Multidimensional Bell Polynomials of Higher Order

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Abstract—We develop some extensions of the classical Bell polynomials, previously obtained, by introducing a further class of these polynomials called multidimensional Bell polynomials of higher order. They arise considering the derivatives of functions $f$ in several variables $\varphi(i)$, ($i = 1, 2, \ldots, m$), where $\varphi(i)$ are composite functions of different orders, i.e. $\varphi(i)(t) = \varphi(i,1)(\varphi(i,2)(\ldots(\varphi(i,n)(t))), (i = 1, 2, \ldots, m)$. We show that these new polynomials are always expressible in terms of the ordinary Bell polynomials, by means of suitable recurrence relations or formal multinomial expansions. Moreover, we give a recurrence relation for their computation.

Keywords—Differentiation of composite functions, Symmetric functions, Bell polynomials.

1. INTRODUCTION

The importance and utility of ordinary Bell polynomials is well known in many different frameworks of mathematics. They are often used, for example, in combinatorial analysis [2], and even in statistics [3], but without explicit reference. Moreover, these polynomials have been applied even in other contexts, such as the Blissard problem (see [2, p. 46]), the representation of Lucas polynomials of the first and second kind [4,5], the representation formulas of Newton sum rules.

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MULTIDIMENSIONAL BELL POLYNOMIALS

for polynomials' zeros [6,7], the recurrence relations for a class of Freud-type polynomials [8], and the representation of symmetric functions of a countable set of numbers, therefore, generalizing the classical algebraic Newton-Girard formulas [9]. Consequently, they were also used [10] in order to find reduction formulas for the orthogonal invariants of a strictly positive compact operator, deriving in a simple way, the so-called Robert formulas [11].

We recall that the Bell polynomials are a classical mathematical tool for representing the \( n \)th derivative of a composite function. In fact, by considering the composite function \( \Phi(t) := f(g(t)) \) of functions \( x = g(t) \) and \( y = f(x) \) defined in suitable intervals of the real axis and \( n \) times differentiable with respect to the relevant independent variables and by using the following notations,

\[
\Phi_h := D_h \Phi(t), \quad f_h := D_h f(x)|_{x=g(t)}, \quad g_h := D_h g(t),
\]

and

\[
([f,g]_n) := (f_1, g_1; f_2, g_2; \ldots; f_n, g_n),
\]

they are defined as follows,

\[
Y_n([f,g]_n) := \Phi_n.
\]

For example, one has

\[
\begin{align*}
Y_1([f,g]_1) &= f_1 g_1, \\
Y_2([f,g]_2) &= f_1 g_2 + f_2 g_1, \\
Y_3([f,g]_3) &= f_1 g_3 + f_2 (3g_2 g_1) + f_3 g_3.
\end{align*}
\]

Further examples can be found in [2, p. 49].

Inductively, we can write

\[
Y_n([f,g]_n) = \sum_{k=1}^{n} A_{n,k} (g_1, g_2, \ldots, g_n) f_k,
\]

where the coefficient \( A_{n,k} \), for any \( k = 1, \ldots, n \), is a polynomial in \( g_1, g_2, \ldots, g_n \), homogeneous of degree \( k \) and isobaric of weight \( n \) (i.e., it is a linear combination of monomials \( g_1^{k_1} g_2^{k_2} \cdots g_n^{k_n} \) whose weight is constantly given by \( k_1 + 2k_2 + \cdots + nk_n = n \)).

For them, the following result holds true.

**Proposition 1.1.** The Bell polynomials satisfy the recurrence relation,

\[
Y_0([f,g]_0) := f_1,
\]

\[
Y_{n+1}([f,g]_{n+1}) = \sum_{k=0}^{n} \binom{n}{k} Y_{n-k}([f_1,g]_{n-k}) g_{k+1}, \tag{1.1}
\]

where

\[
([f_1,g]_{n-k}) := (f_2, g_1; f_3, g_2; \ldots; f_{n-k+1}, g_{n-k}).
\]

An explicit expression for the Bell polynomials is also given by the Faà di Bruno formula [12],

\[
\Phi_n = Y_n([f,g]_n) = \sum_{\pi(n)} \frac{n!}{j_1!j_2! \cdots j_n!} f_j \left[ \frac{g_1}{1!} \right]^{j_1} \left[ \frac{g_2}{2!} \right]^{j_2} \cdots \left[ \frac{g_n}{n!} \right]^{j_n}, \tag{1.2}
\]


where the sum runs over all partitions \( \pi(n) \) of the integer \( n \), i.e., \( n = j_1 + 2j_2 + \cdots + nj_n \). In equation (1.2), \( j_h \) denotes the number of parts of size \( h \), and \( j = j_1 + j_2 + \cdots + j_n \) denotes the number of parts of the considered partition. A proof of the Faà di Bruno formula can be found in [2]. In [13], the proof is based on the umbral calculus (see [14] and the references therein).

Some generalized forms of Bell polynomials already appeared in literature (see e.g., [15–17]). A generalization of the Bell polynomials suitable for the differentiation of multivariable composite functions can also be found in [18].

In this article, after recalling in Section 2, the Bell polynomials of order \( r \) [17] and, in Section 3, the multidimensional Bell polynomials [18], we generalize, in Section 4, both these extensions introducing the so-called multidimensional Bell polynomials of higher order which are suitable for representing the derivative of a composite function of several (say \( m \)) variables \( f(\varphi(1)(t), \varphi(2)(t), \ldots, \varphi(m)(t)) \), where \( \varphi(i)(t) = \phi^{(i,1)}(\phi^{(i,2)}(\cdots (\phi^{(i,r_i)}(t))) \), \( i = 1, 2, \ldots, m \). For these polynomials, we will derive a representation formula in terms of the ordinary Bell polynomials and a recurrence relation.

Since we want to obtain a uniform notation for the Bell polynomials of order \( r \), multidimensional Bell polynomials and multidimensional Bell polynomials of higher order, we use in this article some different notations with respect to those introduced in [17,18]. More precisely, for the \( n^{th} \) derivative of a composite function \( f(\varphi(1)(t), \varphi(2)(t), \ldots, \varphi(m)(t)) \), where \( \varphi(i)(t) = \phi^{(i,1)}(\phi^{(i,2)}(\cdots (\phi^{(i,r_i)}(t))) \), \( i = 1, 2, \ldots, m \), we use the notation

\[
Y_{n}^{[m;r_1,\ldots,r_m]} \left( \left[ f, \varphi(1), \ldots, \varphi(m) \right] \right),
\]

where the first upper index \( m \) refers to the number of independent variables of the function \( f \) and the subsequent indices \( r_1, \ldots, r_m \) indicate the number of nested functions in each \( \varphi(i) \) component. Furthermore, in the above notation, the symbol \( \left[ f, \varphi(1), \ldots, \varphi(m) \right] \) denotes the set of all partial derivatives of the function \( f \) with respect to the independent variables up to the order \( n \), and all derivatives of the component functions \( \phi^{(i,h)} \), \( i = 1, 2, \ldots, m \); \( h = 1, 2, \ldots, r_i \), up to the order \( n \), too.

### 2. BELL POLYNOMIALS OF ORDER \( r \)

In [17], a first extension of the classical Bell polynomials was achieved.

Consider \( \Phi(t) := f(\varphi^{(1,1)}(\varphi^{(1,2)}(\cdots (\varphi^{(1,r)}(t)))) \), i.e., the composition of functions \( x^{(1,r)} = \phi^{(1,r)}(t), \ldots, x^{(1,2)} = \phi^{(1,2)}(x^{(1,3)}), x^{(1,1)} = \phi^{(1,1)}(x^{(1,2)}), y = f(x^{(1,1)}) \) defined in suitable intervals of the real axis, and suppose that the functions \( \phi^{(1,r)}, \ldots, \phi^{(1,2)}, \phi^{(1,1)} \), \( f \) are \( n \) times differentiable with respect to the relevant independent variables, so that by using the chain rule, \( \Phi(t) \) can be differentiated \( n \) times with respect to \( t \).

We use the following notations,

\[
\Phi_h := D_h^h \Phi(t),
\]

\[
f_h := D_h^h f \big|_{x^{(1,1)} = \phi^{(1,1)}(\cdots (\phi^{(1,r)}(t)) \big),}
\]

\[
\phi_h^{(1,1)} := D_h^h \phi^{(1,1)} \big|_{x^{(1,2)} = \phi^{(1,2)}(\cdots (\phi^{(1,r)}(t)) \big),}
\]

\[
\vdots
\]

\[
\phi_h^{(1,r)} := D_h^h \phi^{(1,\ldots, r)}(t).
\]


and
\[
\left[ f, \phi^{(1,1)}, \ldots, \phi^{(1,r)} \right]_n := \left( f_1, \phi^{(1,1)}_1, \ldots, \phi^{(1,r)}_1 ; \ldots ; f_n, \phi^{(1,1)}_n, \ldots, \phi^{(1,r)}_n \right).
\]

Then, the \( n \)th derivative of the function \( \Phi \) allows us to define the (one-dimensional) Bell polynomials of order \( r \), \( Y^{[1;r]}_n \), as follows,
\[
Y^{[1;r]}_n \left( \left[ f, \phi^{(1,1)}, \ldots, \phi^{(1,r)} \right] \right) := \Phi_n. \tag{2.2}
\]

For \( r = 1 \), we obtain the ordinary Bell polynomials
\[
Y^{[1;1]}_n \left( \left[ f, \phi^{(1,1)} \right] \right) = Y_n \left( \left[ f, \phi^{(1,1)} \right] \right). \tag{2.3}
\]

The first polynomials have the following explicit expressions,
\[
Y^{[1;r]}_1 \left( \left[ f, \phi^{(1,1)}, \ldots, \phi^{(1,r)} \right] \right) = f_1 \phi^{(1,1)}_1 \ldots \phi^{(1,r)}_1,
\]
\[
Y^{[1;r]}_2 \left( \left[ f, \phi^{(1,1)}, \ldots, \phi^{(1,r)} \right] \right) = f_2 \left( \phi^{(1,1)}_1 \ldots \phi^{(1,r)}_1 \right)^2 + f_1 \phi^{(1,1)}_1 \left( \phi^{(1,2)}_1 \ldots \phi^{(1,r)}_1 \right)^2
\]
\[+ f_1 \phi^{(1,1)}_1 \phi^{(1,2)}_2 \left( \phi^{(1,3)}_1 \ldots \phi^{(1,r)}_1 \right)^2
\]
\[+ f_1 \phi^{(1,1)}_1 \phi^{(1,2)}_2 \ldots \phi^{(1,r-1)}_1 \phi^{(1,r)}_2.
\]

The polynomials \( Y^{[1;r]}_n \) satisfy the following result.

**Theorem 2.1.** For every integer \( n \), the polynomials \( Y^{[1;r]}_n \) are expressed in terms of the Bell polynomials of lower order, by means of the following equation,
\[
Y^{[1;r]}_n \left( \left[ f, \phi^{(1,1)}, \ldots, \phi^{(1,r)} \right] \right) = Y_n \left( f, Y^{[1;r-1]}_n \left( \left[ \phi^{(1,1)}, \ldots, \phi^{(1,r)} \right] \right) \right), \tag{2.4}
\]

where
\[
\left( f, Y^{[1;r-1]}_n \left( \left[ \phi^{(1,1)}, \ldots, \phi^{(1,r)} \right] \right) \right)_n := \left( f_1, Y^{[1;r-1]}_n \left( \left[ \phi^{(1,1)}, \ldots, \phi^{(1,r)} \right] \right) _1 ; \ldots ; f_n, Y^{[1;r-1]}_n \left( \left[ \phi^{(1,1)}, \ldots, \phi^{(1,r)} \right] \right) _n \right).
\]

The relations (1.1), (1.2), can be generalized as follows.

**Theorem 2.2.** The following recurrence relation for the Bell polynomials \( Y^{[1;r]}_n \) holds true,
\[
Y^{[1;r]}_{n+1} \left( \left[ f, \phi^{(1,1)}, \ldots, \phi^{(1,r)} \right] _{n+1} \right) = \sum_{k=0}^{n} \binom{n}{k} Y^{[1;r]}_{n-k} \left( \left[ f_1, \phi^{(1,1)}, \ldots, \phi^{(1,r)} \right] _{n-k} \right) \tag{2.5}
\]
\[
+ Y^{[1;r-1]}_{k+1} \left( \left[ \phi^{(1,1)}, \ldots, \phi^{(1,r)} \right] _{k+1} \right),
\]

where
\[
\left( \left[ f_1, \phi^{(1,1)}, \ldots, \phi^{(1,r)} \right] _{n-k} \right) := \left( f_2, \phi^{(1,1)}_1, \ldots, \phi^{(1,r)}_1 ; \ldots ; f_{n-k+1}, \phi^{(1,1)}_{n-k}, \ldots, \phi^{(1,r)}_{n-k} \right).
\]
Theorem 2.3. The generalized Faà di Bruno formula holds true,

\[
Y^{[1; r]}_n \left( \left[ f, \phi^{(1,1)}, \ldots, \phi^{(1,r)} \right]_n \right) = \sum_{\sigma(n)} \frac{n!}{j_1! j_2! \ldots j_n!} f_j \left[ \frac{Y^{[1;j-1]}_1 \left( \left[ \phi^{(1,1)}, \ldots, \phi^{(1,r)} \right]_1 \right)}{1!} \right]^{j_1} \cdot \left[ \frac{Y^{[1;j-1]}_2 \left( \left[ \phi^{(1,1)}, \ldots, \phi^{(1,r)} \right]_2 \right)}{2!} \right]^{j_2} \ldots \left[ \frac{Y^{[1;j-1]}_n \left( \left[ \phi^{(1,1)}, \ldots, \phi^{(1,r)} \right]_n \right)}{n!} \right]^{j_n}.
\]

(2.6)

By putting, for every integer \( s \) (1 < \( s \) < \( r - 1 \)),

\[
\phi^{(1,s+1)} \left( \phi^{(1,s+2)} \ldots \left( \phi^{(1,r)} (t) \right) \right) =: g (t), \quad f \left( \phi^{(1,1)} \ldots \left( \phi^{(1,s)} (x) \right) \right) =: f (x),
\]

where \( x = g(t) \), the composite function \( \Phi(t) := f(\phi^{(1,1)}(\ldots (\phi^{(1,r)}(t)))) \) can be written as follows,

\[
\Phi(t) = f(g(t)).
\]

Therefore, the following result holds true.

Theorem 2.4. For every integer \( n \), the polynomials \( Y^{[1;r]}_n \) are expressed in terms of the Bell polynomials of lower order, by means of the following equation,

\[
Y^{[1;r]}_n \left( \left[ f, \phi^{(1,1)}, \ldots, \phi^{(1,r)} \right]_n \right) = Y_n \left( Y^{[1;r]} \left( \left[ f, \phi^{(1,1)}, \ldots, \phi^{(1,s)} \right] \right), Y^{[1;r-s-1]} \left( \left[ \phi^{(1,s+1)}, \ldots, \phi^{(1,r)} \right] \right) \right).
\]

(2.7)

3. MULTIDIMENSIONAL BELL POLYNOMIALS

In [18], a further generalization of Bell polynomials was introduced by means of the so-called multidimensional ones, which we briefly recall here.

Let

\[
\Phi(t) = f \left( \phi^{(1,1)}(t), \phi^{(2,1)}(t), \ldots, \phi^{(m,1)}(t) \right),
\]

be a composite function of \( m \) variables \( x^{(i,1)} = \phi^{(i,1)}(t) (i = 1, \ldots, m) \), defined in suitable intervals of real axis \( t \). Moreover, suppose that the functions \( f \) and \( \phi^{(i,1)} (i = 1, \ldots, m) \) are \( n \) times differentiable with respect to the relevant independent variables so that \( \Phi(t) \) can be differentiated \( n \) times with respect to \( t \), by using the differentiation rule of multidimensional function.

By using the following notations,

\[
\Phi_h = D^h \Phi(t),
\]

\[
f_{x^{(1,1)}, \ldots, x^{(m,1)}} := \frac{\partial^{h_1 + \ldots + h_m}}{\partial x^{(1,1)}^{h_1} \ldots \partial x^{(m,1)}^{h_m}} f \left( x^{(1,1)}, \ldots, x^{(m,1)} \right) \bigg|_{x^{(1,1)} = \phi^{(1,1)}(t), \ldots, x^{(m,1)} = \phi^{(m,1)}(t)} ,
\]

\[
\phi^{(i,1)}_h := D^h \phi^{(i,1)}(t), \quad i = 1, \ldots, m,
\]
and
\[
\left( \left[ f, \phi^{(1,1)}, \ldots, \phi^{(m,1)} \right]_n \right)
\]
\[
:= \left( \left\{ f_{x_{h_1}, \ldots, x_{h_m}} \right\}_1, \phi_1^{(1,1)}, \ldots, \phi_1^{(m,1)}; \ldots; \left\{ f_{x_{h_1}, \ldots, x_{h_m}} \right\}_n, \phi_n^{(1,1)}, \ldots, \phi_n^{(m,1)} \right),
\]
where \( \left\{ f_{x_{h_1}, \ldots, x_{h_m}} \right\}_\nu \) is the set of all partial derivatives of \( f \) of order \( \nu \) with respect to his variables (i.e., such that \( h_1 + \cdots + h_m = \nu \)), we can define the \( n \)th multidimensional Bell polynomial (of first order) of \( m \) variables by means of the derivative of order \( n \) of the function \( \Phi(t) \), and we use the following notation,
\[
\Phi_n^{\left[ m;1 \right]} \left( \left[ f, \phi^{(1,1)}, \ldots, \phi^{(m,1)} \right]_n \right) := \Phi_n.
\]
The one-dimensional Bell polynomials (i.e., \( m = 1 \)) are coincident with the classical ones, i.e.,
\[
\Phi_n^{\left[ 1;1 \right]} \left( \left[ f, \phi^{(1,1)} \right]_n \right) = \Phi_n \left( \left[ f, \phi^{(1,1)} \right]_n \right).
\]
We have, for example,
\[
\Phi_n^{\left[ m;1 \right]} \left( \left[ f, \phi^{(1,1)}, \ldots, \phi^{(m,1)} \right]_1 \right) = f_{x_{h_1}, \ldots, x_{h_m}} + \ldots + f_{x_{h_1}, \ldots, x_{h_m}} \phi_1^{(1,1)} = \sum_{i=1}^m \Phi_i \left( \left[ f_{x_{h_1}, \ldots, x_{h_m}} \right]_1, \phi^{(1,1)}_i \right),
\]
where, with \( f_{x_{h_1}, \ldots, x_{h_m}} \), we intend the function \( f \) depending only by the \( i \)th variable \( x^{(1,1)} \).
\[
\Phi_n^{\left[ m;1 \right]} \left( \left[ f, \phi^{(1,1)}, \ldots, \phi^{(m,1)} \right]_2 \right) = \sum_{i=1}^m \left( f_{x_{h_1}, \ldots, x_{h_m}} \phi_1^{(1,1)} \right)^2 + f_{x_{h_1}, \ldots, x_{h_m}} \phi_2^{(1,1)}
\]
\[
+ 2 \sum_{i,j=1 \atop i<j}^m f_{x_{h_1}, \ldots, x_{h_m}} \phi_1^{(1,1)} \phi_1^{(j,1)}.
\]
By using the following differential operator,
\[
\left( f_{x_{h_1}, \ldots, x_{h_m}} \phi_1^{(1,1)} \right) \circ \left( f_{x_{h_1}, \ldots, x_{h_m}} \phi_1^{(j,1)} \right) := f_{x_{h_1}, \ldots, x_{h_m}} \phi_1^{(j,1)} \phi_1^{(j,1)} \phi_1^{(i,1)},
\]
it is easy to verify that
\[
\Phi_n^{\left[ m;1 \right]} \left( \left[ f, \phi^{(1,1)}, \ldots, \phi^{(m,1)} \right]_2 \right)
\]
\[
= \sum_{i_1, \ldots, i_m=0}^2 \frac{2!}{i_1! \cdots i_m!} \Phi_{i_1} \left( \left[ f_{x^{(1,1)}}, \phi^{(1,1)} \right]_{i_1} \right) \circ \cdots \circ \Phi_{i_m} \left( \left[ f_{x^{(m,1)}}, \phi^{(m,1)} \right]_{i_m} \right),
\]
where the indexes \( i_1, \ldots, i_m \), of the sum to the left-hand side assume the values 0, 1, or 2 and satisfy the relation \( i_1 + \cdots + i_m = 2 \).

This last formula can be generalized.

**THEOREM 3.1.** For every positive integer \( n \), the following relation holds true,
\[
\Phi_n^{\left[ m;1 \right]} \left( \left[ f, \phi^{(1,1)}, \ldots, \phi^{(m,1)} \right]_n \right)
\]
\[
= \sum_{i_1, \ldots, i_m=0}^n \frac{n!}{i_1! \cdots i_m!} \Phi_{i_1} \left( \left[ f_{x^{(1,1)}}, \phi^{(1,1)} \right]_{i_1} \right) \circ \cdots \circ \Phi_{i_m} \left( \left[ f_{x^{(m,1)}}, \phi^{(m,1)} \right]_{i_m} \right).
\]
Moreover, the following result holds true.
THEOREM 3.2. For every \( n = 0, 1, \ldots \), the multidimensional Bell polynomials satisfy the recurrence relation,

\[
Y_{n}^{[m;1]} \left( \left[ \left[ f_{x_{1}^{(1)}}, \phi_{1}^{(1)}, \ldots, \phi_{m}^{(1)} \right]_{n+1} \right] \right) = f_{x_{1}^{(1)}}, \quad i = 1, \ldots, m,
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} \sum_{i=1}^{m} Y_{n-k}^{[m;1]} \left( \left[ f_{x_{1}^{(1)}}, \phi_{1}^{(1)}, \ldots, \phi_{m}^{(1)} \right]_{n-k} \right) \phi_{k+1}^{(i)}.
\]

where \( \forall i = 1, \ldots, m \),

\[
\left( \left[ f_{x_{1}^{(1)}}, \phi_{1}^{(1)}, \ldots, \phi_{m}^{(1)} \right]_{n-k} \right) := \left\{ \left. f_{x_{1}^{(1)}}, \ldots, x_{n-k}^{(1)}, \ldots, x_{m}^{(1)} \right\} \phi_{1}^{(1)}, \ldots, \phi_{1}^{(m,1)} ; \ldots ; \left. f_{x_{1}^{(1)}}, \ldots, x_{n-k}^{(1)}, \ldots, x_{m}^{(1)} \right\} \phi_{n-k}^{(1)}, \ldots, \phi_{n-k}^{(m,1)} \right\},
\]

and \( \{ f_{x_{1}^{(1)}}, \ldots, x_{n-k}^{(1)}, \ldots, x_{m}^{(1)} \} \) is the set of all partial derivatives of \( f_{x_{1}^{(1)}} \) of order \( \nu \) with respect to his variables (i.e., such that \( h_{i} + \ldots + h_{m} = \nu \)).

4. MULTIDIMENSIONAL BELL POLYNOMIALS OF HIGHER ORDER

The results summarized in previously sections can be mixed in natural way, in this section, introducing the so-called multidimensional Bell polynomials of higher order.

Let

\[
\Phi (t) = f \left( \varphi^{(1)} (t), \ldots, \varphi^{(m)} (t) \right)
\]

be a \( m \)-dimensional function depending on the variables,

\[
\varphi^{(1)} (t) = \phi^{(1,1)} \left( \varphi^{(1,2)} \left( \ldots \varphi^{(1,r_{1})} (t) \right) \right),
\]

\[
\vdots
\]

\[
\varphi^{(m)} (t) = \phi^{(m,1)} \left( \varphi^{(m,2)} \left( \ldots \varphi^{(m,r_{m})} (t) \right) \right).
\]

For these composite functions, we use the following notation,

\[
x^{(1,1)} = \phi^{(1,1)} \left( x^{(1,2)} \right), x^{(1,2)} = \phi^{(1,2)} \left( x^{(1,3)} \right), \ldots, x^{(1,r_{1})} = \phi^{(1,r_{1})} (t),
\]

\[
\vdots
\]

\[
x^{(m,1)} = \phi^{(m,1)} \left( x^{(m,2)} \right), x^{(m,2)} = \phi^{(m,2)} \left( x^{(m,3)} \right), \ldots, x^{(m,r_{m})} = \phi^{(m,r_{m})} (t).
\]

The functions \( x^{(i,j)} (i = 1, \ldots, m \text{ and } j = 1, \ldots, r_{i}) \) are defined in suitable intervals of the real axis; moreover, with the function \( f \), are \( n \) times differentiable with respect to the relevant independent variables.
Therefore, we put $\Phi_h = D_h^l \Phi(t)$ and, for $i = 1, \ldots, m$,

$$f_{x_{h_1}^{(1,1)}, \ldots, x_{h_m}^{(m,1)}} := \frac{d^{h_1 + \cdots + h_m}}{dx_{x_{h_1}^{(1,1)}} \cdots dx_{x_{h_m}^{(m,1)}}} f\left(x_{x_{h_1}^{(1,1)}}, \ldots, x_{x_{h_m}^{(m,1)}}\right)_{x_{x_{h_1}^{(1,1)}}^t = \phi_{x_{x_{h_1}^{(1,1)}}^{(1,1)}}(t), \ldots, x_{x_{h_m}^{(m,1)}}^t = \phi_{x_{x_{h_m}^{(m,1)}}^{(m,1)}}(t)}.$$

$$\phi_h^{(i,1)} := D_{x^{(i,2)}} \phi^{(i,1)}\left|_{x^{(i,2)} = \phi_{x^{(i,2)}}^{(i,2)}}(t)\right.$$

$$\vdots$$

$$\phi_h^{(i,r_i)} := D_{x^{(i,r_i)}} \phi^{(i,r_i)}(t).$$

By using the following compact notation,

$$\left([f, \phi_1^{(1)}, \ldots, \phi_m^{(m)}]_n\right) = \left(\left(f_{x_{h_1}^{(1,1)}, \ldots, x_{h_m}^{(m,1)}}\right)_{i=1}^n, \left(\phi_{x_{h_1}^{(1,1)}}^1\right)_{i=1}^{r_1}, \ldots, \left(\phi_{x_{h_m}^{(m,1)}}^m\right)_{i=1}^{r_m}\right)$$

$$\vdots$$

$$\left(\left(f_{x_{h_1}^{(1,1)}, \ldots, x_{h_m}^{(m,1)}}\right)_n, \left(\phi_{x_{h_1}^{(1,1)}}^1\right)_1^{r_1}, \ldots, \left(\phi_{x_{h_m}^{(m,1)}}^m\right)_1^{r_m}\right),$$

we define the $m$-dimensional Bell polynomials of higher order as follows,

$$Y_m^{[r_1, \ldots, r_m]}\left([f, \phi_1^{(1)}, \ldots, \phi_m^{(m)}]\right)_n := \Phi_h.$$

It is clear that for $m = 1$ and $r_1 = \cdots = r_m = r$, we obtain the one-dimensional Bell polynomials of order $r$ defined in Section 2, for $r_1 = \cdots = r_m = 1$, we obtain the $m$-dimensional Bell polynomials of first order defined in Section 3, at last for $m = 1$ and $r_1 = \cdots = r_m = 1$, we obtain the ordinary Bell polynomials.

Just like in the previous section, we want to obtain a representation formula for these polynomials in terms of classical Bell ones. For $n = 1$, we have

$$Y_1^{[r_1, \ldots, r_m]}\left([f, \phi_1^{(1)}, \ldots, \phi_m^{(m)}]\right)_1 = \sum_{i=1}^m f_{x_{h_1}^{(i,1)}, \phi_1^{(i,1)}} \cdots \phi_{x_{h_m}^{(i,1)}, r_i}.\left(\phi_{x_{h_1}^{(i,1)}}\cdots \phi_{x_{h_m}^{(i,1)}}\right)_{i=1}^{r_i}.$$
which, by means of the differential operator (3.1), we can write as follows,

\[
Y_2^{[m_1, \ldots, m_n]} \left( \left[ f, \varphi^{(1)}, \ldots, \varphi^{(m)} \right] \right)_{ij}
\]

\[
= \sum_{i_1, \ldots, i_m=0}^{2} \frac{2!}{i_1! \cdots i_m!} Y_{i_1} \left( \left[ f_{x(i_1,1)}, Y^{[1; r_1-1]} \left( \left[ \varphi^{(1,1)}, \ldots, \varphi^{(1,r_1)} \right] \right) \right]_{i_1} \right)
\]

\[
\cdots \cdots Y_{i_m} \left( \left[ f_{x(m,1)}, Y^{[1; r_m-1]} \left( \left[ \varphi^{(m,1)}, \ldots, \varphi^{(m,r_m)} \right] \right) \right]_{i_m} \right).
\]

The last formula can be generalized as follows.

**Theorem 4.1.** For every positive integer \( n \), the following relation holds true,

\[
Y_n^{[m_1, \ldots, m_n]} \left( \left[ f, \varphi^{(1)}, \ldots, \varphi^{(m)} \right] \right)_{i_1}
\]

\[
= \sum_{i_1, \ldots, i_m=0}^{n} \frac{n!}{i_1! \cdots i_m!} Y_{i_1} \left( \left[ f_{x(i_1,1)}, Y^{[1; r_1-1]} \left( \left[ \varphi^{(1,1)}, \ldots, \varphi^{(1,r_1)} \right] \right) \right]_{i_1} \right)
\]

\[
\cdots \cdots Y_{i_m} \left( \left[ f_{x(m,1)}, Y^{[1; r_m-1]} \left( \left[ \varphi^{(m,1)}, \ldots, \varphi^{(m,r_m)} \right] \right) \right]_{i_m} \right).
\]

**Proof.** We proceed by induction. It is easy to verify that for \( n = 1 \), the right-hand sides of (4.4) and (4.5) are coincident. Now, let (4.5) be true for a fixed value of \( n \), we prove that the same formula holds also true for the value \( n + 1 \). In fact,

\[
Y_{n+1}^{[m_1, \ldots, m_n]} \left( \left[ f, \varphi^{(1)}, \ldots, \varphi^{(m)} \right] \right)_{i_1}
\]

\[
= D^n \left( D_t \Phi(t) \right) = \Phi_n \circ \Phi_1
\]

\[
= Y_n^{[m_1, \ldots, m_n]} \left( \left[ f, \varphi^{(1)}, \ldots, \varphi^{(m)} \right] \right) \circ Y_1^{[m_1, \ldots, m_n]} \left( \left[ f, \varphi^{(1)}, \ldots, \varphi^{(m)} \right] \right)
\]

\[
= \left\{ \sum_{i_1, \ldots, i_m=0}^{n} \frac{n!}{i_1! \cdots i_m!} Y_{i_1} \left( \left[ f_{x(i_1,1)}, Y^{[1; r_1-1]} \left( \left[ \varphi^{(1,1)}, \ldots, \varphi^{(1,r_1)} \right] \right) \right]_{i_1} \right) \right\}
\]

\[
\cdots \cdots Y_{i_m} \left( \left[ f_{x(m,1)}, Y^{[1; r_m-1]} \left( \left[ \varphi^{(m,1)}, \ldots, \varphi^{(m,r_m)} \right] \right) \right]_{i_m} \right)
\]

\[
\circ \sum_{i_1=1}^{n} Y_{i_1} \left( \left[ f_{x(i_1,1)}, Y^{[1; r_1-1]} \left( \left[ \varphi^{(i,1)}, \ldots, \varphi^{(i,r_1)} \right] \right) \right]_{i_1} \right)
\]

\[
= \left\{ \sum_{i_1, \ldots, i_m=0}^{n} \frac{n!}{i_1! \cdots i_m!} Y_{i_1+1} \left( \left[ f_{x(i_1+1,1)}, Y^{[1; r_1+1]} \left( \left[ \varphi^{(1,1)}, \ldots, \varphi^{(1,r_1)} \right] \right) \right]_{i_1+1} \right) \right\}
\]

\[
\cdots \cdots Y_{i_m} \left( \left[ f_{x(m,1)}, Y^{[1; r_m-1]} \left( \left[ \varphi^{(m,1)}, \ldots, \varphi^{(m,r_m)} \right] \right) \right]_{i_m} \right)
\]

\[
+ \cdots + \left\{ \sum_{i_1, \ldots, i_m=0}^{n} \frac{n!}{i_1! \cdots i_m!} Y_{i_1} \left( \left[ f_{x(i_1,1)}, Y^{[1; r_1-1]} \left( \left[ \varphi^{(1,1)}, \ldots, \varphi^{(1,r_1)} \right] \right) \right]_{i_1} \right) \right\}
\]
Remark 4.2. We can note that formula (4.5) reduces to the (2.4),(3.2) ones when \( m = 1 \) and \( r_1 = \ldots = r_m = 1 \), respectively.

At last, a similar remark can be done for the recurrence relations (2.5),(3.3), in fact, we prove the following result.

Theorem 4.3. For every \( n = 0, 1, \ldots \), the \( m \)-dimensional Bell polynomials of higher order satisfy the recurrence relation,

\[
\begin{align*}
Y^{[m; r_1, \ldots, r_m]}_{i_1, \ldots, i_m} & \left( \left[ f_{x_{i_1, i_2}}^{(1)}, \phi^{(1)}, \ldots, \phi^{(m)} \right]_{i_1, \ldots, i_m} \right) = f_{x_{i_1, i_2}}^{(1)}, \\
& \quad i = 1, \ldots, m, \\
Y^{[m; r_1, \ldots, r_m]}_{n+1} & \left( \left[ f^{(1)}, \phi^{(1)}, \ldots, \phi^{(m)} \right]_{n+1} \right) \\
& = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k!} \sum_{i=1}^{m} Y^{[m; r_1, \ldots, r_m]}_{n-k} \left( \left[ f_{x_{i_1, i_2}}^{(1)}, \phi^{(1)}, \ldots, \phi^{(m)} \right]_{n-k} \right) \\
& \quad + \sum_{i_1, \ldots, i_m = 0}^{n+1} \frac{n!}{i_1! \ldots i_m!} \cdot Y_{i_1} \left( \left[ f_{x_{i_1, i_2}}^{(1)}, \phi^{(1)}, \ldots, \phi^{(m)} \right]_{i_1, \ldots, i_m} \right) \\
& \quad + \sum_{i_1, \ldots, i_m = 0}^{n+1} \frac{n!}{i_1! \ldots i_m!} \cdot Y_{i_1} \left( \left[ f_{x_{i_1, i_2}}^{(1)}, \phi^{(1)}, \ldots, \phi^{(m)} \right]_{i_1, \ldots, i_m} \right).
\end{align*}
\]
where $\forall i = 1, \ldots, m,$

$$
\left( f_{x_1^{(1)}, \ldots, x_m^{(m)}}, \varphi^{(1)}, \ldots, \varphi^{(m)} \right)_{n-k}
$$

$$
\left( \begin{array}{c}
\left( f_{x_1^{(1)}, \ldots, x_n^{(1)}}, \ldots, f_{x_1^{(m)}, \ldots, x_n^{(m)}} \right)_{n-k}
\end{array}
\right)
$$

and $\{ f_{x_1^{(1)}, \ldots, x_n^{(1)}}, \ldots, f_{x_1^{(m)}, \ldots, x_n^{(m)}} \}_\nu$ is the set of all partial derivatives of $f_{x_1^{(i)}}$ of order $\nu$ with respect to its variables (i.e., such that $h_1 + \cdots + h_m = \nu$).

**Proof.** We have

$$
\left( f_{x_1^{(1)}, \ldots, x_n^{(1)}}, \varphi^{(1)}, \ldots, \varphi^{(m)} \right)_{n+1} = D^n f \left( \varphi^{(1)}, \ldots, \varphi^{(m)} \right)
$$

$$
= D^n \sum_{i=1}^{m} f_{x_1^{(i)}} \varphi_{1}^{(i)} \cdots \varphi_{n}^{(i)} = \sum_{i=1}^{m} D^n \left( f_{x_1^{(i)}}, Y^{(1)}_{n+1} \left( \left[ \begin{array}{c}
\varphi_{1}^{(i)}, \ldots, \varphi_{n}^{(i)}
\end{array} \right] \right) \right)
$$

By using the Leibniz formula, we obtain

$$
\left( f_{x_1^{(1)}, \ldots, x_n^{(1)}}, \varphi^{(1)}, \ldots, \varphi^{(m)} \right)_{n+1} = \sum_{k=0}^{n} \sum_{i=1}^{m} \binom{n}{k} D^n-k f_{x_1^{(i)}}, Y^{(1)}_{k+1} \left( \left[ \begin{array}{c}
\varphi_{1}^{(i)}, \ldots, \varphi_{k}^{(i)}
\end{array} \right] \right)
$$

Since that

$$
D^n-k f_{x_1^{(i)}} = Y^{(1)}_{n-k} \left( f_{x_1^{(i)}}, \varphi^{(1)}, \ldots, \varphi^{(m)} \right)_{n-k}
$$

from (4.7) (interchanging the sums), we obtain the result. 

**REFERENCES**