Jumping nonlinearities and weighted Sobolev spaces

Adolfo J. Rumbos\textsuperscript{a,\,*}, Victor L. Shapiro\textsuperscript{b}

\textsuperscript{a}Department of Mathematics and Computer Science, Pomona College 610 N, College Avenue, Claremont, CA 91711, USA
\textsuperscript{b}University of California, Riverside, CA, USA

Received 9 May 2004; revised 29 October 2004
Available online 21 December 2004

Abstract

Working in a weighted Sobolev space, a new result involving jumping nonlinearities for a semilinear elliptic boundary value problem in a bounded domain in $\mathbb{R}^N$ is established. The nonlinear part of the equation is assumed to grow at most linearly and to be at resonance with the first eigenvalue of the linear part on the right. On the left, the nonlinearity crosses over (or jumps over) several higher eigenvalues. Existence is obtained through the use of infinite-dimensional critical point theory in the context of weighted Sobolev spaces and appears to be new even for the standard Dirichlet problem for the Laplacian.

© 2004 Elsevier Inc. All rights reserved.

Keywords: Singular elliptic equations; Weighted Sobolev spaces; Resonance; Jumping nonlinearities

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, stand for either the open unit $N$-ball or the open unit $N$-cube and consider a weak solution in $W_0^{1,2}(\Omega)$ to the following Dirichlet problem

$$
\begin{cases}
-\Delta u &= \lambda_1 u - au^- + g(u) + f(x); & x \in \Omega \\
u &= 0 & \text{on } \partial \Omega,
\end{cases}
$$

\*Corresponding author. Fax: +1 909 607 1247.
E-mail address: arumbos@pomona.edu (A.J. Rumbos).

0022-0396/$ - see front matter © 2004 Elsevier Inc. All rights reserved.
doi:10.1016/j.jde.2004.11.005
where $\Delta u$ is the Laplacian of $u$, $\lambda_1$ is the first eigenvalue of the Laplacian in $\Omega$ with Dirichlet boundary conditions, $u^-$ is the negative part of $u$ (i.e., $u^- = \max\{-u, 0\}$), $a$ is a positive constant, $g$ is a real-valued function in $L^\infty(\mathbb{R})$ possessing a limit at $+\infty$ denoted by $g_+$, and $f \in L^2(\Omega)$. We show that a weak solution to this Dirichlet problem in $\Omega$ exists provided that

$$\int_{\Omega} [g_+ + f(x)]\phi_1(x) \, dx > 0,$$

where $\phi_1$ is the positive eigenfunction of the Laplacian in $\Omega$ corresponding to $\lambda_1$. This result is known to hold if $f \in L^p(\Omega)$ with $p > N$ and $\Omega$ has smooth enough boundary; for example, see [1, p. 115] for the case in which $\Omega$ is the unit $N$-ball. However, the result presented in [1] does not apply to the case in which $\Omega$ is the unit $N$-cube even for $f \in L^p(\Omega)$ and $p > N$ (see [1, p. 114]). The main difficulty is that the proof in [1] (see also [2, p. 294]) depends upon the strong maximum principle. We avoid this difficulty by presenting a variational proof using a linking argument combined with the Deformation theorem in [8, p. 82]. Because our proof is variational, it goes over to weighted Sobolev spaces and singular elliptic partial differential equations and we shall present our result in that setting. For a reference to other papers in the literature that deal with a similar type problem to the one presented above, we refer the reader to the bibliography in [1].

For the more general setting, let $\Omega$ denote a bounded domain (open and connected subset) in $\mathbb{R}^N$ and let $p_1, p_2, \ldots, p_N$ and $\rho$ denote continuous functions on $\Omega$. Assume that $\rho > 0$, $p_j > 0$, $j = 1, 2, \ldots, N$ in $\Omega$ and that

$$\int_{\Omega} \rho < \infty \quad \text{and} \quad \int_{\Omega} p_j < \infty, \quad j = 1, 2, \ldots, N. \quad (1)$$

We define the linear differential operator

$$Lu = -D_i(p_i^{1/2} p_j^{1/2} b_{ij} D_j u) + \rho cu \quad (2)$$

acting on real-valued functions $u = u(x)$ defined in $\Omega$, and satisfying the following conditions:

(L-1). $b_{ij}$ and $c$ are functions in $L^\infty(\Omega)$, for $i, j \in \{1, 2, \ldots, N\}$, with $c \geq 0$ a.e. in $\Omega$;
(L-2). the matrix $(b_{ij})$ is symmetric;
(L-3). there exists a constant $c_0 > 0$ for which

$$b_{ij}(x)\xi_i \xi_j \geq c_0 |\xi|^2 \quad \text{for } x \in \Omega \text{ and } \xi \in \mathbb{R}^N. \quad (3)$$

In Eqs. (2) and (3), we have used the summation convention for repeated indices $i, j = 1, 2, \ldots, N$. We shall follow this convention throughout this paper.
Because the $p_j$’s may tend to zero on all or part of the boundary of $\Omega$, $L$ given by (2) may be a singular elliptic operator.

Let $\Gamma \subseteq \partial \Omega$ be a fixed closed set (it may be the empty set). Denote by $p$ the vector function $(p_1, p_2, \ldots, p_N)$. We consider the following pre-Hilbert spaces:

$$C^0_p(\Omega) = \left\{ u \in C^0(\Omega) \mid \int_\Omega |u|^2 \rho < \infty \right\}$$

with inner-product $\langle u, v \rangle_p = \int_\Omega u v \rho$ for all $u, v \in C^0_p(\Omega)$;

$$C^1_{p,\rho}(\Omega, \Gamma) = \left\{ u \in C^0(\overline{\Omega}) \cap C^2(\Omega) \mid u = 0 \text{ on } \Gamma, \text{ and } \int_\Omega (|D_i u|^2 p_i + \rho |u|^2) < \infty \right\}$$

with inner-product

$$\langle u, v \rangle_{p,\rho} = \int_\Omega (D_i u)(D_i v) p_i + \int_\Omega \rho u$$

for all $u, v \in C^1_{p,\rho}(\Omega, \Gamma)$.

Let $L^2_{\rho}(\Omega)$ denote the Hilbert space obtained through completion of $C^0_p(\Omega)$ using the method of Cauchy sequences with respect to the norm $\|u\|_\rho = \langle u, u \rangle_p^{1/2}$. Similarly, let $H^1_{p,\rho}(\Omega, \Gamma)$ denote the completion of the space $C^1_{p,\rho}(\Omega, \Gamma)$ with respect to the norm $\|u\|_{p,\rho} = \langle u, u \rangle_{p,\rho}^{1/2}$. The latter is an example of a weighted Sobolev space (see [6]). It is identical with the space $H^1_{p,q,\rho}(\Omega, \Gamma)$ given in [10, p. 2], with $q = \rho$.

We associate with the linear differential operator $L$ a bilinear form on $H^1_{p,\rho}(\Omega, \Gamma) \times H^1_{p,\rho}(\Omega, \Gamma)$ given by

$$\mathcal{L}(u, v) = \int_\Omega \left[ p_i^{1/2} p_j^{1/2} b_{ij} D_i u D_j v + c \rho u v \right], \quad \forall u, v \in H^1_{p,\rho}(\Omega, \Gamma).$$

The assumptions on $\rho$ and the $p_j$’s in conjunction with (L-1) imply that $\mathcal{L}(\cdot, \cdot)$ is well defined.

We say that $\lambda \in \mathbb{R}$ is an eigenvalue for $L$, with corresponding eigenfunction $\varphi \in H^1_{p,\rho}(\Omega, \Gamma)$, if $\varphi \not\equiv 0$ and $\mathcal{L}(\varphi, v) = \lambda \langle \varphi, v \rangle_{\rho}$ for all $v \in H^1_{p,\rho}(\Omega, \Gamma)$. We shall assume that the given domain $\Omega$ and operator $L$ satisfy the following conditions (O$_1$)–(O$_3$), which we shall refer to as $V_L(\Omega, \Gamma)$:

(O$_1$). There exists a complete orthonormal sequence of functions $\{\varphi_n\}_{n=1}^\infty$ in $L^2_{\rho}(\Omega)$, such that $\varphi_n \in C^2(\Omega) \cap L^\infty(\Omega)$ and $\varphi_n \in H^1_{p,\rho}(\Omega, \Gamma)$ for all $n$. 


There exists a sequence of real eigenvalues, \( \{ \lambda_n \} \), corresponding to the orthonormal sequence \( \{ \varphi_n \} \), and satisfying

\[
0 \leq \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n \leq \lambda_{n+1} \to \infty \quad \text{as} \quad n \to \infty;
\]

thus,

\[
\mathcal{L}(\varphi_n, v) = \lambda_n \langle \varphi_n, v \rangle_p \quad \text{for all} \quad v \in H^1_{p, \rho}(\Omega, I), \quad \text{and} \quad n \geq 1;
\]

\((O_3)\). \( \lambda_1 \) is a simple eigenvalue, and \( \varphi_1 \) can be taken to be positive in \( \Omega \).

The only place where the \( V_L(\Omega, I) \) conditions in this paper differ from the definition of a \( V_L \)-region in \([10, \text{p. 3}]\) is in the assumption that \( \varphi_n \in L^\infty(\Omega) \) for all \( n \).

As a simple example of a domain \( \Omega \) and operator \( L \) for which the conditions \( V_L(\Omega, I) \) hold, consider \( \Omega = (-1, 1) \times (0, 1) \subseteq \mathbb{R}^2 \). We take \( L \) with \( b_{ij} = 1 \) if \( i = j, \) \( b_{ij} = 0 \) if \( i \neq j, i, j = 1, 2, \) \( c = 0, \) \( p_1(x_1, x_2) = (1 - x_1^2)x_2, \) \( p_2(x_1, x_2) = x_2, \) and weight \( \rho(x_1, x_2) = x_2 \) for \( (x_1, x_2) \in \Omega \). Take \( I \) to be the top edge of the rectangle \( \partial \Omega \). Then, the bilinear form in (6) is given by

\[
\mathcal{L}(u, v) = \int_{\Omega} \left[ (1 - x_1^2)x_2D_1uD_1v + x_2D_2uD_2v \right] \quad (7)
\]

for all \( u, v \in H^1_{p, \rho}(\Omega, I) \).

Put \( \varphi_{mn} = P_n(x_1) \hat{J}_o(\eta_m x_2) \) for \( n = 0, 1, 2, \ldots, \) \( m = 1, 2, 3, \ldots, \) and \( (x_1, x_2) \in \Omega \); where, \( P_n(t) \) is the \( n \)th degree Legendre polynomial defined by

\[
P_n(t) = \frac{(-1)^n}{2^n n!} \left( \frac{d}{dt} \right)^n [(1 - t^2)^n], \quad \text{for} \quad n = 0, 1, 2, \ldots;
\]

\[
\hat{J}_o(\eta_m t) = \frac{J_o(\eta_m t)}{\left( \frac{1}{2} J_1(\eta_m) \right)^{1/2}},
\]

\( J_o \) and \( J_1 \) being the familiar Bessel functions and \( \eta_m \) the \( m \)th positive zero of \( J_o(t) \). It is well known (see for example, \([4]\)) that \( P_n(t) \) and \( \hat{J}_o(\eta_m t) \) satisfy the second order ordinary differential equations

\[
\frac{d}{dt} \left[ (1 - t^2) \frac{d}{dt} P_n(t) \right] = -n(n + 1) P_n(t) \quad (8)
\]

and

\[
\frac{d}{dt} \left[ t \frac{d}{dt} \hat{J}_o(\eta_m t) \right] = -\eta_m^2 \hat{J}_o(\eta_m t), \quad (9)
\]
respectively. Moreover, \( \{P_n(t)\}_{n=0}^\infty \) forms a complete orthonormal system (CONS) in \( L^2([-1, 1]) \), and \( \{J_0(\eta_m t)\}_{m=1}^\infty \) forms a CONS with weight \( t \) on \([0, 1]\). Consequently, the system \( \{q_{mn}\}_{m=1,n=0}^\infty \) forms a CONS for \( L^2_\rho(\Omega) \), where the weight is \( \rho(x_1, x_2) = x_2 \). We therefore see that (O_1) holds. Next, observe that, by virtue of (7), for any \( v \in H^1_{p,\rho}(\Omega, \Gamma) \),

\[
\mathfrak{L}(\varphi_{mn}, v) = \int_\Omega \left[ (1 - x_1^2) P_n'(x_1)x_2 \widehat{J}_0(\eta_m x_2) \frac{\partial v}{\partial x_1} + P_n(x_1)x_2 \frac{d}{dx_2} \left( \widehat{J}_0(\eta_m x_2) \right) \frac{\partial v}{\partial x_2} \right].
\]

(10)

Integration by parts, together with (8) and (9), yields from (10) above that

\[
\mathfrak{L}(\varphi_{mn}, v) = [n(n + 1) + \eta_m^2] \int_\Omega x_2 \varphi_{mn} v
\]

for all \( v \in H^1_{p,\rho}(\Omega, \Gamma) \). Therefore, (O_2) holds with \( \lambda_1 = \eta_1^2 \), \( \lambda_2 = \eta_1^2 + 2 \), \( \lambda_3 = \eta_2^2 \), and so on. Finally, \( \varphi_{1,0}(x_1, x_2) = \widehat{J}_0(\eta_1 x_2) > 0 \) on \( \Omega \); thus, (O_3) also holds.

For additional examples of operators and domains for which the conditions \( V_L(\Omega, \Gamma) \) hold, the reader is referred to [9, pp. 1413–1415] and [10, pp. 20–26].

We study the following problem:

\[
\begin{align*}
Lu - \lambda_1 \rho u &= -a(x, u) \rho u^- + \rho g(x, u) + h; \quad x \in \Omega \\
u &= \in H^1_{p,\rho}(\Omega, \Gamma)
\end{align*}
\]

(11)

where \( h \in H^1_{p,\rho}(\Omega, \Gamma)^* \) (the dual of \( H^1_{p,\rho}(\Omega, \Gamma) \)) and \( u^- = \max\{0, -u\} \) in \( \Omega \).

We shall assume the following conditions for \( a \) and \( g \):

(a-1). \( a(x, s) \) is a Carathéodory real-valued function; i.e., for each \( s \in \mathbb{R} \) the function \( x \mapsto a(x, s) \) is measurable in \( \Omega \); and for a.e. \( x \in \Omega \) the map \( s \mapsto a(x, s) \) is continuous on \( \mathbb{R} \).

(a-2). \( q_1(x) \leq a(x, s) \leq q_2(x) \) for a.e. \( x \in \Omega \) and \( s \leq 0 \); where \( q_1, q_2 \in L^\infty(\Omega) \). Assume also that \( q_1(x) \geq a_1 \) for \( x \in \Omega \) and some constant \( a_1 > 0 \).

(a-3). Set \( A(x, t) = \int_0^t a(x, s) s \, ds \) for \( t \leq 0 \), and \( A(x, t) = 0 \) for \( t > 0 \). Assume there exists \( b^* \in L^2_{p,\rho}(\Omega) \) such that \( b^* \geq 0 \) a.e. and

\[
2A(x, t) - a(x, t)t^2 \geq - b^*(x)|t|
\]

for a.e. \( x \in \Omega \) and \( t \leq 0 \).

(g-1). \( g(x, s) \) is Carathéodory real-valued function as in (a-1) above.
(g-2). There exists $b \in L^2_\rho(\Omega)$ with $b \geq 0$ a.e. in $\Omega$ such that

$$|g(x, s)| \leq b(x) \text{ for a.e. } x \in \Omega \text{ and } s \geq 0.$$  

(g-3). For every $\eta > 0$, there exists $b_{\eta} \in L^2_\rho(\Omega)$ with $b_{\eta} \geq 0$ a.e. in $\Omega$ and

$$|g(x, s)| \leq \eta |s| + b_{\eta}(x) \text{ for a.e. } x \in \Omega \text{ and } s < 0.$$  

(g-4). Set $G(x, t) = \int_0^t g(x, s) \, ds$ for $t \in \mathbb{R}$. Suppose there exists $b^{**} \in L^2_\rho(\Omega)$ such that $b^{**} \geq 0$ a.e. and

$$2G(x, t) - g(x, t)t \geq -b^{**}(x)|t|$$

for a.e. $x \in \Omega$ and $t \leq 0$.

(g-5). Assume that there exists $g_+ \in L^\infty(\Omega)$ such that

$$\lim_{s \to +\infty} g(x, s) = g_+(x) \text{ for a.e. } x \in \Omega.$$  

By a weak solution of problem (11) we shall mean a function $u$ in the space $H^{1}_{\rho, \rho}(\Omega, \Gamma)$ for which

$$\mathcal{L}(u, v) - \tilde{\omega}_1 \langle v, \cdot \rangle_\rho = -\langle a(\cdot, u)u^-, v \rangle_\rho + \langle g(\cdot, u), v \rangle_\rho + h(v)$$  

(12)

for all $v \in H^{1}_{\rho, \rho}(\Omega, \Gamma)$.

Since, by (a-2), (g-2) and (g-3),

$$\lim_{s \to +\infty} \frac{-a(x, s)s^- + g(x, s)}{s} = 0$$

for a.e. $x \in \Omega$, and

$$\liminf_{s \to -\infty} \frac{-a(x, s)s^- + g(x, s)}{s} \geq a_1 > 0$$

for a.e. $x \in \Omega$, problem (11) involves a jumping nonlinearity which is resonant on the right. Existence results for this type of problems at resonance are usually obtained by imposing certain solvability conditions on the nonlinearity and on $h$. These conditions usually prescribe the kind of asymptotic interaction that the nonlinearity and $h$ can
have with the eigenspace corresponding to $\lambda_1$. In this work we shall impose on $g$ and $h$ a variant of the condition used by Landesman and Lazer [7] in their 1970 paper:

$$\int_{\Omega} g_+(x)\varphi_1(x)\rho(x)\,dx + h(\varphi_1) > 0, \quad (13)$$

where $g_+$ is as given in (g-5).

The main result in this paper is the following

**Theorem 1.1.** Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain and let $p$ and $\rho$ satisfy (1). Assume also that $\Omega$ and the operator $L$ satisfy the conditions $\mathbf{V}_L(\Omega, \Gamma)$. Suppose that $L$ satisfies (L-1)–(L-3), $a$ satisfies (a-1)–(a-3) with $q_1$ in (a-2) satisfying $q_1(x) \geq a_1$ for some constant $a_1 > 0$. Suppose also that $g$ satisfies (g-1)–(g-5) and that the solvability condition (13) holds. Then, problem (11) has at least one weak solution.

If we assume that, in addition to (g-1)–(g-5), the nonlinearity $g$ also satisfies

(i) (g-6) $g(x, s) < g_+(x)$ for a.e. $x \in \Omega$ and $s \in \mathbb{R}$,

then condition (13) is also necessary for the solvability of (11). We therefore have

**Theorem 1.2.** In addition to the assumptions of Theorem 1, suppose also that $g$ satisfies (g-6). Then, condition (13) is both necessary and sufficient for the solvability of problem (11).

**Proof of the necessity of condition (13).** Suppose that $u \in H^{1}_{p, \rho}(\Omega, \Gamma)$ is a weak solution of (11). Then, by (12) with $v = \varphi_1$, we obtain

$$0 = -\langle a(\cdot, u)u^-, \varphi_1 \rangle_\rho + \langle g(\cdot, u), \varphi_1 \rangle_\rho + h(\varphi_1).$$

Thus, since $\langle a(\cdot, u)u^-, \varphi_1 \rangle_\rho \geq 0$, we get

$$\langle g(\cdot, u), \varphi_1 \rangle_\rho + h(\varphi_1) \geq 0. \quad (14)$$

Using (g-6) we obtain $\langle g(\cdot, u), \varphi_1 \rangle_\rho < \langle g_+, \varphi_1 \rangle_\rho$, from which condition (13) follows by virtue of (14). $\Box$

**Remark 1.3.** Our proof of Theorem 1 is variational in nature; it is based on a linking argument. In the non-singular partial differential equations setting, both Dancer in [2] and Berestycki and De Figueiredo in [1] use degree theory. Even in the non-singular setting, the variational aspect of our approach gives improvements of the results in [2,1]. For example, our result applies to a larger class of domains since we do not require regularity of the boundary. Regularity of the domain is required in both [2,1] in order to apply the strong maximum principle needed to obtain the a priori estimates.
called for in the degree theoretic approach. Also, we do not require that \( h \) be given by an \( L^p \)-function where \( p > N \).

**Remark 1.4.** We now present examples of functions \( a \) and \( g \) to which Theorem 1 applies. An example of a function which meets (a-1)–(a-3) is the following: let \( f \in L^\infty(\Omega) \) be such that \( 0 < a_1 \leq f(x) \leq q_2(x) \) for a.e. \( x \in \Omega \). Let \( \varepsilon_1 \) be a fixed number with \( 0 \leq \varepsilon_1 < 1 \). Then the function

\[
a(x, s) = f(x) \left[ 1 - \frac{\varepsilon_1 |s|}{|s| + 1} \right]
\]

for \( x \in \Omega \) and \( s \in \mathbb{R} \) meets conditions (a-1)–(a-3). (Note that \( \varepsilon_1 \) could be zero, thus, in particular, \( a(x, s) \equiv a_1 \) would also work.)

Examples of functions which meet both (g-3) and (g-4) are the following:

1. \( |g(x, s)| \leq b^{**} \) for a.e. \( x \in \Omega \) and \( s \leq 0 \), where \( b^{**} \in L^2(\Omega) \).
2. \( g(x, s) = -|s|^\beta \) where \( 0 \leq \beta < 1 \) for \( s \leq 0 \).
3. \( g(x, s) = \begin{cases} -1 & \text{for } -1 \leq s \leq 0, \\ s/(1 + \ln |s|) & \text{for } s \leq -1. \end{cases} \)
4. \( g(x, s) = \gamma_1 g_1(x, s) + \gamma_2 g_2(x, s) + \gamma_3 g_3(x, s) \) where \( \gamma_j \geq 0 \) and \( g_j(x, s) \) is a function as in (j) above for \( j = 1, 2, 3 \).

We leave to the reader the details of constructing the corresponding \( g(x, s) \) for \( s > 0 \).

### 2. Preliminary lemmas

In this section, we establish some preliminary lemmas and derive a few important consequences of the conditions \( V_L(\Omega, \Gamma) \) which will be needed in the variational argument of the next section. Throughout this section we will assume that \( \Omega \subseteq \mathbb{R}^N \) is a bounded domain, \( p \) and \( \rho \) satisfy (1), \( L \) satisfy the conditions (L-1)–(L-3), and that the \( V_L(\Omega, \Gamma) \) conditions hold. We will also assume throughout the rest of the paper that

\[
c \geq 1 \quad \text{a.e. in } \Omega,
\]

where \( c \) is as given in (L-1). There is no loss of generality in making this assumption because of the fact that a solution of the equation

\[
Lu + \rho u = (\lambda_1 + 1)\rho u - a(x, u)\rho u^\circ + \rho g(x, u) + h
\]

is also a solution of the equation in problem (11). Denoting \( Lu + \rho u \) by \( L_1 u \), we see from (6) that its corresponding bilinear form \( \mathcal{E}_1 \) is given by \( \mathcal{E}_1(u, v) = \mathcal{E}(u, v) + \langle u, v \rangle_\rho \) for all \( u, v \in H^1_{p, \rho}(\Omega, \Gamma) \). It then follows that the eigenfunctions of \( L_1 \) and \( L \) are the
same, and that the eigenvalues of $L_1$ are those of $L$ shifted by 1. Observe also that, by (2),

$$L_1u = -D_i p_i^{1/2} p_j^{1/2} b_{ij} D_j u + \rho (c + 1) u.$$ 

Thus, the coefficient of $\rho u$ in $L_1$ is $c(x) + 1$ for which (15) clearly holds. We may therefore work with $L_1$ instead of $L$, if necessary.

We begin with the following estimate for $\mathcal{E}(u, u)$:

**Lemma 2.1.** Let $\mathcal{E}$ be as defined in (6) and assume that (15) holds; then,

$$\mathcal{E}(u, u) \geq c_1 ||u||_{p, \rho}^2 \quad \forall u \in H^1_{p, \rho}(\Omega, \Gamma)$$

for some positive constant $c_1$.

**Proof.** The estimate in (16) follows from (6), the ellipticity condition in (L-3), and (15); in fact, for $u \in H^1_{p, \rho}(\Omega, \Gamma)$,

$$\mathcal{E}(u, u) = \int_{\Omega} [b_{ij} (p_j^{1/2} D_j u) (p_i^{1/2} D_i u) + c \rho u^2]$$

$$\geq \int_{\Omega} [c_0 p_j |D_j u|^2 + \rho u^2]$$

$$\geq c_1 \int_{\Omega} [p_j |D_j u|^2 + \rho u^2],$$

where $c_1 = \min \{c_0, 1\}$. □

Similarly, an upper estimate for $\mathcal{E}(u, u)$ follows from (6), the Cauchy–Schwarz inequality, and the facts that $b_{ij}$ and $c$ are $L^\infty(\Omega)$ functions, according to (L-1):

$$\mathcal{E}(u, u) \leq c_2 ||u||_{p, \rho}^2 \quad \forall u \in H^1_{p, \rho}(\Omega, \Gamma)$$

for some positive constant $c_2$. Taken together, (16) and (17) imply that $\mathcal{E}(\cdot, \cdot)$ defines a real inner product in $H^1_{p, \rho}(\Omega, \Gamma)$ equivalent to $\langle \cdot, \cdot \rangle_{p, \rho}$.

The following are important consequences of the $V_L(\Omega, \Gamma)$ conditions:

(i) For any $v \in L^2_\rho(\Omega)$, let $\tilde{v}(n) = \langle v, \varphi_n \rangle_\rho$ for each $n = 1, 2, 3, \ldots$. Then, for every $v, w \in L^2_\rho(\Omega)$

$$\langle v, w \rangle_\rho = \sum_{n=1}^{\infty} \tilde{v}(n) \tilde{w}(n).$$
This follows from the completeness of the orthonormal system \( \{ \varphi_n \}_{n=1}^{\infty} \) in \( L_2^\rho(\Omega) \) given in (O_1).

(ii) It follows from (O_2), (O_1) and Lemma 1 that \( \langle \varphi_n, \varphi_n \rangle = \lambda_n > 0 \) for all \( n = 1, 2, 3, \ldots \). Also, by (O_2), if \( v \in H^1_{p,\rho}(\Omega, \Gamma) \) is such that \( \langle \varphi_n, v \rangle = 0 \) for all \( n \), then \( \langle \varphi_n, v \rangle_\rho = 0 \) for all \( n \). Consequently, since \( \{ \varphi_n \}_{n=1}^{\infty} \) is a CONS in \( L_2^\rho(\Omega) \), we get that \( v = 0 \) a.e. in \( \Omega \). Therefore, \( \{ \varphi_n / \sqrt{\lambda_n} \}_{n=1}^{\infty} \) constitutes a complete orthonormal system for \( H^1_{p,\rho}(\Omega, \Gamma) \) with respect to the inner product \( \langle \cdot, \cdot \rangle \). Consequently, it follows that

\[
\langle \varphi_n, v \rangle_\rho = 0 \quad \text{for all } n = 1, 2, 3, \ldots.
\]

(iii) For future reference, we state the following fact

\[
\lambda_1 = \inf_{u \in H^1_{p,\rho}(\Omega, \Gamma) \setminus \{0\}} \frac{\|u\|_{p,\rho}^p}{\langle u, u \rangle_\rho},
\]

which is a straightforward consequence of (O_2), (18) and (19).

In the variational argument of the next section, a crucial role is played by the compactness of the embedding \( H^1_{p,\rho}(\Omega, \Gamma) \hookrightarrow L_2^\rho(\Omega) \). This result, which can be viewed as a weighted Sobolev spaces version of the Rellich–Kondrasov theorem (see [5, p. 84], [3, p. 305]), is presented as Lemma 2.3 below. In the proof of Lemma 2.3, we will need the following lemma, a version of which may be found as Lemma 1 in [10, p. 37].

**Lemma 2.2.** Assume the \( V_L(\Omega, \Gamma) \) conditions and that \( v \in L_2^\rho(\Omega) \). Put \( \widehat{v}(n) = \langle \varphi_n, v \rangle_\rho \) for each \( n = 1, 2, \ldots \). Then, \( v \in H^1_{p,\rho}(\Omega, \Gamma) \) if and only if

\[
\sum_{n=1}^{\infty} \lambda_n |\widehat{v}(n)|^2 < \infty.
\]

**Proof.** If \( v \in H^1_{p,\rho}(\Omega, \Gamma) \), then \( ||v||_{p,\rho} < \infty \) and so, by (17), \( \|v\|_{p,\rho} < \infty \), which yields (21) through application of (19).

Conversely, suppose that (21) holds for \( v \in L_2^\rho(\Omega) \) and set

\[
w_n = \sum_{k=1}^{n} \widehat{v}(k) \varphi_k \quad \text{for } n = 1, 2, \ldots.
\]
Then, each \( w_n \) is in \( H_{p, \rho}^1(\Omega, \Gamma) \) and for \( m > n \), using (19),

\[
\mathcal{L}(w_m - w_n, w_m - w_n) = \sum_{k=n+1}^{m} \lambda_k |\hat{v}(n)|^2.
\]

Hence, it follows from (21) and Lemma 2.1 that \( \{w_n\} \) is a Cauchy sequence in the Hilbert space \( H_{p, \rho}^1(\Omega, \Gamma) \). Therefore, there exists \( w \in H_{p, \rho}^1(\Omega, \Gamma) \) such that \( ||w_n - w||_{p, \rho} \to 0 \) as \( n \to \infty \), which implies that \( ||w_n - w||_{\rho} \to 0 \) as \( n \to \infty \). Consequently, since \( ||w_n - v||_{\rho} \to 0 \) as \( n \to \infty \), it follows that \( v = w \) almost everywhere and we conclude that \( v \in H_{p, \rho}^1(\Omega, \Gamma) \). □

The next lemma is essentially Lemma 2 in [10, p. 38]. We omit the proof and refer the reader to the last named reference.

**Lemma 2.3.** Assume the \( V_L(\Omega, \Gamma) \) conditions. Then, \( H_{p, \rho}^1(\Omega, \Gamma) \) is compactly embedded in \( L_{\rho}^2(\Omega) \).

### 3. Variational setting

The main goal of this section is to set the stage for the proof of Theorem 1.1, which will be presented in Section 4.

Define a functional \( J \) on \( H_{p, \rho}^1(\Omega, \Gamma) \) as follows:

\[
J(u) = \frac{1}{2} \int_\Omega \rho \left( u - \frac{\lambda_1}{2} \int_\Omega \rho u^2 - \int_\Omega \rho(x)[A(x, u) + G(x, u)] - h(u) \right) \, dx.
\]

for all \( u \in H_{p, \rho}^1(\Omega, \Gamma) \), where \( G \) is as in (g-4) and \( A \) is given by

\[
A(x, s) = \begin{cases} 
\int_0^s a(x, t) \, dt & \text{for } s \leq 0, \\
0 & \text{for } s > 0.
\end{cases}
\]

Observe that by (g-2), (g-3) and the definition of \( G \) in (g-4), for any \( \eta > 0 \) there exists \( b_\eta \in L_{\rho}^2(\Omega) \) with \( b_\eta \geq 0 \) a.e. in \( \Omega \), and

\[
|G(x, s)| \leq \begin{cases} 
\frac{\eta}{2} s^2 + b_\eta(x)|s| & \text{for } s \leq 0, \\
b(x)s & \text{for } s > 0
\end{cases}
\]

for a.e. \( x \in \Omega \), where \( b \) is a non-negative function in \( L_{\rho}^2(\Omega) \).

Consequently, it follows from (a-2), (23), (24), and the Cauchy–Schwarz inequality that \( J \) in (22) is well defined for all \( u \in H_{p, \rho}^1(\Omega, \Gamma) \). One can also show that
\( J \in C^1(H^1_{p, \rho}(\Omega, \Gamma), \mathbb{R}) \) and that

\[
J'(u)v = \mathcal{L}(u, v) - \langle \lambda_1 u - a(\cdot, u)u^- + g(\cdot, u), v \rangle_\rho - h(v)
\]  

(25)

for all \( u, v \in H^1_{p, \rho}(\Omega, \Gamma) \). If \( u \in H^1_{p, \rho}(\Omega, \Gamma) \) is a critical point of \( J \), i.e., \( J'(u)v = 0 \) for all \( v \in H^1_{p, \rho}(\Omega, \Gamma) \), we see from (25) above that Eq. (12) holds, and therefore \( u \) is a weak solution of problem (11). Consequently, we can establish Theorem 1.1 by showing the existence of critical points of \( J \) defined in (22).

To prove the existence of critical points of \( J \), we will use the Deformation theorem in [8, p. 82]. For this purpose we will first need to show that, under the assumptions in Theorem 1.1, the functional \( J \) defined in (22) satisfies the Palais–Smale condition (subsequently referred to as (PS)):

\[
(PS) \left\{ \begin{array}{l}
\text{every sequence } \{u_n\} \subset H^1_{p, \rho}(\Omega, \Gamma) \text{ for which:} \\
(i) \{J(u_n)\}_{n=1}^\infty \text{ is bounded, and} \\
(ii) J'(u_n) \to 0 \text{ in norm as } n \to \infty, \\
\text{has a strongly convergent subsequence.}
\end{array} \right.
\]

Lemma 3.1. Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain and let \( p \) and \( \rho \) satisfy (1). Assume also that \( \Omega \) and the operator \( L \) satisfy the conditions \( \mathbf{V}_L(\Omega, \Gamma) \). Suppose that \( L \) satisfies (L-1)–(L-3), \( a \) satisfies (a-1)–(a-3) with \( q_1 \) in (a-2) satisfying \( q_1(x) \geq a_1 \) for some constant \( a_1 > 0 \). Suppose also that \( g \) satisfies (g-1)–(g-5) and that the solvability condition (13) holds. Then, the functional \( J \) defined in (22) satisfies (PS).

Proof. Let \( \{u_n\}_{n=1}^\infty \) be a sequence in \( H^1_{p, \rho}(\Omega, \Gamma) \) with \( \{J(u_n)\}_{n=1}^\infty \) bounded and \( J'(u_n) \to 0 \) in norm as \( n \to \infty \). Since \( \mathcal{L}(\cdot, \cdot) \) defines an inner product in \( H^1_{p, \rho}(\Omega, \Gamma) \) equivalent to \( \langle \cdot, \cdot \rangle_{p, \rho} \), and \( J'(u) \in H^1_{p, \rho}(\Omega, \Gamma)^* \) for each \( u \), there exists \( \nabla J(u) \in H^1_{p, \rho}(\Omega, \Gamma) \) such that

\[
J'(u)v = \mathcal{L}(\nabla J(u), v) \quad \text{for all } v \in H^1_{p, \rho}(\Omega, \Gamma).
\]  

(26)

The fact that \( J \in C^1(H^1_{p, \rho}(\Omega, \Gamma), \mathbb{R}) \) implies that the map

\[
\nabla J: H^1_{p, \rho}(\Omega, \Gamma) \to H^1_{p, \rho}(\Omega, \Gamma)
\]

is continuous. Similarly, since by virtue of the growth conditions for \( a \) and \( g \) in (a-1), (a-2), (g-1)–(g-3), the map

\[
v \mapsto \langle \lambda_1 u - a(\cdot, u)u^- + g(\cdot, u), v \rangle_\rho + h(v) \quad \text{for all } v \in H^1_{p, \rho}(\Omega, \Gamma)
\]
defines an element in $H_{p,\rho}^1(\Omega, \Gamma)^*$ for each $u$, we obtain $N(u) \in H_{p,\rho}^1(\Omega, \Gamma)$ such that

$$\langle \lambda u - a(\cdot, u)u^- + g(\cdot, u), v \rangle\rho + h(v) = \mathcal{E}(N(u), v) \quad \forall \ v \in H_{p,\rho}^1(\Omega, \Gamma).$$  \hspace{1cm} (27)$$

The compactness of the embedding $H_{p,\rho}^1(\Omega, \Gamma) \hookrightarrow L_p^2(\Omega)$ given by Lemma 2.3 implies, by virtue of (27), that $N$ is a completely continuous operator on $H_{p,\rho}^1(\Omega, \Gamma)$. It therefore follows from (25)–(27) that $\nabla J$ is of the form $I - N$, where $N$ maps bounded sets to relatively compact sets in $H_{p,\rho}^1(\Omega, \Gamma)$. Hence, in order to establish (PS), it suffices to prove that $\{u_n\}_{n=1}^\infty$ is bounded in $H_{p,\rho}^1(\Omega, \Gamma)$ (see Proposition 2.2, p. 71, in [11]).

Suppose to the contrary that, for a subsequence if necessary, $||u_n||_{p,\rho} \to \infty$ as $n \to \infty$. It then follows from Lemma 2.1 that

$$\lim_{n \to \infty} \mathcal{E}(u_n, u_n) = \infty.$$  \hspace{1cm} (28)

Put

$$w_n = u_n[\mathcal{E}(u_n, u_n)]^{\frac{1}{2}} \quad \text{for } n = 1, 2, \ldots.$$  \hspace{1cm} (29)

Then $\mathcal{E}(w_n, w_n) = 1$ for all $n$, which by Lemma 2.1 implies that $||w_n||_{p,\rho}^2 \leq 1/c_1$ for every $n$. Hence, recalling that $\mathcal{E}(\cdot, \cdot)$ defines a real inner product in $H_{p,\rho}^1(\Omega, \Gamma)$, we obtain that, passing to subsequences if necessary, there exists $w \in H_{p,\rho}^1(\Omega, \Gamma)$ for which

$$\lim_{n \to \infty} \mathcal{E}(w_n, v) = \mathcal{E}(w, v) \quad \text{for all } v \in H_{p,\rho}^1(\Omega, \Gamma),$$  \hspace{1cm} (30)

$$w_n \rightharpoonup w \quad \text{weakly in } H_{p,\rho}^1(\Omega, \Gamma),$$  \hspace{1cm} (31)

$$w_n \to w \quad \text{in } L_p^2(\Omega) \text{ and}$$  \hspace{1cm} (32)

$$w_n(x) \to w(x) \quad \text{for a.e. } x \in \Omega,$$  \hspace{1cm} (33)

$$\exists \ W_1 \in L_p^2(\Omega) \quad \text{s.t. } |w_n(x)| \leq W_1(x) \quad \text{for a.e. } x \in \Omega.$$  \hspace{1cm} (34)

From the assumption $J'(u_n) \to 0$ as $n \to \infty$ we obtain that

$$\frac{J'(u_n)v}{\mathcal{E}(u_n, u_n)^{\frac{1}{2}}} \to 0 \quad \text{as } n \to \infty \text{ for all } v \in H_{p,\rho}^1(\Omega, \Gamma),$$
thus, using (25) with \( u = u_n \), after dividing by \( \mathcal{L}(u_n, u_n)^{1/2} \) and letting \( n \to \infty \),

\[
\mathcal{L}(w_n, v) - \langle \lambda_1 w_n - a(\cdot, u_n)w_n^-, v \rangle + \frac{\langle g(\cdot, u_n), v \rangle}{\mathcal{L}(u_n, u_n)^{1/2}} \to 0
\]

(35)
as \( n \to \infty \), for all \( v \in H^1_{p, \rho}(\Omega, \Gamma) \).

Next, we note from (g-2) and (g-3), the Cauchy–Schwarz inequality, Lemma 2.1, (28) and (32), that

\[
\lim_{n \to \infty} \frac{\langle g(\cdot, u_n), v \rangle}{\mathcal{L}(u_n, u_n)^{1/2}} = 0
\]

(36)
for all \( v \in H^1_{p, \rho}(\Omega, \Gamma) \). Setting \( v = \varphi_1 \) and using the fact that \( \mathcal{L}(\varphi_1, w) = \lambda_1 \langle \varphi_1, w \rangle \) for all \( w \in H^1_{p, \rho}(\Omega, \Gamma) \), we obtain from (36) that

\[
\lim_{n \to \infty} \int_{\Omega} \rho(x)a(x, u_n(x))w_n^-(x)\varphi_1(x) \, dx = 0.
\]

(37)

Now, using (a-2) with \( q_1 \) satisfying \( q_1(x) \geq a_1 \) for some constant \( a_1 > 0 \), and the fact that \( \varphi_1 > 0 \) in \( \Omega \), we have that

\[
a(x, u_n(x))w_n^-(x)\varphi_1(x) \geq a_1 w_n^-(x)\varphi_1(x) \geq 0 \quad \text{a.e. in } \Omega.
\]

Thus, from (37) we get that

\[
\lim_{n \to \infty} \int_{\Omega} \rho(x)|w_n^-|^2 \, dx = 0.
\]

(38)

Next, using (32) and (34), we get from (38) and the Lebesgue dominated convergence theorem that \( \langle w^-, \varphi_1 \rangle = 0 \), and so \( \varphi_1 w^- = 0 \) a.e. in \( \Omega \), since \( \varphi_1 \) and \( \rho \) are positive in \( \Omega \). We therefore have that \( w^-(x) = 0 \) for a.e. \( x \in \Omega \); from which it follows that

\[
w \geq 0 \quad \text{a.e. in } \Omega
\]

(39)
and therefore

\[
\lim_{n \to \infty} \int_{\Omega} \rho|w_n^-|^2 = 0.
\]

(40)
Observing now from (a-2) that

$$|\langle a(\cdot, u_n)w_n^{-}, v \rangle_{\rho}| \leq ||q_2||_{L^\infty(\Omega)} ||w_n^{-}||_{\rho} \ ||v||_{\rho} \quad \forall \ v \in H^1_{p, \rho}(\Omega, \Gamma),$$

we obtain from (40) that \( \lim_{n \to \infty} \langle a(\cdot, u_n)w_n^{-}, v \rangle_{\rho} = 0 \) for all \( v \in H^1_{p, \rho}(\Omega, \Gamma) \). Using this fact in (36) we get

$$\lim_{n \to \infty} [\mathcal{E}(w_n, v) - \lambda_1 \langle w_n, v \rangle_{\rho}] = 0 \quad \text{for every} \ v \in H^1_{p, \rho}(\Omega, \Gamma).$$

Hence, from (30) and (32), we have that

$$\mathcal{E}(w, v) - \lambda_1 \int_\Omega \rho w v = 0 \quad \text{for all} \ v \in H^1_{p, \rho}(\Omega, \Gamma),$$

which shows that \( w \in E_{\lambda_1} \), the eigenspace corresponding to \( \lambda_1 \). Since \( \lambda_1 \) is simple, as stipulated in (O3), and \( w \geq 0 \) a.e. by (39), we can write

$$w = \gamma \varphi_1 \quad \text{for some} \ \gamma \geq 0. \quad (41)$$

Next, we show that \( \gamma \neq 0 \). Arguing by contradiction, suppose that \( \gamma = 0 \). Then, by (32) and (41) we would have that

$$\lim_{n \to \infty} \int_\Omega \rho |w_n|^2 = 0. \quad (42)$$

Recall that by Lemma 2.1 and the fact that \( \mathcal{E}(w_n, w_n) = 1 \) for all \( n \), the sequence \( \{||w_n||_{p, \rho}\} \) is uniformly bounded. Hence, from the assumption that \( J'(u_n) \to 0 \) in norm as \( n \to \infty \), we obtain that \( \lim_{n \to \infty} J'(u_n)(w_n) = 0 \). Thus, after dividing by \( \mathcal{E}(u_n, u_n)^{\frac{1}{2}} \) and letting \( n \to \infty \), we obtain by (28) that

$$\lim_{n \to \infty} \frac{J'(u_n)w_n}{\mathcal{E}(u_n, u_n)^{\frac{1}{2}}} = 0. \quad (43)$$

Using the boundedness of the sequence \( \{||w_n||_{p, \rho}\} \) once again, we obtain from (28) that

$$\lim_{n \to \infty} \frac{h(w_n)}{\mathcal{E}(u_n, u_n)^{\frac{1}{2}}} = 0 \quad (44)$$
and by virtue of the Cauchy–Schwarz inequality,

$$\lim_{n \to \infty} \int_{\Omega} \rho(x) b_{\eta}(x) w_n(x) \, dx / E(u_n, u_n)^{1/2} = 0,$$

where $b_{\eta}$ is the function in (g-3). Using this fact in conjunction with (g-2), (g-3) and (42), we obtain that

$$\lim_{n \to \infty} \frac{\langle g(\cdot, u_n), w_n \rangle_{\rho}}{E(u_n, u_n)^{1/2}} = 0. \quad (45)$$

We can also show, using (a-2), (28), (40) and (42), that

$$\lim_{n \to \infty} \frac{\langle a(\cdot, u_n) w_n^- \cdot w_n \rangle_{\rho}}{E(u_n, u_n)^{1/2}} = 0. \quad (46)$$

From (25) with $u = u_n$ and $v = w_n$, after dividing by $E(u_n, u_n)^{1/2}$ and letting $n \to \infty$, we obtain by virtue of (43)–(46) that

$$\lim_{n \to \infty} [E(w_n, w_n) - \lambda_1(w_n, w_n)_{\rho}] = 0.$$

But this limit in conjunction with (42) gives that $\lim_{n \to \infty} E(w_n, w_n) = 0$, which contradicts the fact that $E(w_n, w_n) = 1$ for all $n$. Thus, we must have that $\gamma \neq 0$, and so (41) now reads

$$w = \gamma \varphi_1 \quad \text{for some } \gamma > 0. \quad (47)$$

Before continuing with the rest of the proof we establish the following lemmettes.

**Lemma 3.2.** In addition to the assumptions of Lemma 2.1, suppose also that $\{u_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ are sequences which meet (28)–(34) and (47) where $w = \gamma \varphi_1$ with $\gamma > 0$. Let $\chi_n$ be the indicator function for the set $\{x \in \Omega \mid u_n(x) > 0\}$. Then

(i) $\int_{\Omega} \rho \chi_n(x) g(x, u_n(x)) w_n(x) \, dx \to \gamma \int_{\Omega} \rho g^+(x) \varphi_1(x) \, dx$ \quad as $n \to \infty$,

and

(ii) $\int_{\Omega} \rho \chi_n(x) G(x, u_n(x)) \frac{1}{E(u_n, u_n)^{1/2}} \, dx \to \gamma \int_{\Omega} \rho g^+(x) \varphi_1(x) \, dx$ \quad as $n \to \infty$. 


**Proof.** To establish the lemmette, first observe that, as a consequence of (28), (32), (47) and the fact that $u_n(x) = \mathcal{E}(u_n, u_n)^{1/2} w_n(x)$,

$$
\lim_{n \to \infty} u_n(x) = \infty \quad \text{a.e. in } \Omega.
$$

(48)

Consequently, $\gamma_n(x) \to 1$ for a.e. $x$ in $\Omega$ and $g_n(x, u_n(x)) \to g_+(x)$ for a.e. $x$ in $\Omega$, by (g-5), as $n \to \infty$. Also, $|\gamma_n(x) g(x, u_n(x))| \leq b(x)$ for a.e. $x$ in $\Omega$, where $b \in L^2(\Omega)$ by (g-2). Hence (i) of the lemmette follows from the Lebesgue dominated convergence theorem in conjunction with (33), (34) and (47).

To establish (ii) of the lemmette, we observe from (g-4) that for $u_n(x) > 0$,

$$
\frac{G(x, u_n(x))}{u_n(x)} = \frac{1}{u_n(x)} \int_0^{u_n(x)} g(x, s) \, ds.
$$

(49)

Since in this case $|g(x, s)| \leq b(x)$, we have that

$$
\left| \frac{\gamma_n(x) G(x, u_n(x))}{\mathcal{E}(u_n, u_n)^{1/2}} \right| = \gamma_n(x) \left| \frac{G(x, u_n(x))}{u_n(x)} \right| \left| \frac{u_n(x)}{\mathcal{E}(u_n, u_n)^{1/2}} \right| \leq b(x) W_1(x),
$$

by (34). Consequently,

$$
\left| \frac{\gamma_n(x) G(x, u_n(x))}{\mathcal{E}(u_n, u_n)^{1/2}} \right| \leq b(x) W_1(x) \quad \text{a.e. in } \Omega.
$$

(50)

Also, we see from (49) and (g-5) that if $u_n(x) \to +\infty$ as $n \to \infty$, then the left-hand side of (49) tends to $g_+(x)$ as $n \to \infty$. But then it follows from (48), (33) and (47) that

$$
\gamma_n(x) \frac{G(x, u_n(x))}{\mathcal{E}(u_n, u_n)^{1/2}} \to g_+(x) \gamma \varphi_1(x) \quad \text{a.e. in } \Omega
$$

(51)

as $n \to \infty$. Thus, (ii) of the lemmette follows immediately from the Lebesgue dominated convergence theorem, (50) and (51) above. □

**Lemmette 3.3.** Under the same conditions in the hypothesis as Lemmette 3.2, the following holds

$$
\liminf_{n \to \infty} \int_{\Omega} \rho[1 - \gamma_n] \left[ 2 G(\cdot, u_n) - g(\cdot, u_n) u_n \right] \geq 0.
$$

(52)
Proof. Let \( Z_n(x) \) designate the integrand in (52). Then we see from (g-4) and the definition of \( f_n \) that

\[
\frac{Z_n(x)}{\mathcal{E}(u_n, u_n)^{1/2}} \geq -\rho(x)b^*(x)w_n^-(x) \quad \text{a.e. in } \Omega,
\]

where \( b^* \in L^2(\Omega) \). Since \( \rho(x)b^*(x)w_n^-(x) \to 0 \) for a.e. \( x \in \Omega \), (52) in Lemma 3.3 follows immediately from (34) and the Lebesgue dominated convergence theorem. \( \square \)

Proof of Lemma 3.1 (continued). From the assumption that the sequence \( \{J(u_n)\} \) is bounded and (28), we get that

\[
\frac{2J(u_n)}{\mathcal{E}(u_n, u_n)^{1/2}} \to 0 \quad \text{as } n \to \infty.
\]

Likewise, from the assumption that \( J'(u_n) \to 0 \) in norm as \( n \to \infty \) and the boundedness of the sequence \( \{w_n\} \), we obtain \( \lim_{n \to \infty} J'(u_n)(w_n) = 0 \). Thus

\[
\lim_{n \to \infty} \left\{ \frac{2J(u_n)}{\mathcal{E}(u_n, u_n)^{1/2}} - J'(u_n)(w_n) \right\} = 0.
\]

Put \( M_n = \frac{2J(u_n)}{\mathcal{E}(u_n, u_n)^{1/2}} - J'(u_n)(w_n) \) for all \( n = 1, 2, 3, \ldots \), then

\[
\lim_{n \to \infty} M_n = 0 \quad (53)
\]

and, by (22) and (25),

\[
M_n = \int_{\Omega} -2[A(x, u_n) + G(x, u_n)][\mathcal{E}(u_n, u_n)^{-1/2} \rho(x) dx - h(w_n)
+ \int_{\Omega} [-a(x, u_n)u_n^- + g(x, u_n)]w_n \rho(x) dx.
\]

Next, we observe from (a-3) and (23) that

\[
\frac{2A(x, u_n) + a(x, u_n)u_n^-(x)u_n(x)}{\mathcal{E}(u_n, u_n)^{1/2}} \geq -b^*(x)w_n^- \quad \text{a.e. in } \Omega,
\]

where \( b^* \) is a non-negative function in \( L^2(\Omega) \). Thus, since \( \rho(x)b^*(x)w_n^-(x) \to 0 \) a.e. in \( \Omega \), we obtain from (34), (55) and the Lebesgue dominated convergence theorem
that

$$\lim_{n \to \infty} \frac{\int_{\Omega} \rho(x)[2A(x, u_n) + a(x, u_n)u_n^{-}(x)u_n(x)]dx}{\mathcal{L}(u_n, u_n)^{1/2}} \geq 0. \quad (56)$$

But then from (31), (47), (53), (54) and (56) we have that

$$\limsup_{n \to \infty} \int_{\Omega} \rho \left[ \frac{2G(\cdot, u_n) - g(\cdot, u_n)w_n}{\mathcal{L}(u_n, u_n)^{1/2}} \right] \leq -\gamma h(\varphi_1). \quad (57)$$

We next apply Lemma 3.3 to the expression in (57) with \( \chi_n \) being the indicator function of the set \( \{ x \in \Omega \mid u_n(x) > 0 \} \) and obtain from an easy calculation that

$$\lim_{n \to \infty} \int_{\Omega} \rho \chi_n \left[ \frac{2G(\cdot, u_n) - g(\cdot, u_n)w_n}{\mathcal{L}(u_n, u_n)^{1/2}} \right] \leq -\gamma h(\varphi_1). \quad (58)$$

We next apply Lemma 3.2 to the expression on the left-hand side of the inequality in (58) to obtain that

$$\gamma \int_{\Omega} \rho(x)g_+(x)\varphi_1(x)dx \leq -\gamma h(\varphi_1) \quad \text{where} \quad \gamma > 0.$$

This is in direct contradiction of the solvability condition (13). Hence (28) does not hold and \( \{ ||u_n||_{p, \rho} \}_{n=1}^{\infty} \) is a bounded sequence. As we observed earlier, this fact proves Lemma 3.1. \( \square \)

Next write \( W = \{ u \in H^{1}_{p, \rho}(\Omega, \Gamma) \mid u = t\varphi_1 \text{ for some } t \in \mathbb{R} \} \).

Lemma 3.4. Let \( \Omega \subseteq \mathbb{R}^N \) be a bounded domain and let \( p \) and \( \rho \) satisfy (1). Assume also that \( \Omega \) and the operator \( L \) satisfy the conditions \( V_L(\Omega, \Gamma) \). Suppose that \( L \) satisfies (L-1)-(L-3), \( a \) satisfies (a-1) and (a-2) with \( q_1 \) satisfying \( q_1(x) \geq a_1 \) for some constant \( a_1 > 0 \). Suppose also that \( g \) satisfies (g-1)-(g-3) and (g-5) and that the solvability condition (13) holds. Let \( J \) be as defined in (22). Then

$$\lim_{||u||_{p, \rho} \to \infty} J(u) = -\infty \quad \text{for } u \in W.$$  

Proof. Since \( W = \text{span}\{\varphi_1\} \), the result will follow if we can prove that

$$\lim_{t \to \pm \infty} J(t\varphi_1) = -\infty.$$
It follows from the fact that $\mathcal{L}(\varphi_1, \varphi_1) = \lambda_1(\varphi_1, \varphi_1)\rho$ and from (22) that

$$J(t\varphi_1) = -\int_{\Omega} \rho(x)[A(x, t\varphi_1) + G(x, t\varphi_1)] \, dx - th(\varphi_1) \tag{59}$$

for any $t \in \mathbb{R}$.

We shall first consider the case $t \to -\infty$. Observe from (23) and (a-2) with $q_1$ satisfying $q_1(x) \geq a_1$ for some constant $a_1 > 0$ that

$$A(x, s) \geq \frac{a_1}{2}s^2$$

for $s \leq 0$ and a.e. $x \in \Omega$.

Consequently, it follows from (24) that there exists $b_\ast \in L^2_\rho(\Omega)$ such that

$$A(x, t\varphi_1) + G(x, t\varphi_1) \geq \frac{a_1}{4}t^2\varphi_1^2 + t|b_\ast|\varphi_1$$

for $t < 0$ and a.e. $x \in \Omega$. We conclude from this last inequality and (59) that

$$J(t\varphi_1) \leq -\frac{a_1t^2}{4} + K_1|t| + |t|h(\varphi_1)$$

for $t < 0$ and some constant $K_1$. So that, since $a_1 > 0$,

$$J(t\varphi_1) \to -\infty \quad \text{as} \quad t \to -\infty. \tag{60}$$

Next suppose that $t \geq 0$ in (59), then using (23) we get

$$-J(t\varphi_1) = \int_{\Omega} \rho G(x, t\varphi_1) + h(t\varphi_1).$$

Thus, in view of (60) and this last equality, to complete the proof of the lemma we have to show that

$$\lim_{t \to \infty} \left\{ \int_{\Omega} \rho \frac{G(x, t\varphi_1)}{t} \, dx + h(\varphi_1) \right\} > 0. \tag{61}$$

In view of the solvability condition (13), (61) will follow once we can show that

$$\lim_{t \to \infty} \int_{\Omega} \rho \frac{G(x, t\varphi_1)}{t} \, dx = \int_{\Omega} \rho g_+(x)\varphi_1(x) \, dx. \tag{62}$$
From (g-5), the fact that \( \phi_1 > 0 \) a.e. in \( \Omega \), and the definition of \( G(x, s) \), it is clear that

\[
\lim_{t \to \infty} \phi_1(x) \frac{G(x, t\phi_1)}{t\phi_1(x)} = \varphi_1(x)g_+(x) \quad \text{a.e. in } \Omega.
\]

Also, from (g-2) we see that

\[
\left| \frac{G(x, t\phi_1)}{t\phi_1(x)} \right| \leq b(x) \quad \text{a.e. in } \Omega.
\]

Hence, (62) follows from the Lebesgue dominated convergence theorem, and the proof of Lemma 3.4 is now complete. \( \square \)

Define \( V = W^\perp \), the orthogonal complement of \( W \) in \( H^1_{p, \rho}(\Omega, \Gamma) \) with respect to the inner product \( \langle \cdot, \cdot \rangle \). Then \( V = \{ u \in H^1_{p, \rho}(\Omega, \Gamma) | \langle u, \phi_1 \rangle = 0 \} \) and \( H^1_{p, \rho}(\Omega, \Gamma) = W \oplus V \).

For every \( \gamma > 0 \) put

\[
H_\gamma = \{ u \in H^1_{p, \rho}(\Omega, \Gamma) | u = \gamma \{ \Phi(v, v) \}^{1/2} \phi_1 + v, \text{ where } v \in V \}.
\]

Observe that, for each \( u = w + v \in H_\gamma \),

\[
\Phi(v, v) = \frac{\Phi(u, u)}{\gamma^2 \dot{\lambda}_1 + 1}.
\]  \( \text{(63)} \)

**Lemma 3.5.** Let \( \Omega \subseteq \mathbb{R}^N \) be a bounded domain and let \( p \) and \( \rho \) satisfy (1). Assume also that \( \Omega \) and the operator \( L \) satisfy the conditions \( V_L(\Omega, \Gamma) \). Suppose that \( L \) satisfies (L-1)–(L-3), \( a \) satisfies (a-1) and (a-2) with \( q_1 \) satisfying \( q_1(x) \geq a_1 \) for some constant \( a_1 > 0 \). Suppose also that \( g \) satisfies (g-1)–(g-3). Let \( J \) be as defined in (22). Then there exists a positive constant \( \gamma_1 \) such that

\[
\lim_{\Phi(u, u) \to +\infty} J(u) = +\infty \quad \text{for } u \in H_{\gamma_1}.
\]

**Proof.** Let \( a_2 \) be a positive constant for which \( q_2(x) \leq a_2 \) for a.e. \( x \in \Omega \), where \( q_2 \in L^\infty(\Omega) \) is as given in (a-2). Then, using (23) and (a-2) we get that

\[
|A(x, s)| \leq \frac{a_2}{2} (s^-)^2 \quad \text{for a.e. } x \in \Omega \text{ and all } s \in \mathbb{R}.
\]  \( \text{(64)} \)

Let \( k \in \mathbb{N} \) be such that for \( k \geq 3 \),

\[
a_2 + \dot{\lambda}_1 \leq \dot{\lambda}_{k-1}
\]  \( \text{(65)} \)
and

\[ \lambda_1 < \lambda_2 < \lambda_3 \leq \cdots \leq \lambda_{k-1} < \lambda_k, \]

where \( \{\lambda_j\} \) are the eigenvalues of \( L \) given in (O2). Since the corresponding eigefunctions \( \{\varphi_j\} \) are in \( C^2(\Omega) \cap L^\infty(\Omega) \) by (O1), there exists a positive number \( R \) such that

\[ |\varphi_2(x)|, |\varphi_3(x)|, \ldots, |\varphi_{k-1}(x)| \leq R \quad \text{for all } x \in \Omega. \tag{66} \]

For each \( u \in H^1_\infty(\Omega, \Gamma) \), write \( u = w + v \) where \( w \in W \) and \( v \in V \). Then

\[ v = \sum_{n=2}^{\infty} \hat{u}(n) \varphi_n, \tag{67} \]

where

\[ \hat{u}(n) = \int_\Omega \rho u \varphi_n \quad \text{for all } n = 1, 2, 3, \ldots. \]

Put

\[ v_1 = \sum_{n=2}^{k-1} \hat{u}(n) \varphi_n \tag{68} \]

and

\[ v_2 = \sum_{n=k}^{\infty} \hat{u}(n) \varphi_n. \tag{69} \]

Thus, \( v = v_1 + v_2 \).

Using (19) and (67) we get that

\[ \mathcal{E}(v, v) = \sum_{n=2}^{\infty} \lambda_n |\hat{u}(n)|^2. \]

Thus, for all \( n \geq 2 \),

\[ \lambda_n |\hat{u}(n)|^2 \leq \mathcal{E}(v, v). \]
and consequently
\[ |\hat{u}(n)|^2 \leq \frac{1}{\lambda_2} \langle \xi, v \rangle \quad \text{for all } n \geq 2. \]

It follows from this last inequality in conjunction with (66) and (68) that
\[
|v_1(x)|^2 \leq R^2 (|\hat{u}(2)| + |\hat{u}(3)| + \cdots + |\hat{u}(k-1)|)^2 \\
\leq R^2 (k-2) \sum_{n=2}^{k-1} |\hat{u}(n)|^2 \\
\leq \frac{R^2 (k-2)^2}{\lambda_2} \langle \xi, v \rangle
\]
for all \( x \in \Omega \). Thus
\[
|v_1(x)| \leq \frac{R(k-2)}{\sqrt{\lambda_2}} \{\xi(v, v)\}^{1/2} \quad \text{for all } x \in \Omega. \tag{70}
\]

Next, let
\[
\tilde{J}(u) = \frac{1}{2} \{\xi(u, u) - \lambda_1 \langle u, u \rangle_\rho \} - \int_\Omega \rho A(x, u) \tag{71}
\]
for \( u \in H_{1, \rho}^1(\Omega, \Gamma) \), and put
\[
\beta = \min \left\{ 1 - \frac{\lambda_{k-1}}{\lambda_k}, 1 - \frac{\lambda_1}{\lambda_2} \right\}. \tag{72}
\]

We shall first prove that there exists \( \gamma > 0 \), depending on \( \beta \), such that, for every \( u \in H_{\gamma} \),
\[
\tilde{J}(u) \geq \frac{\beta}{4} \xi(v, v), \tag{73}
\]
where \( u = \gamma \{\xi(v, v)\}^{1/2} \varphi_1 + v \) and \( v \in V \).

With \( \varepsilon = \min \left\{ \left( \frac{\beta}{8C} \right)^2, \frac{1}{2} \right\} \), where
\[
C = \frac{a_2 R(k-2)}{\sqrt{\lambda_2}} \left( \frac{R(k-2)}{2 \sqrt{\lambda_2}} + \frac{1}{\sqrt{\lambda_k}} \right),
\]
find $\Omega_\varepsilon \subset \subset \Omega$ such that
\[
\int_{\Omega \setminus \Omega_\varepsilon} \rho < \varepsilon.
\] (74)

Since $\varphi_1 > 0$ on $\Omega$ and $\Omega$ is bounded, there exists $q_5 > 0$ such that
\[
\varphi_1(x) \geq q_5 \quad \text{on } \overline{\Omega_\varepsilon}.
\] (75)

To establish (73), we first show that if $u \in H_\gamma$ and $u = \gamma \langle \xi(v, v) \rangle^{1/2} \varphi_1 + v_1 + v_2$, where $v$, $v_1$ and $v_2$ are as given in (67)–(69) respectively, then there exists a $\gamma > 0$ such that
\[
\gamma \langle \xi(v, v) \rangle^{1/2} \varphi_1 + v_1 > 0 \quad \text{on } \overline{\Omega_\varepsilon}.
\] (76)

To see this, use (70) to get
\[
\gamma \langle \xi(v, v) \rangle^{1/2} \varphi_1(x) + v_1(x) \geq \left( \gamma \varphi_1(x) - \frac{R(k - 2)}{\sqrt{\lambda_2}} \right) \langle \xi(v, v) \rangle^{1/2}
\]
for all $x \in \Omega$. Consequently, using (75), we see that (76) will hold for
\[
\gamma > \frac{R(k - 2)}{q_5 \sqrt{\lambda_2}}.
\]

Put
\[
\gamma_1 = \frac{R(k - 2)}{q_5 \sqrt{\lambda_2}} + 1.
\] (77)

Next, let $u \in H_{\gamma_1}$. Write $u = w + v_1 + v_2$ where $w = \gamma \langle \xi(v, v) \rangle^{1/2} \varphi_1$ and $v$, $v_1$ and $v_2$ are as given in (67)–(69) respectively. It then follows from (76) with $\gamma = \gamma_1$ that
\[
\frac{a_2}{2} \int_{\overline{\Omega_\varepsilon}} \rho(u^-)^2 = \frac{a_2}{2} \int_{\overline{\Omega_\varepsilon}} \rho((w + v_1 + v_2)^-)^2
\leq \frac{a_2}{2} \int_{\overline{\Omega_\varepsilon}} \rho(v_2^-)^2.
\] (78)

On the other hand, since $\varphi_1 > 0$ on $\Omega$ and $(v_1 + v_2)^- \leq (v_1)^- + (v_2)^-$,
\[
\frac{a_2}{2} \int_{\Omega \setminus \overline{\Omega_\varepsilon}} \rho(u^-)^2 = \frac{a_2}{2} \int_{\Omega \setminus \overline{\Omega_\varepsilon}} \rho((w + v_1 + v_2)^-)^2
\leq \frac{a_2}{2} \int_{\Omega \setminus \overline{\Omega_\varepsilon}} \rho((v_1 + v_2)^-)^2.
\]
\[ \leq \frac{a_2}{2} \left( \int_{\Omega \setminus \overline{\Omega}_e} \rho(v_1^-)^2 + 2 \int_{\Omega \setminus \overline{\Omega}_e} \rho|v_1^-||v_2^-| + \int_{\Omega \setminus \overline{\Omega}_e} \rho(v_2^-)^2 \right). \]

Using this last inequality in conjunction with (71), (64) and (78) we get that, if \( u \in H_{\gamma_1}^{1,p} \) for \( \gamma_1 \) as in (77), then

\[
\tilde{J}(u) \geq \frac{1}{2} \left\{ E(u, u) - \lambda_1 \langle u, u \rangle_{\rho} \right\} - \frac{a_2}{2} \int_{\Omega} \rho(u^-)^2 \\
\geq \left( \frac{1}{2} \left\{ E(u, u) - \lambda_1 \langle u, u \rangle_{\rho} \right\} - \frac{a_2}{2} \int_{\Omega} \rho(v_2^-)^2 \right) \\
- \frac{a_2}{2} \int_{\Omega \setminus \overline{\Omega}_e} \rho(v_1^-)^2 - a_2 \int_{\Omega \setminus \overline{\Omega}_e} \rho|v_1^-||v_2^-|.
\] (79)

Proceeding with the proof of (73), next we observe that, for any \( u \in H_{p, \rho}^{1,1}(\Omega, \Gamma) \), by virtue of (18) and (19),

\[ E(u, u) = \sum_{n=1}^{\infty} \lambda_n |\hat{u}(n)|^2 \]

and

\[ \lambda_1 \langle u, u \rangle_{\rho} = \sum_{n=1}^{\infty} \lambda_1 |\hat{u}(n)|^2, \]

so that

\[ \frac{1}{2} \left\{ E(u, u) - \lambda_1 \langle u, u \rangle_{\rho} \right\} = \sum_{n=2}^{\infty} (\lambda_n - \lambda_1)|\hat{u}(n)|^2. \]

On the other hand, from (65), (69) and (18), we obtain that

\[ a_2 \int_{\Omega} \rho(v_2^-)^2 \leq (\lambda_{k-1} - \lambda_1) \sum_{n=k}^{\infty} |\hat{u}(n)|^2. \]

Consequently,

\[
\frac{1}{2} \left\{ E(u, u) - \lambda_1 \langle u, u \rangle_{\rho} \right\} - \frac{a_2}{2} \int_{\Omega} \rho(v_2^-)^2 \\
\geq \frac{1}{2} \left( \sum_{n=2}^{\infty} (\lambda_n - \lambda_1)|\hat{u}(n)|^2 - \sum_{n=k}^{\infty} (\lambda_{k-1} - \lambda_1)|\hat{u}(n)|^2 \right)
\]
\[= \frac{1}{2} \left( \sum_{n=2}^{k-1} (\lambda_n - \lambda_1) |\hat{u}(n)|^2 + \sum_{n=k}^{\infty} (\lambda_n - \hat{\lambda}_{k-1}) |\hat{u}(n)|^2 \right)\]

\[\geq \frac{1}{2} \left[ \sum_{n=2}^{k-1} \left( 1 - \frac{\lambda_1}{\lambda_2} \right) \lambda_n |\hat{u}(n)|^2 + \sum_{n=k}^{\infty} \left( 1 - \frac{\lambda_{k-1}}{\lambda_k} \right) \lambda_n |\hat{u}(n)|^2 \right] \]

\[\geq \frac{\beta}{2} \sum_{n=2}^{\infty} \lambda_n |\hat{u}(n)|^2,\]

where we have used (72). It therefore follows from (19), (67), (79) and the last inequality that

\[\tilde{J}(u) \geq \frac{\beta}{2} \tilde{\varepsilon}(v, v) - \frac{a_2}{2} \int_{\Omega \setminus \overline{\Omega}_\varepsilon} \rho(v_1^-)^2 - a_2 \int_{\Omega \setminus \overline{\Omega}_\varepsilon} \rho|v_1^-| |v_2^-|,\]  

(80)

where \(u \in H_{\gamma_1}\) with \(\gamma_1\) as in (77) and \(v\) is as given in (67).

To complete the proof of (73), we next estimate the integrals in (80). First, use (70) and (74) to obtain that

\[\int_{\Omega \setminus \overline{\Omega}_\varepsilon} \rho(v_1^-)^2 \leq \frac{R^2(k - 2)^2}{\lambda_2} \varepsilon \tilde{\varepsilon}(v, v).\]  

(81)

Next, apply Cauchy–Schwarz inequality together with (81) to get

\[\int_{\Omega \setminus \overline{\Omega}_\varepsilon} \rho|v_1^-| |v_2^-| \leq \frac{R(k - 2)}{\sqrt{\lambda_2}} \sqrt{\varepsilon \tilde{\varepsilon}(v, v)} \sqrt{\int_{\Omega \setminus \overline{\Omega}_\varepsilon} \rho(v_2^-)^2} \right)^{1/2},\]

where, using (18), (19), (69) and (67),

\[\int_{\Omega \setminus \overline{\Omega}_\varepsilon} \rho(v_2^-)^2 \leq \int_{\Omega} \rho v_2^2 = \sum_{n=k}^{\infty} |\hat{u}(n)|^2 \]

\[\leq \frac{1}{\lambda_k} \sum_{n=k}^{\infty} \lambda_n |\hat{u}(n)|^2 \]

\[\leq \frac{1}{\lambda_k} \sum_{n=2}^{\infty} \lambda_n |\hat{u}(n)|^2 = \frac{1}{\lambda_k} \tilde{\varepsilon}(v, v).\]

Hence,

\[\int_{\Omega \setminus \overline{\Omega}_\varepsilon} \rho|v_1^-| |v_2^-| \leq \frac{R(k - 2)}{\sqrt{\lambda_k} \sqrt{\lambda_2}} \sqrt{\varepsilon \tilde{\varepsilon}(v, v)}.\]

This last inequality, together with (81), yields from (80) that, since \(0 < \varepsilon < 1\),
\[ J(u) \geq \frac{\beta}{2} \langle v, v \rangle - C \varepsilon^{1/2} \langle v, v \rangle, \]  
(82)

where \( u \in H_{\gamma_1} \) with \( \gamma_1 \) as in (77), \( v \) is as given in (67), and

\[ C = \frac{a_2 R(k - 2)}{\sqrt{\lambda_k}} \left( \frac{R(k - 2)}{2 \sqrt{\lambda_k}} + \frac{1}{\sqrt{\lambda_k}} \right). \]

But \( \varepsilon^{1/2} C \leq \frac{\beta}{8} \), so (73) follows from (82).

Next, let \( u \in H_{\gamma_1} \); that is, \( u = \gamma_1 \{ \varepsilon (v, v) \}^{1/2} \varphi_1 + v \) where \( v \in V \). Then, it follows from (22), (71) and (73) that

\[ J(u) \geq \frac{\beta}{4} \langle v, v \rangle - \int_{\Omega} \rho G(x, u) - h(u). \]  
(83)

Put

\[ \eta = \frac{\beta \lambda_1}{4(1 + \gamma_1^2 \lambda_1)}, \]  
(84)

where \( \beta \) is given in (72) and \( \gamma_1 \) in (77), and use (g-3) to obtain \( b_\eta \in L^2_{\rho}(\Omega) \), with \( b_\eta \geq 0 \) a.e. in \( \Omega \), such that

\[ |g(x, s)| \leq \eta |s| + b_\eta(x) \text{ for a.e. } x \in \Omega \text{ and } s < 0. \]

It then follows from (g-2) and (29) that, for any \( u \in H^1_{p, \rho}(\Omega, \Gamma) \),

\[ \int_{\Omega} \rho G(x, u) \ dx \leq \frac{\eta}{2} \int_{\Omega} \rho u^2 + \int_{\Omega} \rho (b_\eta + b) |u|, \]

where \( b \in L^2_{\rho}(\Omega) \) is as given in (g-2). Consequently, it follows from (83), the Cauchy–Schwarz inequality, and this last inequality, that there exists a constant \( K_4 \) such that, if \( u \in H_{\gamma_1} \),

\[ J(u) \geq \frac{\beta}{4} \langle v, v \rangle - \frac{\eta}{2} \int_{\Omega} \rho u^2 - K_4 ||u||_{p, \rho}, \]
where we have also used the fact that $h$ is a bounded linear functional on $H^1_{p,\mu}(\Omega, \Gamma)$. It then follows from (20), Lemma 2.1, (63) with $\gamma = \gamma_1$, and the last estimate, that

$$J(u) \geq \frac{\beta}{4(\gamma_1^2 \lambda_1 + 1)} E(u, u) - \frac{\eta}{2 \lambda_1} E(u, u) - K_4 \sqrt{\frac{1}{c_1} \{E(u, u)\}^{1/2}}$$

for all $u \in H_{\gamma_1}$, or

$$J(u) \geq \frac{\beta}{8(\gamma_1^2 \lambda_1 + 1)} E(u, u) - K_4 \sqrt{\frac{1}{c_1} \{E(u, u)\}^{1/2}} \quad \forall u \in H_{\gamma_1},$$

because of our choice of $\eta$ in (84). The conclusion of Lemma 3.5 follows immediately from this last inequality. □

4. A linking argument and proof of Theorem 1.1

Let $W$, $V$ and $H_{\gamma_1}$ be as defined in the previous section. Using Lemma 3.4 we can find a constant $K_5$ such that

$$J(u) \leq K_5 \quad \forall u \in W. \quad (85)$$

Applying Lemma 3.5 and (63), there exists a constant $r_o > 0$ such that, if $u = \gamma_1 \{E(v, v)\}^{1/2} \varphi_1 + v \in H_{\gamma_1}$ and $E(v, v) \geq r_o^2$, then $J(u) \geq K_5 + 1$. We define

$$S_1 = \{ \gamma_1 \{E(v, v)\}^{1/2} \varphi_1 + v \mid v \in V \text{ and } E(v, v) \geq r_o^2 \};$$

then,

$$J(u) \geq K_5 + 1 \quad \forall u \in S_1. \quad (86)$$

Put

$$S_2 = \{ u \in H^1_{p,\mu}(\Omega, \Gamma) \mid u = \gamma_1 r_o \varphi_1 + v \text{ where } v \in V \text{ and } E(v, v) \leq r_o^2 \}.$$

Observe that, as a consequence of Lemma 2.1, the set $S_2$ is bounded in the space $H^1_{p,\mu}(\Omega, \Gamma)$. Therefore, by the definition of $J$ in (22) and the estimates in (17), (24), and (64), there exists a constant $K_6 > 0$ with

$$J(u) \geq - K_6 \quad \forall u \in S_2. \quad (87)$$
Also, as a consequence of (L-1) and the definition of $\mathcal{E}(\cdot, \cdot)$ in (6), both $S_1$ and $S_2$ are closed sets in $H^1_{p, \rho}(\Omega, \Gamma)$.

Next, apply Lemma 3.4 again to obtain $t_o \geq \gamma_1 r_0 + 1$ with

$$\max\{J(t_o \varphi_1), J(-t_o \varphi_1)\} \leq -K_0 - 1.$$ (88)

Set $S = H^1_{p, \rho}(\Omega, \Gamma) \setminus S_1$ and define

$$\Psi = \{\sigma \in C([0, 1], S) \mid \sigma(0) = -t_o \varphi_1 \text{ and } \sigma(1) = t_o \varphi_1\}.$$ (89)

**Lemma 4.1.** For every $\sigma \in \Psi$, $\sigma([0, 1]) \cap S_2 \neq \emptyset$; in fact, there exists $s_o \in [0, 1]$ such that $\sigma(s_0) \in S_2$, and $\sigma(s_0) = \gamma_1 r_0 \varphi_1 + v$, where $v \in V$, and $\mathcal{E}(v, v) < r_o^2$.

**Proof.** Let $\sigma \in \Psi$ and consider the orthogonal projection $\pi: H^1_{p, \rho}(\Omega, \Gamma) \to W$ given by

$$\pi(u) = \frac{1}{\lambda_1} \mathcal{E}(u, \varphi_1) \varphi_1 \quad \forall u \in H^1_{p, \rho}(\Omega, \Gamma).$$

Put $\pi(\sigma(s)) = f(s) \varphi_1$ for all $s \in [0, 1]$, where $f(s) = \frac{\mathcal{E}(\sigma(s), \varphi_1)}{\lambda_1}$. Then $f: [0, 1] \to \mathbb{R}$ is a continuous function satisfying $f(0) = -t_o$ and $f(1) = t_o$. Since $-t_o < \gamma_1 r_0 < t_o$, the intermediate value theorem implies that there exists $s^* \in (0, 1)$ such that $f(s^*) = \gamma_1 r_0$. Thus, for some $s^* \in (0, 1)$, $\pi(\sigma(s^*)) = \gamma_1 r_0 \varphi_1$. Put $s_o = \sup\{s \mid f(s) = \gamma_1 r_0\}$. It is clear that $s_o < 1$ and $f(s) > \gamma_1 r_0$ for $s_o < s \leq 1$. We claim that $\sigma(s_0) \in S_2$. Suppose not and set

$$g(s) = \{\mathcal{E}(\sigma(s) - \pi(\sigma(s)), \sigma(s) - \pi(\sigma(s)))\}^{1/2} \quad \text{for } s \in [0, 1].$$

Observe that $g(s_o) \neq r_o$ because $\sigma(s_o) \in S = H^1_{p, \rho}(\Omega, \Gamma) \setminus S_1$. Therefore, since we are also assuming that $\sigma(s_o) \notin S_2$, we must have that $g(s_o) > r_o$. Now, let $\zeta(s) = \frac{g(s)}{f(s)}$ for $s_o \leq s \leq 1$. Then $\zeta$ is continuous on $[s_o, 1]$ with $\zeta(s_o) = \frac{g(s_o)}{\gamma_1 r_o} > \frac{1}{\gamma_1}$ and $\zeta(1) = 0$. Consequently, by the intermediate value theorem, there exists $s_1 \in (s_o, 1)$ such that $\zeta(s_1) = \frac{1}{\gamma_1}$. Then,

$$\sigma(s_1) = \gamma_1 g(s_1) \varphi_1 + v_1,$$

where $v_1 = \sigma(s_1) - \pi(\sigma(s_1)) \in V$ is such that

$$\mathcal{E}(v_1, v_1)^{1/2} = g(s_1) = \frac{f(s_1)}{\gamma_1} > r_o.$$
Hence \( \sigma(s_1) \in S_1 \), which contradicts \( \sigma \in \Psi \). Therefore \( \sigma(s_0) \in S_2 \), and since \( \sigma(s_0) \notin S_1 \), it also follows that \( g(s_0) < r_0 \). This concludes the proof of the lemma. \( \square \)

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** We shall show that the functional \( J \) defined in (22) has a critical point in \( H^1_{p, \rho}(\Omega, \Gamma) \). In fact, we will prove that the number

\[
c^* = \inf_{\sigma \in \Psi} \sup_{s \in [0,1]} J(\sigma(s))
\]

is a critical value of \( J \), where \( \Psi \) is as defined in (89).

We first observe that \( c^* \) is indeed a real number. To see this first note that, for any \( \sigma \in \Psi \), it follows from Lemma 4.1 that \( \sigma(s) \in S_2 \) for some \( s \in [0, 1] \). Consequently, \( J(\sigma(s)) \geq -K_6 \) by (87). Thus

\[
\sup_{s \in [0,1]} J(\sigma(s)) \geq -K_6 \quad \text{for all } \sigma \in \Psi
\]

and therefore \( c^* \geq -K_6 \) by (90). On the other hand, the continuous path \( \sigma_o : [0, 1] \to H^1_{p, \rho}(\Omega, \Gamma) \setminus S_1 \) given by

\[
\sigma_o(s) = 2st_0 \phi_1 - t_0 \phi_1 \quad \forall s \in [0, 1],
\]

is in \( \Psi \). Also, \( \sigma_o(s) \in W \) for all \( s \in [0, 1] \). It then follows from (85) that \( J(\sigma_o(s)) \leq K_5 \) for all \( s \in [0, 1] \), and hence \( c^* \leq K_5 \) by (90). We therefore have that

\[
-K_6 \leq c^* \leq K_5.
\]

To prove that \( c^* \) is a critical value of \( J \) we argue by contradiction. So suppose that \( c^* \) is not a critical value. We can then use the facts that \( c^* \in \mathbb{R} \) and that \( J \) satisfies the Palais–Smale condition by Lemma 3.1, to invoke the Deformation theorem (see Theorem A.4 in [8, p. 82]) with \( \varepsilon = \frac{1}{2} \). We then obtain \( \varepsilon \in (0, 1/2) \) and \( \eta \in C([0, 1] \times H^1_{p, \rho}(\Omega, \Gamma), H^1_{p, \rho}(\Omega, \Gamma)) \) satisfying:

\[
\eta(0, u) = u,
\]

\[
\eta(t, u) = u \quad \text{for all } t \in [0, 1] \text{ if } |J(u) - c^*| > \frac{1}{2}
\]

and

\[
J(\eta(1, u)) \leq c^* - \varepsilon \quad \text{for all } u \text{ such that } J(u) \leq c^* + \varepsilon.
\]

Now, by the definition of \( c^* \) in (90), there exists \( \sigma \in \Psi \) such that

\[
J(\sigma(s)) \leq c^* + \varepsilon \quad \text{for } 0 \leq s \leq 1.
\]
Define \( \tau(s) = \eta(1, \sigma(s)) \) for all \( s \in [0, 1] \). We claim that \( \tau \in \mathcal{P} \). To prove the claim, first observe that \( \tau = \eta(1, \cdot) \circ \sigma \in C([0, 1], H^1_{p, \rho} (\Omega, \Gamma)) \). Next, by (88) and (91),

\[
J(\pm t_0 \varphi_1) \leq - K_6 - 1 \leq c^* - 1,
\]

so that

\[
J(\pm t_0 \varphi_1) - c^* \leq - 1.
\]

Consequently, it follows from (92) that

\[
\eta(1, \pm t_0 \varphi_1) = \pm t_0 \varphi_1.
\]

Thus, \( \tau(0) = - t_0 \varphi_1 \) and \( \tau(1) = t_0 \varphi_1 \). It remains to show now that \( \tau(s) \notin S_1 \) for all \( 0 \leq s \leq 1 \). Now by 5° in [8, Theorem A.4, p. 83]

\[
J(\eta(1, \sigma(s))) \leq J(\sigma(s)) \quad \text{for all } s \in [0, 1].
\]

Thus, by (94) and (91),

\[
J(\eta(1, \sigma(s))) < K_5 + \frac{1}{2} \quad \text{for all } s \in [0, 1].
\]

It therefore follows from (86) that \( \tau(s) \notin S_1 \) for any \( s \in [0, 1] \), and hence \( \tau \in \mathcal{P} \). We then have by (90) that

\[
c^* \leq \sup_{s \in [0, 1]} J(\tau(s)). \tag{95}
\]

On the other hand, it follows from (94) and (93) that \( J(\tau(s)) \leq c^* - \varepsilon \) for all \( 0 \leq s \leq 1 \), and so

\[
\sup_{s \in [0, 1]} J(\tau(s)) \leq c^* - \varepsilon,
\]

which contradicts (95). Thus, \( c^* \) is a critical value of \( J \) and so \( J \) has a critical point, and hence Theorem 1.1 is proved. \( \square \)

References