). $\square$

Under the same assumptions the SSOR method converges if and only if [21]

$$\bar{\mu} < 1 \quad \text{and} \quad 0 < \omega < 2.$$

The condition $\bar{\mu} < 1$ was therefore expected to hold for the PSD method also. Note, however, that the PSD method presents no restriction on ω for its convergence.

In an attempt to obtain a method with only one parameter like the SSOR method the preconditioned Jacobi (PJ) method was introduced in [3] by letting $\tau = 1$ in the PSD method.

Theorem 3.2. *If the matrix B of (15) has real eigenvalues, then the PJ method converges if and only if*

$$\bar{\mu} < 1 \quad \textit{and} \quad \omega_\ell < \omega < \omega_r, \tag{21}$$

where

$$\omega_\ell = 1 - \frac{1}{\bar{\mu}}\left(\frac{1+\bar{\mu}^2}{2}\right)^{1/2}, \quad \omega_r = 1 + \frac{1}{\bar{\mu}}\left(\frac{1+\bar{\mu}^2}{2}\right)^{1/2}. \tag{22}$$

Proof. Applying Lemma 2.1 of [21] to (14) it follows that the PJ method converges if and only if

$$|\mu^2(\hat{\omega}-1)| < 1, \quad |\hat{\omega}\mu^2| < 1 + \bar{\mu}^2(\hat{\omega}-1)$$

or

$$\bar{\mu} < 1, \quad \frac{1}{2}\left(1 - \frac{1}{\bar{\mu}^2}\right) < \hat{\omega} < 1 + \frac{1}{\bar{\mu}^2}.$$

In view of (11) it is easy to prove that the latter inequalities are equivalent to (21). $\square$

The convergence conditions (21) for the PJ method were also found in [20] via a different approach.

3.2. The imaginary case

Here we assume that the Jacobi iteration matrix has imaginary eigenvalues $\{\pm i\mu_j\}_{j=1}^N$ with $\bar{\mu} = \max_{1\leqslant j\leqslant N}|\mu_j|$ and $\underline{\mu} = \min_{1\leqslant j\leqslant N}|\mu_j|$. This case occurs, for example, when A is a skew-Hermitian matrix.

Theorem 3.3. *If the matrix B of (15) has imaginary eigenvalues, then the PSD method converges if and only if the parameters ω, τ lie in any of the corresponding domains given by Table 1, where*

Table 1
Necessary and sufficient conditions for the convergence of the PSD method.

Case	τ-Domain	ω-Domain
1	$\frac{2}{\lambda_-(\underline{\mu})} < \tau < 0$	$\omega \leqslant \omega_2(\underline{\mu})$ $\omega_1(\underline{\mu}) \leqslant \omega$
2	$h(\omega,\underline{\mu}) < \tau < 0$	$\omega_2(\underline{\mu}) \leqslant \omega < \omega_3(\underline{\mu})$ $\omega_4(\underline{\mu}) < \omega \leqslant \omega_1(\underline{\mu})$
3	$0 < \tau < h(\omega,\bar{\mu})$	$\omega_3(\bar{\mu}) < \omega \leqslant \omega_5(\bar{\mu})$ $\omega_6(\bar{\mu}) \leqslant \omega < \omega_4(\bar{\mu})$
4	$0 < \tau < \frac{2}{\lambda_+(\bar{\mu})}$	$0 < \omega_5(\bar{\mu}) \leqslant \omega \leqslant \omega_6(\bar{\mu})$

$$\omega_1(\mu) = \frac{2}{1+\mu-(1+\mu^2)^{1/2}}, \quad \omega_2(\mu) = \frac{2}{1-\mu-(1+\mu^2)^{1/2}},$$
$$\omega_3(\mu) = 1 - \frac{1}{\mu}(2+\mu^2)^{1/2}, \qquad \omega_4(\mu) = 1 + \frac{1}{\mu}(2+\mu^2)^{1/2},$$
$$\omega_5(\mu) = \frac{2}{1+\mu+(1+\mu^2)^{1/2}}, \quad \omega_6(\mu) = \frac{2}{1-\mu+(1+\mu^2)^{1/2}},$$
$$h(\omega,\mu) = \frac{2+\omega(2-\omega)\mu^2}{1+\mu^2} \tag{23}$$

and $\lambda_{\pm}(\mu)$ *is given by* (25).

Proof. If B has imaginary eigenvalues, then substituting μ^2 with $-\mu^2$, it follows from (16) that

$$\lambda^2 - (2+\hat{\omega}\mu^2)\lambda + 1 + \mu^2 = 0. \tag{24}$$

We remark that the eigenvalue relationship (24) is the same as the one satisfied by the eigenvalues of the preconditioned matrix (see (2.6) of [13]) of the extrapolated SOR (ESOR) method and B, where $\hat{\omega}$ plays the role of ω. It follows that the ESOR method's convergence analysis developed in [13] can be applied in the present case. As a result, the convergence conditions of PSD are given by Table 2.1 of [13] or by Table 2, with

$$\lambda_{\pm}(\mu) = \frac{2+\hat{\omega}\mu^2 \pm \widehat{\Delta}^{1/2}}{2}, \tag{25}$$
$$\hat{\omega}_6(\mu) = \frac{2}{1-(1+\mu^2)^{1/2}} \quad \text{and} \quad \hat{\omega}_7(\mu) = \frac{2}{1+(1+\mu^2)^{1/2}}, \tag{26}$$

where

$$\widehat{\Delta} \equiv \widehat{\Delta}(\mu) = \mu^2(\hat{\omega}^2\mu^2 + 4\hat{\omega} - 4) \tag{27}$$

with $\hat{\omega}_6(\mu) < 0 < \hat{\omega}_7(\mu) < 1$ and $h(\hat{\omega},\mu) = \frac{2+\hat{\omega}\mu^2}{1+\mu^2}$. Note, that the τ-ranges of Table 2 are the same as the ones given in Table 1.

In the following, we express the ranges of $\hat{\omega}$, for each case of Table 2, in terms of ω using (11).
Case 1: In this case $\hat{\omega} \leqslant \hat{\omega}_6(\underline{\mu})$, which is equivalent to $\omega(2-\omega) \leqslant \hat{\omega}_6(\underline{\mu})$, it follows that

$$\omega \leqslant \omega_2(\underline{\mu}) \quad \text{or} \quad \omega_1(\underline{\mu}) \leqslant \omega,$$

where $\omega_1(\mu)$ and $\omega_2(\mu)$ are given by (23), with $\omega_2(\mu) < 0 < \omega_1(\mu)$. Hence, case 1 of Table 1 is proved.
Case 2: Here $\hat{\omega}_6(\underline{\mu}) \leqslant \hat{\omega} < -2/\underline{\mu}^2$ which is equivalent to

$$\hat{\omega}_6(\underline{\mu}) \leqslant \omega(2-\omega) < -2/\underline{\mu}^2. \tag{28}$$

The left-hand side inequality of (28) is equivalent to

$$\omega_2(\underline{\mu}) \leqslant \omega \leqslant \omega_1(\underline{\mu}), \tag{29}$$

whereas the right-hand side is equivalent to

$$\omega < \omega_3(\underline{\mu}) \quad \text{or} \quad \omega_4(\underline{\mu}) < \omega, \tag{30}$$

Table 2
Necessary and sufficient conditions for the convergence of the PSD method.

Case	τ-Domain	ω-Domain
1	$\frac{2}{\lambda_-(\underline{\mu})} < \tau < 0$	$-\infty < \hat{\omega} \leqslant \hat{\omega}_6(\underline{\mu})$
2	$h(\hat{\omega},\underline{\mu}) < \tau < 0$	$\hat{\omega}_6(\underline{\mu}) \leqslant \hat{\omega} < \frac{-2}{\underline{\mu}^2}$
3	$0 < \tau < h(\hat{\omega},\bar{\mu})$	$\frac{-2}{\bar{\mu}^2} < \hat{\omega} \leqslant \omega_7(\bar{\mu})$
4	$0 < \tau < \frac{2}{\lambda_+(\bar{\mu})}$	$\omega_7(\bar{\mu}) \leqslant \hat{\omega} < +\infty$

where $\omega_3(\mu)$ and $\omega_4(\mu)$ are given by (23). From (29) and (30) it follows that

$$\omega_2(\underline{\mu}) \leqslant \omega < \min\{\omega_1(\underline{\mu}), \omega_3(\underline{\mu})\} \tag{31}$$

or

$$\max\{\omega_2(\underline{\mu}), \omega_4(\underline{\mu})\} < \omega \leqslant \omega_1(\underline{\mu}). \tag{32}$$

It is easily verified that (31) and (32), are equivalent to

$$\omega_2(\underline{\mu}) \leqslant \omega < \omega_3(\underline{\mu})$$

or

$$\omega_4(\underline{\mu}) < \omega \leqslant \omega_1(\underline{\mu}),$$

respectively, with $\omega_2(\underline{\mu}) < \omega_3(\underline{\mu}) < 0 < 2 < \omega_4(\underline{\mu}) < \omega_1(\underline{\mu})$. So case 2 of Table 1 is proved.
Case 3: In this case $-2/\bar{\mu}^2 < \hat{\omega} \leqslant \hat{\omega}_7(\bar{\mu})$, which is equivalent to

$$-2/\bar{\mu}^2 < \omega(2-\omega) \leqslant \hat{\omega}_7(\bar{\mu}). \tag{33}$$

By the left-hand side inequality of (33) it follows that

$$\omega_3(\bar{\mu}) < \omega < \omega_4(\bar{\mu}), \tag{34}$$

whereas the right-hand side of (33) is equivalent to

$$\omega \leqslant \omega_5(\bar{\mu}) \quad \text{or} \quad \omega_6(\bar{\mu}) \leqslant \omega, \tag{35}$$

where $\omega_5(\mu)$ and $\omega_6(\mu)$ are given by (23). From (34) and (35) it follows that

$$\omega_3(\bar{\mu}) < \omega \leqslant \min\{\omega_4(\bar{\mu}), \omega_5(\bar{\mu})\}$$

or

$$\max\{\omega_3(\bar{\mu}), \omega_6(\bar{\mu})\} \leqslant \omega < \omega_4(\bar{\mu}).$$

The above inequalities are equivalent to

$$\omega_3(\bar{\mu}) < \omega \leqslant \omega_5(\bar{\mu})$$

or

$$\omega_6(\bar{\mu}) \leqslant \omega < \omega_4(\bar{\mu}),$$

respectively, with $\omega_3(\bar{\mu}) < 0 < \omega_5(\bar{\mu}) < \omega_6(\bar{\mu}) < 2 < \omega_4(\bar{\mu})$. Consequently, case 3 of Table 1 is proved.
Case 4: In this case $\hat{\omega}_7(\bar{\mu}) \leqslant \hat{\omega}$, which is equivalent to $\hat{\omega}_7(\bar{\mu}) \leqslant \omega(2-\omega)$, from which it follows that

$$0 < \omega_5(\bar{\mu}) \leqslant \omega \leqslant \omega_6(\bar{\mu}).$$

Therefore, case 4 of Table 1 is proved. $\square$

Corollary 3.1. *Under the hypothesis of Theorem 3.3 and if $\underline{\mu} = 0$, then the PSD method converges if and only if*

$$\omega_3(\bar{\mu}) < \omega < \omega_4(\bar{\mu}) \quad \textit{and} \quad 0 < \tau < \frac{2}{\lambda_+(\bar{\mu})}, \tag{36}$$

where $\omega_3(\mu)$ and $\omega_4(\mu)$ are given by (23) and $\lambda_+(\bar{\mu})$ by (25).

Proof. From Corollary 2.3 of [13], we have that $S(\mathscr{D}_{\tau,\omega}) < 1$ if and only if

$$-\frac{2}{\bar{\mu}^2} < \hat{\omega} < +\infty \quad \text{and} \quad 0 < \tau < \frac{2}{\lambda_+(\bar{\mu})}. \tag{37}$$

It is readily verified that, because of (11), the left-hand side inequality of (37) yields the first part of (36). $\square$

Table 3
Sufficient and necessary conditions for the convergence of the SSOR method.

$\bar{\mu}$-Condition	$\underline{\mu}$-Condition	Case	ω-Domain
$\bar{\mu} \leqslant 1$	–	1	$0 < \omega < 2$
$1 < \bar{\mu}$	$\underline{\mu} < 1$	2	$0 < \omega < 2$
		3	$0 < \omega < \omega_{13}(\bar{\mu})$
	$1 \leqslant \underline{\mu}$	4	$\omega_{14}(\bar{\mu}) < \omega < 2$

Theorem 3.4. *If the matrix B of* (15) *has imaginary eigenvalues, then the SSOR method converges if and only if any case of Table* 3 *holds, where*

$$\omega_{13}(\mu) = \frac{2}{1+\mu+(\mu^2-1)^{1/2}} \quad \textit{and} \quad \omega_{14}(\mu) = \frac{2}{1+\mu-(\mu^2-1)^{1/2}}.$$

Proof. If B has imaginary eigenvalues the functional relationship for the SSOR method is obtained by letting $\mu^2 = -\mu^2$ in (12), which gives

$$\nu^2 + (\hat{\omega}^2\mu^2 + 2\hat{\omega} - 2)\nu + (\hat{\omega}-1)^2 = 0.$$

By Lemma 2.1 of [21] it follows that $|\nu| < 1$ if and only if

$$0 < \hat{\omega} < 2 \tag{38}$$

and

$$\hat{\omega}^2(1-\mu^2) - 4\hat{\omega} + 4 > 0. \tag{39}$$

In the following, we consider the three cases: case 1: $\bar{\mu} \leqslant 1$, case 2: $1 \leqslant \underline{\mu}$ and case 3: $\underline{\mu} < 1 < \bar{\mu}$.
Case 1: $\bar{\mu} \leqslant 1$. In this case inequality (39) is satisfied for

$$\hat{\omega} < \frac{2}{1+\bar{\mu}} \quad \text{or} \quad \hat{\omega} > \frac{2}{1-\bar{\mu}}. \tag{40}$$

Combining (40) and (38) we have that

$$0 < \hat{\omega} < \frac{2}{1+\bar{\mu}}. \tag{41}$$

Expressing (41) in terms of ω yields

$$0 < \omega < 2. \tag{42}$$

Hence, case 1 of Table 3 is proved.
Case 2: $1 \leqslant \underline{\mu}$. In this case (39) and (38) hold for

$$0 < \hat{\omega} < \frac{2}{1+\bar{\mu}}$$

which in terms of ω yields either

$$0 < \omega < \omega_{13}(\bar{\mu}) \tag{43}$$

or

$$\omega_{14}(\bar{\mu}) < \omega < 2, \tag{44}$$

where $0 < \omega_{13}(\bar{\mu}) < \omega_{14}(\bar{\mu}) < 2$. Hence, cases 3 and 4 of Table 3 are proved.
Case 3: $\underline{\mu} < 1 < \bar{\mu}$. Let α, β be two positive integers such that $\mu_\alpha = \max\{|\mu| \mid |\mu| < 1\}$, $\mu_\beta = \min\{|\mu| \mid |\mu| > 1\}$ and consider the following two cases: (i) $\underline{\mu} \leqslant |\mu| \leqslant \mu_\alpha$ and (ii) $\mu_\beta \leqslant |\mu| \leqslant \bar{\mu}$.

(i) By case 1, we have that (42) holds. (ii) By case 2, we have that either (43) or (44) holds. Combining (42) and (43) we find that the SSOR method converges when $0 < \omega < 2$. Similarly (42) and (44) yield the same range of ω. Therefore, case 2 of Table 3 is proved. □

Table 4
Sufficient and necessary conditions for the convergence of the PJ method.

Condition	Case	ω-Domain
$\bar{\mu} < 1$	1	$\omega_7(\bar{\mu}) < \omega < \omega_8(\bar{\mu})$
	2	$\omega_7(\bar{\mu}) < \omega < \omega_9(\bar{\mu})$
$1 \leqslant \bar{\mu} < \sqrt{3}$	3	$\omega_{10}(\bar{\mu}) < \omega < \omega_8(\bar{\mu})$

Theorem 3.5. *If the matrix B of* (15) *has imaginary eigenvalues, then the PJ method converges if and only if any case of Table* 4 *holds, where*

$$\omega_7(\mu) = 1 - \frac{1}{|\mu|}, \quad \omega_8(\mu) = 1 + \frac{1}{|\mu|},$$
$$\omega_9(\mu) = 1 - \frac{1}{|\mu|}\left(\frac{\mu^2-1}{2}\right)^{1/2}, \quad \omega_{10}(\mu) = 1 + \frac{1}{|\mu|}\left(\frac{\mu^2-1}{2}\right)^{1/2}. \tag{45}$$

Proof. Since the matrix B has imaginary eigenvalues the functional relationship for the PJ method is obtained by letting $\mu^2 = -\mu^2$ in (14) to give

$$\nu^2 + \hat{\omega}\mu^2\nu + \mu^2(1-\hat{\omega}) = 0.$$

By Lemma 2.1 of [21] it follows that $|\nu| < 1$ if and only if

$$\frac{\mu^2-1}{\mu^2} < \hat{\omega} < \frac{\mu^2+1}{\mu^2}$$

and

$$\hat{\omega} < \frac{1}{2}\frac{\mu^2+1}{\mu^2},$$

which are equivalent to

$$\bar{\mu} < \sqrt{3}, \quad \frac{\bar{\mu}^2-1}{\bar{\mu}^2} < \hat{\omega} < \frac{1}{2}\frac{\bar{\mu}^2+1}{\bar{\mu}^2}. \tag{46}$$

Expressing (46) in terms of ω yields

$$\omega^2 - 2\omega + \frac{\bar{\mu}^2-1}{\bar{\mu}^2} < 0 \tag{47}$$

and

$$\omega^2 - 2\omega + \frac{\bar{\mu}^2+1}{2\bar{\mu}^2} > 0. \tag{48}$$

Further (47) is satisfied for

$$\omega_7(\bar{\mu}) < \omega < \omega_8(\bar{\mu}), \tag{49}$$

where $\omega_7(\bar{\mu})$ and $\omega_8(\bar{\mu})$ are given by (45). Inequality (48) is satisfied if either $\bar{\mu} < 1$, or if $1 \leqslant \bar{\mu} < \sqrt{3}$ and either

$$\omega < \omega_9(\bar{\mu}) \quad \text{or} \quad \omega > \omega_{10}(\bar{\mu}), \tag{50}$$

where $\omega_9(\bar{\mu}), \omega_{10}(\bar{\mu})$ are given by (45). If $\bar{\mu} < 1$ then, in view of (49), case 1 of Table 4 is proved. If $1 \leqslant \bar{\mu} < \sqrt{3}$ then combining (49) and (50) we have either

$$\omega_7(\bar{\mu}) < \omega < \omega_9(\bar{\mu})$$

or

$$\omega_{10}(\bar{\mu}) < \omega < \omega_8(\bar{\mu})$$

and cases 2 and 3 of Table 4 are proved, respectively. $\square$

4. Optimum parameters

In the present section, we determine the optimum values τ_0, ω_0 of τ, ω, respectively, for which the rate of convergence of the PSD method is maximized.

4.1. The real case

Let us assume that all the eigenvalues $\{\mu_j\}_{j=1}^N$ of the associated Jacobi iteration matrix are real. If we let $S(\mathscr{D}_{\tau,\omega})$ denote the spectral radius of $\mathscr{D}_{\tau,\omega}$, we can state the following.

Theorem 4.1. *If the matrix B of (15) has real eigenvalues and $\bar{\mu} < 1$, then $S(\mathscr{D}_{\tau,\omega})$ is minimized at*

$$\omega_0 = 1 \quad \textit{and} \quad \tau_0 = \frac{2}{2 - \bar{\mu}^2} \tag{51}$$

and its corresponding value is given by

$$S(\mathscr{D}_{\tau_0,\omega_0}) = \frac{\bar{\mu}^2}{2 - \bar{\mu}^2}. \tag{52}$$

Proof. By Lemma 3.1 and Theorem 3.1 $\lambda_\pm(\mu)$, the eigenvalues of $\mathscr{B}_\omega$, are real and positive. In this case the optimum value for τ is given by [18,21]

$$\tau_0 = \frac{2}{\lambda_+(\bar{\mu}) + \lambda_-(\bar{\mu})} \tag{53}$$

and

$$S(\mathscr{D}_{\tau_0,\hat{\omega}}) = \frac{P(\mathscr{B}_{\hat{\omega}}) - 1}{P(\mathscr{B}_{\hat{\omega}}) + 1} \quad \text{with } P(\mathscr{B}_{\hat{\omega}}) = \frac{\lambda_+(\bar{\mu})}{\lambda_-(\bar{\mu})}. \tag{54}$$

Since $S(\mathscr{D}_{\tau_0,\hat{\omega}})$ is an increasing function of $P(\mathscr{B}_{\hat{\omega}})$ it suffices to minimize $P(\mathscr{B}_{\hat{\omega}})$ with respect to $\hat{\omega}$ in order to find the optimum value of $\hat{\omega}$. Using (17) in (54) it is easily verified that

$$P(\mathscr{B}_{\hat{\omega}}) = \frac{4(1 - \bar{\mu}^2)}{\varphi(\hat{\omega}, \bar{\mu})},$$

where $\varphi(\hat{\omega}, \bar{\mu}) = \left(2 - \hat{\omega}\bar{\mu}^2 - \sqrt{\widehat{\Delta}(\bar{\mu})}\right)^2$ and $\widehat{\Delta}(\bar{\mu})$ is given by (27). A study of the behavior of φ as a function of $\hat{\omega}$ reveals that $P(\mathscr{B}_{\hat{\omega}})$ is a decreasing function of $\hat{\omega}$. Moreover (11) yields

$$\omega^2 - 2\omega + \hat{\omega} = 0,$$

which for $\omega \in \mathbb{R}$ we must have

$$\hat{\omega} \leqslant 1. \tag{55}$$

Since $P(\mathscr{B}_{\hat{\omega}})$ is a decreasing function of $\hat{\omega}$, then its maximum value is achieved at $\hat{\omega}_0 = 1$, which in turn yields $\omega = 1$. In addition (53), because of (17), yields $\tau_0 = \frac{2}{2 - \hat{\omega}_0\bar{\mu}^2}$, which proves (51) since $\hat{\omega}_0 = 1$. Moreover, for $\hat{\omega}_0 = 1$ (54) yields (52). $\square$

In other words Theorem 4.1 states that the PSD method coincides with the extrapolated Gauss–Seidel (EGS) method [10] for optimum parameter values. An analogous result holds also for the SSOR method with $\omega_0 = 1$ and [1]

$$S(\mathscr{D}_{\omega_0,\omega_0}) = \bar{\mu}^2.$$

Theorem 4.2. *If the matrix B of* (15) *has real eigenvalues and* $\bar{\mu} < 1$, *then* $S(\mathscr{D}_{1,\omega})$ *is minimized at* $\omega_0 = 1$ *and its corresponding value is given by*

$$S(\mathscr{D}_{1,\omega_0}) = \bar{\mu}^2. \tag{56}$$

Proof. The eigenvalue relationship (14) connecting the eigenvalues of $\mathscr{D}_{1,\hat{\omega}}$ with B is the same as the one found in [14] between the preconditioned matrix of the ESOR method and B. In addition, the eigenvalues of $\mathscr{D}_{1,\hat{\omega}}$ are real since $\mathscr{D}_{\tau,\hat{\omega}}$ possesses real eigenvalues. Hence, from (14) it follows that

$$S(\mathscr{D}_{1,\hat{\omega}}) = \max_{\mu^2} \frac{|\hat{\omega}|\bar{\mu}^2 + \widehat{\Delta}^{1/2}}{2}. \tag{57}$$

For $\hat{\omega} \in (\omega_\ell, 0)$, where ω_ℓ is given by (22), $S(\mathscr{D}_{1,\hat{\omega}})$ is an increasing function of $\hat{\omega}$, whereas for $0 < \hat{\omega} < \omega_r$ $\mathscr{D}_{1,\hat{\omega}}$ is a decreasing function of $\hat{\omega}$. Therefore, because of (55) and (22), its minimum value is attained at $\hat{\omega}_0 = 1$ and its corresponding value, because of (57), is given by (56). $\square$

As a result we have that the SSOR method and the PJ method coincide with the Gauss–Seidel method for optimum parameter values. Note that the convergence rate of the EGS method is twice that of the GS method [10]. As a consequence the PSD method will converge asymptotically twice as fast compared to the SSOR and PJ methods.

4.2. The imaginary case

In the present case, the associated Jacobi iteration matrix has imaginary eigenvalues. Since the eigenvalue relationship between $\mathscr{D}_{\tau,\omega}$ and B is (24), which coincides with (2.6) of [13], it follows that we can apply Theorem 3.3 of [13] for determining the optimum values for ω and τ.

Theorem 4.3. *If the matrix B of* (15) *has imaginary eigenvalues, then* $S(\mathscr{D}_{\tau,\omega})$ *is minimized at*

$$\omega_0 = \omega_5(\bar{\mu}) \quad or \quad \omega_0 = \omega_6(\bar{\mu}) \quad and \quad \tau_0 = \frac{2 + \omega_0(2 - \omega_0)\underline{\mu}^2}{2(1 + \underline{\mu}^2)} \tag{58}$$

and its corresponding value is given by the expression

$$S(\mathscr{D}_{\tau_0,\omega_0}) = \frac{\underline{\mu}(\bar{\mu}^2 - \underline{\mu}^2)^{1/2}}{(1 + \underline{\mu}^2)^{1/2}(1 + (1 + \bar{\mu}^2)^{1/2})}, \tag{59}$$

where $\omega_5(\bar{\mu}), \omega_6(\bar{\mu})$ *are given by* (23).

Proof. According to Theorem 3.3 of [13] the optimum values for $\hat{\omega}$ and τ occur at $\hat{\omega}_0 = \hat{\omega}_7(\bar{\mu})$ where $\hat{\omega}_7(\bar{\mu})$ is given by (26) with τ_0 given by (58), which on letting $\hat{\omega}_0 = \omega_0(2 - \omega_0)$ yields the two optimum values of ω given in (58). $\square$

Remark. The expression in the right-hand side of (59) is the minimum value of the spectral radius of the ESOR method (see (3.18) of [13]). Therefore, in the imaginary case the PSD method and the ESOR method coincide for optimum values of their parameters.

Corollary 4.1. *Under the hypothesis of Theorem* 4.3 *and if* $\underline{\mu} = 0$, *then* $S(\mathscr{D}_{\tau,\omega})$ *is minimized at* $\tau_0 = \hat{\omega}_7(\bar{\mu})$ *and* $\omega_0 = \omega_5(\bar{\mu})$ *or* $\omega_0 = \omega_6(\bar{\mu})$ *and its corresponding value is given by*

$$S(\mathscr{D}_{\omega_0,\omega_0}) = 1 - \hat{\omega}_7(\bar{\mu}) = \left(\frac{\bar{\mu}}{1 + (1 + \bar{\mu}^2)^{1/2}}\right)^2. \tag{60}$$

Proof. According to Corollary 3.4 of [13] the optimum values for τ and $\hat{\omega}$ occur at $\tau_0 = \hat{\omega}_0 = \hat{\omega}_7(\bar{\mu})$, which by letting $\hat{\omega}_0 = \omega_0(2 - \omega_0)$ yields the two optimum values of $\omega, \omega_5(\bar{\mu})$ and $\omega_6(\bar{\mu})$. $\square$

Table 5
Optimum values for $\omega_0,\ \tau_0$ and $S(\mathscr{D}_{\tau_0,\omega_0})$.

Condition	Case	ω_0	τ_0	$S(\mathscr{D}_{\tau_0,\omega_{\text{opt}}})$
$\underline{\mu}=0$	1	$\omega_5(\bar{\mu})$ or $\omega_6(\bar{\mu})$	$\frac{2}{1+(1+\bar{\mu}^2)^{1/2}}$	$\frac{\bar{\mu}^2}{(1+(1+\bar{\mu}^2)^{1/2})^2}$
$\underline{\mu}\neq 0$	2	$\omega_5(\bar{\mu})$ or $\omega_6(\bar{\mu})$	$\frac{1+\underline{\mu}^2+(1+\bar{\mu}^2)^{1/2}}{(1+\underline{\mu}^2)(1+(1+\bar{\mu}^2)^{1/2})}$	$\frac{\underline{\mu}(\bar{\mu}^2-\underline{\mu}^2)^{1/2}}{(1+\underline{\mu}^2)^{1/2}(1+(1+\bar{\mu}^2)^{1/2})}$
	3	$\omega_5(\bar{\mu})$ or $\omega_6(\bar{\mu})$	$\frac{1}{(1+\mu^2)^{1/2}}$	0
$\underline{\mu}=\bar{\mu}=\mu$	4	$\omega_1(\bar{\mu})$ or $\omega_2(\bar{\mu})$	$-\frac{1}{(1+\mu^2)^{1/2}}$	0

As a result we have that if $\underline{\mu}=0$, then the PSD method coincides with the SOR method for optimum parameter values.

Corollary 4.2. *Under the hypothesis of Theorem 4.3 and if $\underline{\mu}=\bar{\mu}=\mu$, then for either*

$$\text{(i) } \omega_0=\omega_5(\mu) \text{ or } \omega_0=\omega_6(\mu) \quad \text{and} \quad \tau_0=\frac{1}{(1+\mu^2)^{1/2}}$$

or

$$\text{(ii) } \omega_0=\omega_1(\mu) \text{ or } \omega_0=\omega_2(\mu) \quad \text{and} \quad \tau_0=-\frac{1}{(1+\mu^2)^{1/2}},$$

we have $S(\mathscr{D}_{\tau_0,\omega_0})=0$.

Proof. It follows immediately from Corollary 3.5 of [13]. □

Table 5 summarizes the results of Theorem 4.3 and Corollaries 4.1 and 4.2.

Theorem 4.4. *If the matrix B of (15) has imaginary eigenvalues and $S(\mathscr{D}_{\omega(2-\omega),\omega})<1$, then $S(\mathscr{D}_{\omega(2-\omega),\omega})$ is minimized at*

$$\omega_0=\omega_5(\bar{\mu}) \tag{61}$$

and its corresponding value is given by the expression

$$S(\mathscr{D}_{\omega_0(2-\omega_0),\omega_0})=1-\omega_5(\bar{\mu})=\left(\frac{\bar{\mu}}{1+(1+\bar{\mu}^2)^{1/2}}\right)^2. \tag{62}$$

Proof. The SSOR method has the same functional relationship as the SOR method where instead of ω we now have $\hat{\omega}$. Following the SOR theory [15] we can determine the optimum value $\hat{\omega}_0=\hat{\omega}_7(\bar{\mu})$ which in terms of ω yields (61). Therefore, $S(\mathscr{D}_{\hat{\omega}_0,\hat{\omega}_0})=1-\hat{\omega}_0$ or $S(\mathscr{D}_{\hat{\omega}_0,\hat{\omega}_0})=\frac{(1+\bar{\mu}^2)^{1/2}-1}{(1+\bar{\mu}^2)^{1/2}+1}$, which yields (62). □

Accordingly, the SSOR method coincides with the SOR method for optimum parameter values. Also, from the above theorem and Corollary 4.1 we have that if $\underline{\mu}=0$, then the PSD method coincides with the SSOR and SOR methods for optimum parameter values. Furthermore, note that the optimum parameter value $\omega_5(\bar{\mu})$ in the SSOR method is equal to one of the optimum parameter values of ω of the PSD method.

Theorem 4.5. *If the matrix B of (15) has imaginary eigenvalues and $S(\mathscr{D}_{1,\omega})<1$, then $S(\mathscr{D}_{1,\omega_0})$ is minimized at either*

$$\omega_0=\omega_5(\bar{\mu}) \quad \text{or} \quad \omega_0=\omega_6(\bar{\mu}) \tag{63}$$

and its corresponding value is given by

$$S(\mathscr{D}_{1,\omega_0})=\frac{\bar{\mu}^2}{1+(1+\bar{\mu}^2)^{1/2}}. \tag{64}$$

Proof. From Theorem 3.6 of [13], $S(\mathscr{D}_{1,\hat{\omega}})$ is minimized at

$$\hat{\omega}_0 = \hat{\omega}_7(\bar{\mu}) \tag{65}$$

and its corresponding value is given by the expression

$$S(\mathscr{D}_{1,\hat{\omega}_0}) = \bar{\mu}\left(\frac{(1+\bar{\mu}^2)^{1/2}-1}{(1+\bar{\mu}^2)^{1/2}+1}\right)^{1/2}, \tag{66}$$

where $\hat{\omega}_7(\bar{\mu})$ is given by (26). From (65) we obtain the optimum values of ω given by (63), whereas (66) yields (64). $\square$

5. Comparisons

Recall that for any convergent iterative method of the form $u^{(n+1)} = Gu^{(n)} + k, n \geqslant 0$

$$R(G) = -\log S(G) \tag{67}$$

is the rate of convergence whereas

$$RR(G) = 1/R(G) \tag{68}$$

is its reciprocal rate of convergence which is associated with the number of iterations [21].

In the following, we compare the reciprocal rates of convergence for the PSD, SSOR and PJ methods.

5.1. The real case

With (52) and (67), it follows from $\log(1+x) = x + O(x^2)$ as $x \to 0$ that for $\bar{\mu}$ close to 1, we have

$$R(\mathscr{D}_{\tau_0,\omega_0}) = -\log S(\mathscr{D}_{\tau_0,\omega_0}) = -\log\frac{\bar{\mu}^2}{2-\bar{\mu}^2} \simeq 2(1-\bar{\mu}^2). \tag{69}$$

Similarly, from (60) and (64) we have

$$R(\mathscr{D}_{\omega_0(2-\omega_0),\omega_0}) = R(\mathscr{D}_{1,\omega_0}) \simeq 1-\bar{\mu}^2. \tag{70}$$

From (70), (69) and using (68) we conclude that

$$RR(\mathscr{D}_{\tau_0,\omega_0}) = \frac{1}{2}RR(\mathscr{D}_{\omega_0(2-\omega_0),\omega_0}) \simeq \frac{1}{2}RR(\mathscr{D}_{1,\omega_0}). \tag{71}$$

Accordingly, the reciprocal rate of convergence for the PSD method is approximately half the reciprocal rate of either the SSOR method or the PJ method.

5.2. The imaginary case

From (62) and (64) it follows that

$$S(\mathscr{D}_{\omega_0(2-\omega_0),\omega_0}) = \frac{S(\mathscr{D}_{1,\omega_0})}{1+(1+\bar{\mu}^2)^{1/2}}. \tag{72}$$

From (72), using (67), we obtain

$$\begin{aligned} R(\mathscr{D}_{\omega_0(2-\omega_0),\omega_0}) &= -\log S(\mathscr{D}_{\omega_0(2-\omega_0),\omega_0}) = -\log S(\mathscr{D}_{1,\omega_0}) + \log(1+(1+\bar{\mu}^2)^{1/2}) \\ &= R(\mathscr{D}_{1,\omega_0}) + \log(1+(1+\bar{\mu}^2)^{1/2}). \end{aligned} \tag{73}$$

From (73) it follows that the rate of convergence of the SSOR method will exceed that of the PJ method as $\bar{\mu}$ increases. As regards the optimum PSD method, if $\underline{\mu} = 0$ it coincides with the optimum SOR method, since both have the same spectral radius (Corollary 4.1). Furthermore, if $\underline{\mu} = \bar{\mu}$ (Corollary 4.2), then $S(\mathscr{D}_{\tau_0,\omega_0}) = 0 < S(\mathscr{D}_{\omega_0(2-\omega_0),\omega_0})$.

In the following, we compare the PSD and SSOR methods when $0 < \underline{\mu} < \bar{\mu}$. Setting $x = 1 + \underline{\mu}^2$ and $y = 1 + \bar{\mu}^2$ (62) yields

$$S(\mathscr{D}_{\omega_0(2-\omega_0),\omega_0}) = \frac{\sqrt{y} - 1}{\sqrt{y} + 1}$$

and (59) becomes

$$S(\mathscr{D}_{\tau_0,\omega_0}) = \frac{\sqrt{x-1}\sqrt{y-x}}{\sqrt{x}(1+\sqrt{y})},$$

where $1 < x < y$. As it is easily shown that

$$\frac{\sqrt{x-1}\sqrt{y-x}}{\sqrt{x}(1+\sqrt{y})} < \frac{\sqrt{y}-1}{\sqrt{y}+1},$$

it follows that

$$S(\mathscr{D}_{\tau_0,\omega_0}) < S(\mathscr{D}_{\omega_0(2-\omega_0),\omega_0}). \tag{74}$$

From (71), (72) and (74) it follows that

$$RR(\mathscr{D}_{\tau_0,\omega_0}) < RR(\mathscr{D}_{\omega_0(2-\omega_0),\omega_0}) < RR(\mathscr{D}_{1,\omega_0}).$$

In addition, a simple study of the behavior of $S(\mathscr{D}_{\tau_0,\omega_0})$ with respect to $\underline{\mu}^2$ reveals that it is a decreasing function if $\sqrt{1+\bar{\mu}^2} < 1 + \underline{\mu}^2$. This means that the PSD method will produce a fast rate of convergence for large values of $\underline{\mu}^2$ whereas this is not the case for the SSOR method as its rate of convergence is independent of $\underline{\mu}$. Finally, recall that the reciprocal rates of the PSD and ESOR methods are the same in the imaginary case.

The aforementioned results serve to justify the claim to superiority of the PSD method when compared to the SSOR and PJ methods where the eigenvalues of the associated Jacobi iteration matrix are either all real or all imaginary.

6. Remarks and conclusions

In this paper, we analysed and compared the rates of convergence of the iterative methods PSD, SSOR and PJ under the assumptions that the associated Jacobi iteration matrix B is of a two-cyclic block form (15) and all its eigenvalues are either real or imaginary. Applying the results of [13,14] we were able to find sufficient and necessary conditions for the aforementioned iterative schemes to converge. In addition, we determined the optimum values of their parameters such that they attain their optimum rate of convergence. The conclusions from our analysis are: (i) the PSD method attains a faster rate of convergence than the SSOR and PJ methods and (ii) while the rate of convergence of the PSD method is similar to that of the EGS method if B possesses real eigenvalues, it increases by an order of magnitude and becomes equal to that of the ESOR method if B possesses imaginary eigenvalues.

The PSD method can be combined with semi-iterative methods in a way that is analogous to the development of the SSOR method. This raises the possibility that the rate of convergence of the PSD method can be further improved by the application of such acceleration devices. Where the eigenvalues of the associated Jacobi matrix are all real (the real case) it has been shown in [18,21] that this possibility does indeed occur and yields a performance equivalent to the semi-iterative Gauss–Seidel method (SI-GS). However, in the imaginary case, the problem has to be investigated further.

The use of the PSD method with complex parameters τ, ω with $\sigma(B)$, the complex spectrum of the associated Jacobi matrix B, belonging to a compact subset Σ of the complex plane $\mathbb{C}$ symmetric with respect to the origin is a problem that is yet to be investigated. An equivalent problem has been solved recently in [7] for the SOR method, where it was shown that if the outer boundary of Σ is not an ellipse then the SI-SOR method is always advantageous over the sole use of the SOR method and that if $0 \in \Sigma$ then a best choice would be an asymptotically optimum (AO) SI method based on the Gauss–Seidel method. If Σ is an ellipse, then any AOSI-SOR method is equivalent to the optimal SOR method, which means that SI acceleration does not improve the rate of convergence in this case. Furthermore, for the ellipse $E_{a,b,\theta}$ with major and minor axes $[-ae^{i\theta}, ae^{i\theta}]$ and $[-be^{i\theta+\pi/2}, be^{i\theta+\pi/2}]$, respectively, where $a \geqslant b \geqslant 0, a > 0$ and $\theta \in [0, \pi]$, which is symmetric with respect to the origin and does not contain 1, the optimal value of the SOR parameter ω_{opt} and the spectral radius $S(\mathscr{L}_{\omega_{\text{opt}}})$ of the associated SOR iteration matrix $\mathscr{L}_{\omega_{\text{opt}}}$ can be obtained via the formulas:

$$\omega_{\text{opt}} = \frac{2}{1+\sqrt{1-\mu_0^2}}, \quad S(\mathscr{L}_{\omega_{\text{opt}}}) = \left(\frac{a+b}{1+\sqrt{1-\mu_0^2}}\right)^2, \tag{75}$$

where $\Re\left(\sqrt{\cdot}\right) > 0$ and $\pm\mu_0 = \pm ce^{i\theta}$, with $c = \sqrt{a^2 - b^2}$.

Let us finally indicate how the investigation of the PSD method with complex parameters might proceed in the light of this analysis of the SOR method. Initially, one has to consider the ESOR method since both methods share the same eigenvalue relationship. As the ESOR method is a first order extrapolation of the SOR method it follows that AOSI-ESOR will be equivalent to AOSI-SOR except in case of the ellipse symmetric with respect to the origin where the ESOR method might produce a better rate of convergence than the SOR method. However, it should be noted that if $0 \in \Sigma$, then the optimal ESOR method is equivalent to the optimal SOR method as has been proved in [13,14] for the real and imaginary cases. Further, it is conjectured that the optimum ω for the ESOR method will be the same equal to that given by (75) since both methods coincide if $\tau = \omega$.

As the PSD method is a more general method than the ESOR method one would expect that its convergence would be better. Yet its optimal behavior suppresses the factor $(I - \omega U)^{-1}$ which appears in its preconditioned matrix $\mathscr{B}_\omega$ (see (5)) which reduces the optimal PSD method to the optimal ESOR method. It is conjectured that this phenomenon appears due to the cyclic nature of the matrix A and that the method will show a better convergence for non-cyclic matrices which can arise from the use of high order discretization, e.g. 9-point, 13-point stencils, in the numerical solution of partial differential equations or from the discretization of mixed derivatives.

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