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Note

A Simpler Proof and a Generalization of the Zero-Trees Theorem

A. SCHRIJVER*

Mathematical Centre, Kruislaan 413, 1098 SJ Amsterdam, The Netherlands and University of Amsterdam, The Netherlands

AND

P. D. SEYMOUR

Bellcore, 445 South Street, Morristown, New Jersey 07960

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Z. Füredi and D. J. Kleitman proved that if an integer weight is assigned to each edge of a complete graph on $p+1$ vertices, then some spanning tree has total weight divisible by p. We obtain a simpler proof by generalizing the result to hypergraphs. © 1991 Academic Press, Inc.

1. INTRODUCTION

The following theorem is due to Z. Füredi and D. J. Kleitman [2]. (It was conjectured by A. Bialostocki and P. Dierker [1], who proved the case when p is prime.)

THEOREM (1.1). *Let Γ be a finite abelian group of order p, and let w: $E(K_{p+1}) \to \Gamma$ be some function. Then there is a spanning tree T of K_{p+1} with $w(T)=0$.*

(K_n denotes the complete graph with n vertices; $E(G)$ denotes the set of edges of a graph G; $w(T)$ means $\Sigma(w(e)\colon e \in E(T))$, where the summation is in Γ.)

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We shall give a simpler proof of (1.1). For inductive purposes, it is advantageous to prove a version of (1.1) for complete uniform hypergraphs, because it is then easy to reduce the general problem to the case when p is prime.

Thus, let V be a finite set. A *hypergraph* in V is a collection of subsets of V; and it is *r-uniform* if each of these subsets has cardinality r. (In this paper, all our hypergraphs will be r-uniform for some r.) If H is a hypergraph, we denote $\bigcup\ (e\colon e \in H)$ by $V(H)$. A hypergraph T is *connected* if $T \neq \emptyset$ and for every partition (A, B) of $V(T)$ such that A and B are both nonempty there is a member $e \in T$ with $e \cap A, e \cap B$ both nonempty. It is easy to see that if T is connected and r-uniform then $|V(T)| \leqslant (r-1)\,|T| + 1$; and if equality holds we say that T is a *tree.* (If $r = 2$, this coincides with the usual definition of a tree for graphs, except for trees with $\leqslant 1$ vertex.) If H is r-uniform, and $T \subseteq H$ is a tree, we call it a *tree* of H; and if $V(T) = V(H)$ we call it a *spanning tree* of H. If V is a finite set with $|V| \geqslant r$, we denote by $\binom{V}{r}$ the collection of all r-element subsets of V. We shall prove the following generalization of (1.1).

THEOREM (1.2). *Let Γ be a finite abelian group of order p, let $r \geqslant 2$ be an integer, let V be a set of cardinality $p(r-1)+1$, and let $w\colon \binom{V}{r} \to \Gamma$ be some function. Then there is a spanning tree T of $\binom{V}{r}$ with $w(T) = 0$.*

($w(T)$ means $\Sigma(w(e)\colon e \in T)$.)

2. THE PROOF OF (1.2)

We require several lemmas. First, we shall need the following, which is a special case of the Cauchy–Davenport theorem (see [3]). (It can also be proved directly in a couple of lines, as the reader may verify.)

LEMMA (2.1). *Let p be prime, let $A \subseteq \mathbf{Z}_p$, and let $b, c \in \mathbf{Z}_p$ be distinct. If $1 \leqslant |A| \leqslant p-1$ then*

$$|\{a+b : a \in A\} \cup \{a+c : a \in A\}| > |A|.$$

If T is an r-uniform tree, we say that $f \in T$ is a *leaf* of T if there exists $u \in f$ such that $e \cap f \subseteq \{v\}$ for every $e \in T - \{f\}$. We call such an element u a *root* of the leaf e. If T, T' are trees in $\binom{V}{r}$ with leaves e, e', respectively, and $T - \{e\} = T' - \{e'\}$, we say that T' is obtained from T by *shifting a leaf.* If $T, T' \subseteq \binom{V}{r}$ are trees, we say that T is *shiftable* to T' if there is a sequence

$$T = T_1, T_2, \ldots, T_k = T'$$

of trees in $\binom{V}{r}$ such that T_{i+1} is obtained from T_i by shifting a leaf for $1 \leqslant i \leqslant k-1$. This is evidently an equivalence relation, and in fact all trees in $\binom{V}{r}$ of the same cardinality are shiftable to one another, but we only need a weaker result, the following.

LEMMA (2.2). *Let $r \geqslant 2$, $k \geqslant 1$ be integers, let $|V| \geqslant k(r-1)+2$, and let $v_0 \in V$. Let T_0 be a tree in $\binom{V}{r}$ with $|T_0| = k$. Then T_0 is shiftable to a tree T with $v_0 \notin V(T)$.*

Proof. We may assume that $k \geqslant 2$, for the result is clear if $k=1$. If T is a tree in $\binom{V}{r}$ with $v_0 \in V(T)$ and f is a leaf of T, we define $d(T, f)$ to be the unique $d \geqslant 1$ such that there is a sequence

$$v_0 = v_1, e_1, v_2, e_2, ..., v_d, e_d = f$$

satisfying

(i) $v_1, v_2, ..., v_d \in V(T)$ are all distinct, and so are $e_1, e_2, ..., e_d \in T$

(ii) $v_i \in e_{i-1}$ for $2 \leqslant i \leqslant d$, and $v_i \in e_i$ for $1 \leqslant i \leqslant d$.

Let us choose a tree T in $\binom{V}{r}$ such that T_0 is shiftable to T and $v_0 \in V(T)$, and a leaf f of T, in such a way that $d(T, f)$ is maximum. Let u be a root of f. Since $|T| \geqslant 2$ it follows that T has at least two leaves; let f' be another leaf, with root u'. Since $d(T, f') \leqslant d(T, f)$ it follows that $v_0 \notin f - \{u\}$. Choose $v \in f - \{u\}$, and let $e = (f' - \{u'\}) \cup \{v\}$. Now $T' = (T - \{f'\}) \cup \{e\}$ is shiftable from T and hence from T_0, and e is a leaf of it, and if $v_0 \notin f' - \{u'\}$ then $d(T', e) > d(T, f)$, a contradiction. Thus $v_0 \in f' - \{u'\}$ and, since $V(T) \neq V$, the result follows. ∎

Again, let $r \geqslant 2$, $k \geqslant 1$ and let $|V| \geqslant k(r-1)+1$. We say that $S \subseteq \binom{V}{r}$ is a *(V, k)-blocker* if $|S \cap T| \neq \varnothing$ for every tree T in $\binom{V}{r}$ with $|T| = k$. Our third lemma is the following.

LEMMA (2.3). *Let $r \geqslant 2$, $k \geqslant 1$ be integers, and let $|V| = k(r-1)+1$. If $S \subseteq \binom{V}{r}$ is a (V, k)-blocker then S includes a spanning tree of $\binom{V}{r}$.*

Proof. The result holds if $k=1$, and so we may assume that $k \geqslant 2$ and proceed by induction on k. Since there is a spanning tree and we may assume that it is not included in S, it follows that $\varnothing \neq S \neq \binom{V}{r}$. Thus, we may choose $e, f \in \binom{V}{r}$ with $|e \cap f| = r-1$ and $e \in S, f \notin S$. Let $V - (e \cap f) = V'$. If T' is a spanning tree of $\binom{V'}{r}$ then $T' \cup \{f\}$ is a spanning tree of $\binom{V}{r}$, and so $S \cap (T' \cup \{f\}) \neq \varnothing$, that is, $S' \cap T' \neq \varnothing$, where $S' = S \cap \binom{V'}{r}$. Hence S' is a $(V', k-1)$-blocker, and so S' includes a spanning tree T' of $\binom{V'}{r}$, from the inductive hypothesis. Then $T' \cup \{e\} \subseteq S$ is a spanning tree of $\binom{V}{r}$, as required. ∎

We shall use (2.1)–(2.3) to prove the following, which is the main step in the proof of (1.2).

LEMMA (2.4). *Let p be prime, let $k \geqslant 1$, $r \geqslant 2$ be integers with $k \leqslant p$, let V be a set of cardinality $k(r-1)+1$, and let $w\colon \binom{V}{r} \to \mathbf{Z}_p$ be some function. Then either*

(i) *there are k spanning trees $T_1, ..., T_k$ with $w(T_1), ..., w(T_k)$ all distinct, or*

(ii) *$k \geqslant 2$ and there is a monochromatic $(V, k-1)$-blocker.*

(A subset $S \subseteq \binom{V}{r}$ is *monochromatic* if the restriction of w to S is constant.)

Proof. The result holds if $k=1$, and so we may assume that $k \geqslant 2$ and proceed by induction on k. We say that $X \subseteq V$ is *joint* if $|X| = r-1$ and $X = f_1 \cap f_2$ for some $f_1, f_2 \in \binom{V}{r}$ with $w(f_1) \neq w(f_2)$. We assume that (i) is false. We may assume that

(1) *Some set $X \subseteq V$ is joint.* For $\binom{V}{r}$ is a $(V, k-1)$-blocker since $k \geqslant 2$, and so we may assume that w is non-constant on $\binom{V}{r}$, for otherwise (ii) holds. The claim follows.

(2) *If X is joint then $k \geqslant 3$ and there exists a monochromatic $(V-X, k-2)$-blocker.* For let $X \subseteq V$ be joint. Suppose that there are $k-1$ spanning trees $T_1, ..., T_{k-1}$ of $\binom{V-X}{r}$ with $w(T_1), ..., w(T_{k-1})$ all distinct. Choose $f_1, f_2 \in \binom{V}{r}$ with $f_1 \cap f_2 = X$ and $w(f_1) \neq w(f_2)$. Now $T_i \cup \{f_1\}$ and $T_i \cup \{f_2\}$ are spanning trees of $\binom{V}{r}$ for $1 \leqslant i \leqslant k-1$, and

$$|\{w(T_i) + w(f_1) : 1 \leqslant i \leqslant k-1\} \cup \{w(T_i) + w(f_2) : 1 \leqslant i \leqslant k-1\}| \geqslant k$$

by (2.1). Hence (i) holds, a contradiction. Thus, there do not exist $k-1$ such spanning trees. From our inductive hypothesis applied to $V-X$ the claim follows.

In particular, from (1) and (2) we deduce that $k \geqslant 3$. For each joint set X, let $S(X)$ be a monochromatic $(V-X, k-2)$ blocker, and let $w(e) = q(X)$ for all $e \in S(X)$.

(3) *There exists $q \in \mathbf{Z}_p$ such that $q(X) = q$ for every joint set X.* For let X_1, X_2 be joint; we shall show that $q(X_1) = q(X_2)$. Let $X_1 \cup X_2 \subseteq Z \subseteq V$, where $|Z| = 2r-2$. Now $S(X_1)$ is a $(V-X_1, k-2)$-blocker, and so $S(X_1) \cap \binom{V-Z}{r}$ is a $(V-Z, k-2)$-blocker. By (2.3), there is a spanning tree T of $\binom{V-Z}{r}$ with $T \subseteq S(X_1)$. Similarly, $S(X_2) \cap \binom{V-Z}{r}$ is a $(V-Z, k-2)$-blocker, and so $S(X_2) \cap T \neq \varnothing$. Hence, $S(X_1) \cap S(X_2) \neq \varnothing$, and the claim follows.

Let us say a tree $T \subseteq \binom{V}{r}$ is *bad* if $|T| = k-1$ and $w(e) \neq q$ for all $e \in T$.

(4) *If* f_1 *is a leaf of a bad tree* T, *and* $f_2 \in \binom{V}{r}$ *with* $|f_2 \cap V(T - \{f_1\})| \leqslant 1$, *then* $w(f_2) = w(f_1)$. For let $V' = V(T - \{f_1\})$. If $X \subseteq V - V'$ is joint then $S(X) \cap (T - \{f_1\}) \neq \varnothing$, which is impossible by (3), since T is bad. Thus no subset of $V - V'$ is joint, and the claim follows.

In particular,

(5) *If* T *is a bad tree and* T *is shiftable to* T' *then* T' *is bad.*

Now by (1), there is a joint set X. If there is a bad tree, then by $(r-1)$ applications of (2.2), it is shiftable to a tree T with $X \cap V(T) = \varnothing$; and by (5), T is bad. But then $T \cap S(X) \neq \varnothing$, a contradiction as before. We deduce that there is no bad tree, and so $\{e \in \binom{V}{r} : w(e) = q\}$ is a $(V, k-1)$-blocker. Thus (ii) holds, as required. ∎

Finally, we use (2.4) to prove (1.2).

Proof of (1.2). We proceed by induction on p. If p is prime, then $\Gamma \cong \mathbf{Z}_p$ and by (2.4) with $k = p$, either

(i) there are p spanning trees $T_1, ..., T_p$ with $w(T_1), ..., w(T_p)$ all distinct; but then one of them is zero, as required, or

(ii) for some $q \in \Gamma$ there is a $(V, p-1)$-blocker S such that $w(e) = q$ for all $e \in S$; but then S is a (V, p)-blocker and hence includes a spanning tree T, and $w(T) = \Sigma(q \colon e \in T) = 0$ as required.

We may assume then that p is not prime, and so Γ has a proper subgroup Γ', of order p' say. Let Γ'' be the quotient group Γ/Γ', of order p'' say, where $p = p'p''$, and let $\phi \colon \Gamma \to \Gamma''$ be the homomorphism with kernel Γ'. For each $e \in \binom{V}{r}$, we define $w''(e) = \phi(w(e)) \in \Gamma''$. Let $r' = p''(r-1) + 1$. For each $f \subseteq V$ with $|f| = r'$, we define $w'(f)$ as follows. From our inductive hypothesis applied to $\binom{f}{r}$, Γ'' and w'', there is a spanning tree $T(f)$ of $\binom{f}{r}$ such that $w''(T(f)) = 0$; that is, $w(T(f)) \in \Gamma'$. We define $w'(f) = w(T(f))$. From our inductive hypothesis applied to $\binom{V}{r'}$, Γ' and w', there is a spanning tree T' of $\binom{V}{r'}$ with $w'(T') = 0$. Let $T = \bigcup (T(f) : f \in T')$; then T is a spanning tree of $\binom{V}{r}$ and

$$w(T) = \sum_{f \in T'} \sum_{e \in T(f)} w(e) = \sum_{f \in T'} w'(f) = 0$$

as required. ∎

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