Limiting case for the regularity criterion to the 3-D Magneto-hydrodynamics equations

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A B S T R A C T

We prove that any weak solution \((u, b)\) of three-dimensional incompressible Magneto-hydrodynamics equations is regular if 
\[ u \in L^\infty(0, T; L^3(\mathbb{R}^3)) \] and 
\[ b \in L^\infty(0, T; \text{VMO}^{-1}(\mathbb{R}^3)). \]

The proof is based on the blow-up analysis and backward uniqueness for the parabolic operator developed by Escauriaza–Seregin–Šverák.

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1. Introduction

We consider the 3-D incompressible Magneto-hydrodynamics (MHD) equations:

\[
\begin{aligned}
    u_t - \nu \Delta u + u \cdot \nabla u &= -\nabla p + b \cdot \nabla b, \\
    b_t - \eta \Delta b + u \cdot \nabla b &= b \cdot \nabla u, \\
    \nabla \cdot u &= \nabla \cdot b = 0.
\end{aligned}
\]  

(1.1)

Here \(u, b\) describe the fluid velocity field and the magnetic field respectively, \(p\) is a scalar pressure, \(\nu > 0\) is the kinematic viscosity, \(\eta > 0\) is the magnetic diffusivity. If \(\nu = \eta = 0\), (1.1) is so-called the ideal MHD equations. In the absence of the magnetic field, (1.1) becomes the incompressible Navier–Stokes equations. We take \(\nu = \eta = 1\) for the simplicity of notation throughout this paper.

The global existence of weak solution and local existence of strong solution to the MHD equations (1.1) were proved by Duvaut and Lions [6]. The same as the incompressible Navier–Stokes equations, the regularity and uniqueness of weak solutions remains a challenging open problem. We refer to [15] for some mathematical questions related to the MHD equations.

It is well known that if the weak solution of the Navier–Stokes equations satisfies the Ladyzhenskaya–Prodi–Serrin condition

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u \in L^q(0, T; L^p(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} \leq 1, \quad p \geq 3,

then it is regular on \((0, T) \times \mathbb{R}^3\). Note that the limiting case \(L^\infty(0, T; L^3(\mathbb{R}^3))\) does not fall into the framework of energy method, which was proved by Escauriaza, Seregin and Šverák [7]. Wu [18,19] extended Ladyzhenskaya–Prodi–Serrin type criterions to the MHD equations in terms of both the velocity field \(u\) and the magnetic field \(b\), see [12] for the limiting case \(L^\infty(0, T; L^3(\mathbb{R}^3))\). However, some numerical experiments seem to indicate that the velocity field should play a more important role than the magnetic field in the regularity theory of solutions to the MHD equations [13]. Recently, He and Xin [8] and Zhou [22] have presented some regularity criterions to the MHD equations in terms of the velocity field only. Chen, Miao and Zhang [3,4] extend and improve their results as follows: if the weak solution of the MHD equations (1.1) satisfies

\[ u \in L^q(0, T; B_p^{s}(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} = 1 + s, \quad \frac{3}{1+s} < p \leq \infty, \quad -1 < s \leq 1, \]

then it is regular on \((0, T) \times \mathbb{R}^3\). Here \(B_p^{s}(\mathbb{R}^3)\) is the Besov space. We refer to [2,9,17,20,21,23] and references therein for more relevant results. However, whether the condition on \(b\) can be removed remains unknown in the limiting case (i.e., \((p, q, s) = (3, \infty, 0)\)). The following theorem will give a partial answer to this question.

**Theorem 1.1.** Let \((u, b)\) be a weak solution of the MHD equations (1.1). Assume that \(u \in L^\infty(\mathbb{R}^3)\), \(b \in L^\infty(-1, 0, BMO^{-1}(\mathbb{R}^3))\) and \(b(t) \in VMO^{-1}(\mathbb{R}^3)\) for \(t \in (-1, 0]\).

Then \((u, b)\) is Hölder continuous on \(\mathbb{R}^3 \times (-1, 0]\).

Recall that a local integrable function \(f(x) \in BMO(\mathbb{R}^n)\) if it satisfies

\[ \|f\|_{BMO} = \sup_{R>0, x_0 \in \mathbb{R}^n} \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |f(x) - f_{B_R(x_0)}| \, dx. \]

A remarkable property of \(BMO\) function is

\[ \sup_{R>0, x_0 \in \mathbb{R}^n} \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |f(x) - f_{B_R(x_0)}|^q \, dx < \infty \]

for any \(1 \leq q < \infty\), if the left hand side is finite for some \(q\). Moreover, \(f(x) \in VMO(\mathbb{R}^n)\) if \(f(x) \in BMO(\mathbb{R}^n)\) and for any \(x_0 \in \mathbb{R}^n\),

\[ \limsup_{R \downarrow 0} \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |f(x) - f_{B_R(x_0)}| \, dx = 0. \]

We say that a function \(u \in BMO^{-1}(\mathbb{R}^n)\) if there exist \(U_j \in BMO(\mathbb{R}^n)\) such that \(u = \sum_{j=1}^n \partial_j U_j\), and \(VMO^{-1}(\mathbb{R}^n)\) is defined similarly.

**Remark 1.2.** Due to the inclusion relation: \(L^3(\mathbb{R}^3) \subseteq VMO^{-1}(\mathbb{R}^3)\) [10], this result is an improvement of that given by Mahalov, Nicolaenko and Shilkin [12].
Compared with the case when \((u, b) \in L^\infty(-1, 0; L^2(\mathbb{R}^3))\), the main difficulty is that the function in \(VMO^{-1}(\mathbb{R}^3)\) has no decay at infinity, which ensures that the solution is smooth outside a big ball centered at origin so that the backward uniqueness theorem can be applied. Our key observation is that if \(u \in L^3(\mathbb{R}^3 \times (-1, 0))\), we have
\[
\int_{Q_1(z_0)} |u(x, t)|^3 \, dx \, dt \to 0 \quad \text{as} \quad |z_0| \to \infty,
\]
from which we can deduce that the scaled quantity
\[
r^{-2} \int_{Q_r(z_0)} |u|^3 + |b|^3 + |p|^2 \, dx \, dt
\]
is small for some \(r > 0\) and \(|z_0| \gg 1\). Then the regularity of the solution outside a big ball centered at origin follows from the classical small energy regularity theorem.

2. Preliminaries

Let us first introduce the definitions of weak solution and suitable weak solution.

**Definition 2.1.** Let \(T > 0\) and \(\Omega \subset \mathbb{R}^3\). We say that \((u, b)\) is a weak solution of the MHD equations (1.1) in \(\Omega_T = \Omega \times (-T, 0)\) if

(a) \((u, b) \in L^\infty(-T, 0; L^2(\Omega)) \cap L^2(-T, 0; H^1_0(\Omega))\);
(b) \((u, b)\) satisfies Eqs. (1.1) in \(\mathcal{D}'(\Omega_T)\);
(c) \((u, b)\) satisfies the energy inequality: for a.e. \(t \in [-T, 0]\)
\[
\left\| u(t) \right\|^2_{L^2} + \left\| b(t) \right\|^2_{L^2} + 2 \int_0^t \left\| \nabla u(s) \right\|^2_{L^2} + \left\| \nabla b(s) \right\|^2_{L^2} \, ds \leq \| u_0 \|^2_{L^2} + \| b_0 \|^2_{L^2};
\]
We say that \((u, b, p)\) is a suitable weak solution if (c) is replaced by

(d) \(p \in L^2(\Omega_T)\) and the following local energy inequality holds: for a.e. \(t \in [-T, 0]\)
\[
\int_\Omega \left( |u(x, t)|^2 + |b(x, t)|^2 \right) \phi \, dx + 2 \int_{-T}^t \int_\Omega (|\nabla u|^2 + |\nabla b|^2) \phi \, dx \, ds
\leq \int_{-T}^t \int_\Omega \left( (|u|^2 + |b|^2)(\Delta \phi + \partial_t \phi) + u \cdot \nabla \phi (|u|^2 + |b|^2 + 2p) - (b \cdot u)(b \cdot \nabla \phi) \right) \, dx \, ds,
\]
for any nonnegative \(\phi \in C_c^\infty(\mathbb{R}^3 \times \mathbb{R})\) vanishing in a neighborhood of the parabolic boundary of \(\Omega_T\).

**Remark 2.2.** In general, we don’t know whether the weak solution is suitable. However, this is true if \((u, b) \in L^4(\Omega_T)\).

We define a solution \((u, b)\) to be regular at \(z_0 = (x_0, t_0)\) if \((u, b) \in L^\infty(Q_r(z_0))\) with \(Q_r(z_0) = (-r^2 + t_0, t_0) \times B_r(x_0)\), and \(B_r(x_0)\) is a ball of radius \(r\) centered at \(x_0\). We also denote \(Q_r\) by \(Q_r(0)\)
and $B_r$ by $B_r(0)$. For a function $u$ defined on $Q_{r}(z_0)$, the mixed space–time norm $\|u\|_{L^{p,q}(Q_{r}(z_0))}$ is defined by

$$\|u\|_{L^{p,q}(Q_{r}(z_0))}^{q} := \left(\int_{t_0}^{t_0+r^2} \left( \int_{B_r(x_0)} |u(x,t)|^p \, dx \right)^{q/p} \, dt \right)^{q/p}.$$  

The following small energy regularity result is well known, see [8,12]. Similar result was proved by Lin [11] for the Navier–Stokes equations.

**Proposition 2.3.** Assume that $(u, b, p)$ is a suitable weak solution of (1.1) in $Q_{1}(z_0)$. There exists an absolute constant $\varepsilon > 0$ such that if

$$r^{-2} \int_{Q_{r}(z_0)} |u|^3 + |b|^3 + |p|^3 \, dx \, dt \leq \varepsilon$$

for some $r > 0$, then $(u, b)$ is regular at the point $z_0$.

We also need the small energy regularity result in terms of the velocity proved by the authors [17].

**Proposition 2.4.** Assume that $(u, b, p)$ is a suitable weak solution of (1.1) in $Q_{1}(z_0)$. There exists an absolute constant $\varepsilon > 0$ such that if $u \in L^{p,q}$ near $z_0$ and

$$\lim_{r \to 0} \sup_{r > 0} r^{-\frac{3}{2} + \frac{1}{q} + \frac{3}{p} - 1} \|u\|_{L^{p,q}(Q_{r}(z_0))} < \varepsilon,$$  

with $p, q$ satisfying $1 \leq 3/p + 2/q \leq 2$, $1 \leq q \leq \infty$ and $(p, q) \neq (\infty, 1)$. Then $(u, b)$ is regular at the point $z_0$.

The following is a weak-strong uniqueness result.

**Proposition 2.5.** Let $(u, b)$ be a weak solution of (1.1) in $\mathbb{R}^3 \times (0, T)$. Assume that the initial data $(u_0, b_0) \in L^3(\mathbb{R}^3)$. Then there exists a positive constant $\tau$ such that $(u, b) \in C([0, \tau]; L^3(\mathbb{R}^3))$. Especially, $(u, b)$ is smooth in $(0, \tau) \times \mathbb{R}^3$.

Next we introduce some notations. Let $(u, p, b)$ be a solution of (1.1) and introduce the following scaling:

$$u^\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t), \quad b^\lambda(x, t) = \lambda b(\lambda x, \lambda^2 t), \quad p^\lambda(x, t) = \lambda^2 p(\lambda x, \lambda^2 t),$$  

for any $\lambda > 0$, then the family $(u^\lambda, b^\lambda, p^\lambda)$ is also a solution of (1.1). For $z_0 = (x_0, t_0)$, we define some invariant quantities under the scaling (2.2):

$$A(u, r, z_0) = \sup_{-r^2 + t_0 \leq t < t_0} r^{-1} \int_{B_r(x_0)} |u(y, t)|^2 \, dy, \quad A(b, r, z_0) = \sup_{-r^2 + t_0 \leq t < t_0} r^{-1} \int_{B_r(x_0)} |b(y, t)|^2 \, dy,$$

$$C(u, r, z_0) = r^{-2} \int_{Q_{r}(z_0)} |u(y, s)|^3 \, dy \, ds, \quad C(b, r, z_0) = r^{-2} \int_{Q_{r}(z_0)} |b(y, s)|^3 \, dy \, ds,$$
\[ E(u, r, z_0) = r^{-1} \int_{Q_r(z_0)} |\nabla u(y, s)|^2 \, dy \, ds, \quad E(b, r, z_0) = r^{-1} \int_{Q_r(z_0)} |\nabla b(y, s)|^2 \, dy \, ds, \]

\[ K(u, r, z_0) = r^{-3} \int_{Q_r(z_0)} |u(y, s)|^2 \, dy \, ds, \quad K(b, r, z_0) = r^{-3} \int_{Q_r(z_0)} |b(y, s)|^2 \, dy \, ds, \]

\[ D(p, r, z_0) = r^{-2} \int_{Q_r(z_0)} |p(y, s)|^{\frac{3}{2}} \, dy \, ds. \]

Moreover,

\[ A(u, b; r, z_0) = \sup_{-r^2+t_0 \leq t < 0} r^{-1} \int_{B_r(x_0)} |u(y, t)|^2 + |b(y, t)|^2 \, dy. \]

\[ A(u, r, (0, 0)) = A(u, r), \quad A(u, b; r, (0, 0)) = A(u, b; r) \]

and so on.

### 3. Some technical lemmas

Throughout this section, we denote by \( C \) a constant independent of \( r, \rho \) and different from line to line.

**Lemma 3.1.** Let \((u, b)\) be a weak solution of (1.1) in \( \mathbb{R}^3 \times (-1, 0) \). Assume that,

\[ \|u\|_{L^\infty(-1, 0; L^1)} + \|b\|_{L^\infty(-1, 0; BMO^{-1})} \leq C_0. \]

Then for any \( 0 < r < 1/2 \),

\[ A(u, b; r, z_0) + E(u, b; r, z_0) + D(p, r, z_0) \leq C(0, C(b, 1/2, z_0), D(p, 1/2, z_0)), \]

for any \( z_0 = (x_0, t_0) \in \mathbb{R}^3 \times [-\frac{1}{2}, 0] \).

**Proof.** Due to Lemma 6.5 in [16], we have

\[ \|b\|_{L^4}^4 = \|b\|_{L^2}^2 \leq c\|\nabla b\|_{L^2}^2 \|b\|_{BMO^{-1}}^2, \]

which gives \( b \in L^4(-1, 0; L^4(\mathbb{R}^3)) \), thus there exists a pressure \( p \) such that \((u, b, p)\) is a suitable weak solution of (1.1).

Without loss of generality, we consider \( z_0 = (0, 0) \). Let \( \xi(x, t) \) be a smooth function with \( \xi \equiv 1 \) in \( Q_r \) and \( \xi = 0 \) in \( Q_r^c \). Since \( b \in L^\infty(-1, 0; BMO^{-1}(\mathbb{R}^3)) \), there exists \( U(x, t) \in L^\infty(-1, 0; BMO(\mathbb{R}^3)) \) such that \( b = \nabla \cdot U \). By Hölder inequality, we have

\[
r^{-2} \int_{Q_{2r}} |b|^2 \xi^2 \, dx \, dt = r^{-2} \int_{Q_{2r}} \sum_{j=1}^3 \partial_j U_j \cdot b |\xi|^2 \, dx \, dt \leq 6r^{-2} \int_{Q_{2r}} |U - U_{B_{2r}}|(|\nabla b| + |b|^2 |\nabla \xi|) \, dx \, dt
\]
\[ 6r^{-2} \left( \int_{Q_{2r}} |U - U_{B_{2r}}|^6 dx \, dt \right)^{1/6} \left( \int_{Q_{2r}} |\nabla b|^2 dx \, dt \right)^{1/2} \left( \int_{Q_{2r}} |b|^3 \, dx \, dt \right)^{1/3} + 12r^{-3} \left( \int_{Q_{2r}} |U - U_{B_{2r}}|^2 dx \, dt \right)^{1/3} \left( \int_{Q_{2r}} |b|^3 \, dx \, dt \right)^{2/3}, \]

which implies that

\[ C(b, r) \leq c \left( E(b, 2r)^{1/2} C(b, 2r)^{1/3} + C(b, 2r)^{2/3} \right). \tag{3.1} \]

Since \((u, b, p)\) is a suitable weak solution, for any \(0 < r \leq \rho < 1/2\),

\[ A(u, b; r) + E(u, b; r) \leq c \left( K(u, b; 2r) + C(u, 2r) \int_{r}^{1} D(p, 2r) \frac{1}{2} + C(u, 2r)^{1/4} C(b, 2r)^{1/2} \right), \tag{3.2} \]

and for the pressure term,

\[ D(p, r) \leq c \left( \frac{r}{\rho} D(p, \rho) + \left( \frac{\rho}{r} \right)^{2} C(u, b; \rho) \right). \tag{3.3} \]

Similar computations can be found in [14].

Let \(r = \theta \rho\) with \(\theta \leq \frac{1}{2}\). Then it follows from (3.1)-(3.3) that

\[ C(b, r) \leq c \left( 1 + C(b, 4r)^{2/3} + C(b, 4r)^{5/6} + C(b, 4r)^{1/2} D(p, 4r)^{1/3} \right) \leq c \left( 1 + C(b, 4r)^{2/3} + C(b, 4r)^{5/6} + \theta^{1/3} C(b, 4r)^{1/2} D(p, \rho)^{1/3} \right) \]

\[ + \theta^{-2/3} C(b, 4r)^{1/3} C(b, \rho)^{1/3} \leq \left( \eta C(b, \rho) + \theta \left( D(p, \rho)^{5/6} + C(b, \rho) \right) \right) + c(\eta, \theta), \]

for any \(\eta > 0\). Here we used

\[ K(u, b; 2r) \leq c \left( 1 + C(b, 2r) \right)^{3/2}. \]

Set \(F(r) = C(b, r) + D(p, r)^{5/6}\). We get by (3.3) again that

\[ F(r) \leq c \left( \eta + c \theta \right) F(\rho) + c(\eta, \theta). \]

Choose \(\eta\) and \(\theta\) small enough such that \((\eta + c \theta) < 1/4\), and by a standard iteration argument we infer that

\[ F(r) \leq c F(1/2) + C, \]

for all \(0 < r < 1/2\). This proves the lemma. \(\Box\)
Lemma 3.2. Let \((u, b)\) be a weak solution of (1.1) in \(\mathbb{R}^3 \times (-1, 0)\). Assume that
\[
\rho^{-2} \|u\|_{L^{3,1}(Q_\rho(z_0))} \leq \sigma.
\]
Then we have
\[
K(b, r, z_0) \leq c \left( \frac{\rho}{r} \right)^3 \sigma^2 (A(b, \rho, z_0) + E(b, \rho, z_0)) + c \left( \frac{r}{\rho} \right)^2 K(b, \rho, z_0),
\]
where \(0 < 4r < \rho\) and \(c\) is a constant independent of \(\rho, r\) and \(\sigma\).

Proof. Without loss of generality, we consider \(z_0 = (0, 0)\). Let \(\zeta\) be a cutoff function, which vanishes outside of \(Q_\rho\) and is 1 in \(Q_\rho/2\). Set \(b = \hat{b} + \tilde{b}\), where
\[(\partial_t - \Delta) \hat{b}_i = -\partial_j \left( (u_j b_i - u_i b_j) \zeta^2 \right),
(\partial_t - \Delta) \tilde{b} = 0 \text{ in } Q_\rho/2.
\]
Hence, \(\hat{b}\) satisfies
\[
\hat{b}_i(x, t) = -\int_{-\rho^2}^0 ds \int_{\mathbb{R}^3} \partial_j \Gamma(x - y, t - s) \left[ (u_j b_i - u_i b_j) \zeta^2 \right] dy,
\]
where \(\Gamma(x, t)\) is the heat kernel. By Young’s inequality, we have
\[
\left( \int_{\mathbb{R}^3} |\hat{b}|^2 dy \right)^{1/2} \leq \int_{-\rho^2}^0 \left( \int_{\mathbb{R}^3} |\nabla \Gamma(\cdot, t - s)|^\alpha dy \right)^{1/\alpha} \left( \int_{\mathbb{R}^3} |u(\cdot, s)|^3 dy \right)^{1/3} \left( \int_{\mathbb{R}^3} |b(\cdot, s)|^p dy \right)^{1/p} ds,
\]
where \(\frac{1}{2} = \frac{1}{\alpha} + \frac{1}{3} + \frac{1}{p} - 1\). And using Young’s inequality again, we get
\[
\|\hat{b}\|_{L^2(Q_\rho)} \leq c \|\nabla \Gamma\|_{L^{\alpha,\beta}(Q_{2\rho})} \|u\|_{L^3(Q_\rho)} \|b\|_{L^{p,q}(Q_\rho)},
\]
where \(\frac{1}{2} = \frac{1}{\beta} + \frac{1}{3} + \frac{1}{q} - 1\). Now we choose \(\alpha = \frac{9}{7}, \beta = 1, p = \frac{18}{7}\) and \(q = 6\) so that \(\frac{3}{\alpha} + \frac{2}{\beta} = \frac{13}{3}\) and
\[
\|\nabla \Gamma\|_{L^{\alpha,\beta}(Q_{2\rho})} \leq c \rho^{\frac{3}{\alpha} + \frac{2}{\beta} - 4} \leq c \rho^{1/3}.
\]
Then by Sobolev’s interpolation inequality from [1], we get
\[
K(\hat{b}, \rho) \leq c \sigma^2 (A(b, \rho) + E(b, \rho)).
\]
On the other hand, since \(\tilde{b}\) satisfies the heat equation, we have
\[
K(\tilde{b}, r) \leq c \left( \frac{r}{\rho} \right)^2 K(\tilde{b}, \rho).
\]
Hence, it follows that

\[ K(b, r) \leq K(\hat{b}, r) + K(\hat{b}, r) \]
\[ \leq c \left( \frac{\rho}{r} \right)^3 K(\hat{b}, \rho/2) + c \left( \frac{r}{\rho} \right)^2 K(\hat{b}, \rho/2) \]
\[ \leq c \left( \frac{\rho}{r} \right)^3 \sigma^2 (A(b, \rho) + E(b, \rho)) + c \left( \frac{r}{\rho} \right)^2 K(b, \rho). \]

The proof is finished. \( \square \)

**Lemma 3.3.** Let \((u, p, b)\) be a suitable weak solution of (1.1) in \(\mathbb{R}^3 \times (-2, 0)\). Assume that \(u \in L^2(\mathbb{R}^3 \times (-2, 0))\) and \(C(u, b; r, z_0) + D(p, 1, z_0) \leq C_0 \) for \(0 < r < 1\) and \(z_0 \in \mathbb{R}^3 \times (-1, 0)\). Then there exists \(R_0 > 0\) such that when \(|x| > R_0\) and \(-1 < t < 0\), \(u(x, t)\) is regular and uniformly bounded.

**Proof.** Due to \(C(u, b; r, z_0) \leq C_0\), using the local energy inequality, we can get by a standard iterative argument that (see also the proof of Lemma 3.1)

\[ A(u, b; r, z_0) + E(u, b; r, z_0) + D(p, r, z_0) \leq c(C_0) \quad \text{for } 0 < r < 1. \] \( (3.4) \)

Since \(u \in L^2(\mathbb{R}^3 \times (-2, 0))\), for \(z_0 = (x_0, t_0)\) with \(-1 < t_0 < 0,\)

\[ \int_{Q_{\rho}(z_0)} |u|^3 dz \to 0 \quad \text{as } |x_0| \to \infty. \]

Hence, there exists \(R_0 > 0\) such that \(\int_{Q_{\rho}(z_0)} |u|^3 dz \leq \delta\) for \(|x_0| \geq R_0\), where \(\delta\) is to be chosen. Applying Lemma 3.2 with \(r = \theta\) and \(\rho = 1\) to get

\[ K(b, \theta, z_0) \leq c\theta^{-3} \delta^{2/3} (A(b, 1, z_0) + E(b, 1, z_0)) + c\theta^2 K(b, 1, z_0) \leq c\theta^{-3} \delta^{2/3} + c\theta^2. \]

This together with (3.4) gives by Sobolev’s interpolation inequality that

\[ C(b, \theta, z_0) \leq c \left( \theta^{-5/3} \int_{Q_{\rho}(z_0)} |b| \frac{1}{2} dz \right)^{3/4} \left( \theta^{-3} \int_{Q_{\rho}(z_0)} |b|^2 dz \right)^{1/4} \]
\[ \leq cK(b, \theta, z_0)^{1/4} \leq c\theta^{-3/4} \delta^{1/6} + c\theta^{1/2}. \]

Choosing \(r = \theta \gamma\) and \(\rho = \theta\) in (3.3), by (3.4) we get

\[ D(p, \theta \gamma, z_0) \leq c\gamma D(p, \theta, z_0) + \gamma^{-2} C(u, b; \theta, z_0) \]
\[ \leq c\gamma + \gamma^{-2} \left( \delta^2 + c\theta^{-3/4} \delta^{1/6} + c\theta^{1/2} \right). \]

Consequently,

\[ D(p, \theta \gamma, z_0) + C(u, b; \theta \gamma, z_0) \]
\[ \leq c\gamma + \gamma^{-2} \left[ \delta^2 + c\theta^{-3/4} \delta^{1/6} + c\theta^{1/2} \right] + c\theta^{-3/4} \delta^{1/6} \gamma^{-2} + c\theta^{1/2} \gamma^{-2} + \delta \theta^{-2} \gamma^{-2} \]
\[ \leq c\gamma + c\theta^{1/2} \gamma^{-2} + c \left[ \gamma^{-2} \theta^{-2} \delta + \gamma^{-2} \theta^{-3/4} \delta^{1/6} \right]. \]
Now we first choose $\gamma$, $\theta$ such that $c\gamma \leq \varepsilon/3$, $c\theta^{1/2}\gamma^{-2} \leq \varepsilon/3$, and then choose $\delta$ such that $c[\gamma^{-2}\theta^{-2}\delta + \gamma^{-2}\theta^{-3/4}\delta^{1/6}] \leq \varepsilon/3$. Then by Proposition 2.3, we have $|u(z_0)| \leq c$ uniformly for $|z_0| > R_0$. □

4. Proof of Theorem 1.1

Following [7], the proof of Theorem 1.1 is based on the blow-up analysis and unique continuation theorem. By the assumption of Theorem 1.1, we have

$$
\|u\|_{L^\infty(-1, 0; L^1)} + \|b\|_{L^\infty(-1, 0; \text{BMO}^{-1})} \leq C_0.
$$

Using re-scaling argument, it is enough to prove that the point $(0, 0)$ is regular. Since $(u, b, p)$ is a suitable weak solution, there is a constant $C_0$ such that $C(b, 1) + D(p, 1) \leq C_0$. Assume that $(0, 0)$ is not a regular point. Then by Proposition 2.4, there exists $R_k \downarrow 0$ such that

$$
R_k^{-2} \int_{Q_{R_k}} |u|^3 \, dx \, dt > \varepsilon. \tag{4.1}
$$

We denote

$$
u_k = R_k u(R_k y, R_k^2 s), \quad \theta_k = R_k b(R_k y, R_k^2 s), \quad p_k = R_k^2 p(R_k y, R_k^2 s),
$$

where $(y, s) \in \mathbb{R}^3 \times (-\frac{1}{R_k}, 0)$. Then it follows from Lemma 3.1 that for any $a > 0$ with $a R_k \leq 1/2$,

$$
a^{-2} \int_{Q_a} |u|^3 \, dy = (a R_k)^{-2} \int_{Q_{a R_k}} |u|^3 \, dx \leq c, \tag{4.2}
$$

$$
a^{-2} \int_{Q_a} |b|^3 \, dy = (a R_k)^{-2} \int_{Q_{a R_k}} |b|^3 \, dx \leq c, \tag{4.3}
$$

$$
a^{-2} \int_{Q_a} |p|^{3/2} \, dy = (a R_k)^{-2} \int_{Q_{a R_k}} |p|^{3/2} \, dx \leq c. \tag{4.4}
$$

Here $c$ is a constant depending only on $C_0$. Since $(\nu_k, p_k)$ is still a suitable weak solution, we infer from the local energy inequality and (4.2)–(4.4) that for any $a > 0$ and $T < \frac{1}{R_k^2}$,

$$
\|u_k\|_{L^\infty((-T, 0) \times B_a)} + \|b_k\|_{L^\infty((-T, 0) \times B_a)}
$$

$$
+ \|\nabla u_k\|_{L^2((-T, 0) \times B_a)} + \|\nabla b_k\|_{L^2((-T, 0) \times B_a)} \leq c(a, T). \tag{4.5}
$$

Hence, $u_k \cdot \nabla u_k, u_k \cdot \nabla b_k, b_k \cdot \nabla u_k, b_k \cdot \nabla b_k \in L^{9/8}_x L^{3/2}_t (Q_a)$. This gives by the linear Stokes theory [7] that

$$
|\partial_t u_k| + |\Delta u_k| + |\partial_t b_k| + |\Delta b_k| + |\nabla p_k| \in L^{9/8}_x L^{3/2}_t (Q_{3a/4}).
$$

Then Lions–Aubin’s lemma ensures that there exists $(v, \tilde{b}, q)$ such that for any $a, T > 0$ (up to subsequence),
\[ u^k \to v, \quad b^k \to \tilde{b}, \quad \text{in} \ L^3((-T, 0) \times B_a), \]
\[ u^k \to v, \quad b^k \to \tilde{b}, \quad \text{in} \ C([-T, 0]; L^{9/8}(B_a)), \]
\[ p^k \to q \quad \text{in} \ L^3((-T, 0) \times B_a), \]
as \( k \to +\infty \). Hence by (4.1), we have
\[
\int_{Q_1} |v|^3 \ dx \ dt = \lim_{k \to \infty} \int_{Q_1} |u^k|^3 = \lim_{k \to \infty} R_k^{-2} \int_{Q_1} |u|^3 \ dx \ dt \geq \varepsilon_1. \tag{4.6}
\]
Moreover,
\[
\sup_{-T < x < 0} \int_{R^3} |u^k(y, t)|^{3} \ dy + \| b^k \|_{L^{\infty}(-1, 0; BMO^{-1})} \leq C(0),
\]
and for any \( z_0 \in (-T, 0] \times \mathbb{R}^3 \) and \( R_k|z_0| \leq 1/2 \), Lemma 3.1 gives
\[
\limsup_{k \to \infty} D(p^k, 1/2, z_0) \leq C(0),
\]
hence \((v, \tilde{b}, q)\) satisfies
\[
C(v, \tilde{b}; r, z_0) + D(q, 1/2, z_0) + \|v\|_{L^{\infty}((-T, 0] \times \mathbb{R}^3)} \leq C(0), \tag{4.7}
\]
where \( 0 < r < 1 \) and \( z_0 \in (-T + 1, 0] \times \mathbb{R}^3 \).

Due to \( u(x, 0) \in L^3(\mathbb{R}^3) \), then for any \( a > 0 \),
\[
\int_{B_a} |v(x, 0)| \ dx \leq c \int_{B_a} |v(x, 0) - u^k(x, 0)| \ dx + \int_{B_a} |u^k(x, 0)| \ dx
\]
\[
\leq c \int_{B_a} |v(x, 0) - u^k(x, 0)| \ dx + c a^2 \int_{B_{aR_k}} |u(x, 0)|^3 \ dy
\]
\[
\to 0 \quad \text{as} \quad k \to \infty,
\]
which implies \( v(x, 0) = 0 \) for a.e. \( x \in \mathbb{R}^3 \). On the other hand, \( b(x, 0) = \sum_{j=1}^{n} \partial_j U_j \) with \( U_j \in VMO(\mathbb{R}^3) \), then for any \( \varphi \in C_\infty^\infty(B_a) \), we have
\[
\left| \int_{B_a} \tilde{b}(x, 0) \varphi \ dx \right| \leq c \left| \int_{B_a} \tilde{b}(x, 0) - b^k(x, 0) \ dx \right| + \left| \int_{B_a} b^k(x, 0) \varphi \ dx \right|
\]
\[
\leq c \left| \int_{B_a} \tilde{b}(x, 0) - b^k(x, 0) \ dx + R_k^{-3} \int_{B_{aR_k}} U(y, 0)(\nabla \varphi)(y/R_k) \ dy \right|
\]
\[
\leq c \left| \int_{B_a} \tilde{b}(x, 0) - b^k(x, 0) \ dx + c R_k^{-3} \int_{B_{aR_k}} |U(y, 0) - U_{B_{aR_k}}| \ dy \right|
\]
\[
\to 0 \quad \text{as} \quad k \to \infty.
\]
This implies \( \tilde{b}(x, 0) = 0 \) for a.e. \( x \in \mathbb{R}^3 \).
In the following arguments, we follow the route as in [5]. By Lemma 3.3 and (4.7), there exists $R > 0$ such that $(v, \tilde{b})$ is regular on $\mathbb{R}^3 \setminus B_R \times [-T/2, 0]$. Hence $|v|, |\nabla v|, |\tilde{b}|, |\nabla \tilde{b}| \leq c$ by Theorem 3.3 in [12]. Hence,

$$\left( \partial_t - \Delta \right) \tilde{b} \leq c \left( |\tilde{b}| + |\nabla \tilde{b}| \right), \quad \text{in } \mathbb{R}^3 \setminus B_R \times [-T/2, 0],$$

then by backward uniqueness theorem of parabolic operator (see [7]), we can conclude that $\tilde{b}$ vanishes in $\mathbb{R}^3 \setminus B_R \times [-T/2, 0]$. On the other hand, for almost every $t_0 \in (-T/2, 0)$, $(v(t_0), \tilde{b}(t_0)) \in L^3(\mathbb{R}^3)$, and by Proposition 2.5, we have $(v, \tilde{b})$ are smooth in $\mathbb{R}^3 \times (t_0, t_0 + \delta)$ for some $\delta > 0$. Hence by unique continuation of parabolic operator, we conclude $\tilde{b} = 0$ a.e. in $\mathbb{R}^3 \times (-T/2, 0)$.

Consequently, $v$ satisfies

$$(\partial_t - \Delta) v = -v \cdot \nabla v - \nabla q, \quad \text{in } \mathbb{R}^3 \times (-T/2, 0),$$

and let $w = \nabla \times v$, then

$$(\partial_t - \Delta) w = -v \cdot \nabla w + w \cdot \nabla v.$$

Again using backward uniqueness and unique continuation of parabolic operator, we have $w \equiv 0$ in $\mathbb{R}^3 \times (-T/2, 0)$. Then $\Delta v \equiv 0$ in $\mathbb{R}^3 \times (-T/2, 0)$, and we obtain $v \equiv 0$ in $\mathbb{R}^3 \times (-T/2, 0)$, since $v \in L^{3, \infty}$. This is a contradiction with (4.6). \qed

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**References**