Global regularity of the 2D micropolar fluid flows with zero angular viscosity

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In this paper we prove the global existence and uniqueness of smooth solutions to the 2D micropolar fluid flows with zero angular viscosity.

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1. Introduction

In this paper, we consider the Cauchy problem to the viscous incompressible micropolar fluid flows, which can be viewed as a non-Newtonian fluid model with asymmetric stress tensor [11,12]. From the model viewpoint, the micropolar fluid model is coupled with the incompressible Navier–Stokes equations, micro-rotational effects and micro-rotational inertia. They are so-called non-Newtonian fluids with nonsymmetric stress tensor. Physically it may represent the fluids consisting of bar-like elements. Certain anisotropic fluids, e.g. liquid crystals which are made up of dumbbell molecules, are of this type. The three-dimensional mathematical model of the incompressible micropolar fluid motion in whole spaces (see [12, Eqs. (4.15), (6.9), (6.10)]) is expressed as

\[
\begin{align*}
\partial_t v - (\nu + \kappa) \Delta v - 2\kappa \nabla \times w + \nabla \pi + v \cdot \nabla v &= 0, \\
\partial_t w - \gamma \Delta w - (\alpha + \beta) \nabla \nabla \cdot w + 4\kappa w - 2\kappa \nabla \times v + v \cdot \nabla w &= 0, \\
\nabla \cdot v &= 0.
\end{align*}
\] (1.1)

This motion represents the conservation of linear momentum, the conservation of angular momentum, and the incompressibility of the fluid, respectively. Here $v = (v_1, v_2, v_3)$, $\pi$, and $w = w_1$.
(w_1, w_2, w_3) stand for the divergence free velocity field, the scalar pressure and non-divergence free micro-rotation field (angular velocity of the rotation of the particles of the fluid) respectively. And ν ≥ 0 is the Newtonian kinetic viscosity and κ > 0 is the dynamics micro-rotation viscosity, α, β, γ ≥ 0 are the angular viscosity (see, for example, Łukaszewicz [19]).

When micro-rotation effects are neglected (i.e., w = 0), (1.1) reduces to the incompressible Navier–Stokes equations. Therefore, from the mathematical viewpoint, (1.1) can be viewed as the modification of Navier–Stokes equations. Especially the dynamics micro-rotation viscosity κ > 0 is essential for the micropolar fluid flows, otherwise the velocity and the micro-rotation are uncoupled and the global motion is unaffected by the micro-rotations. When α = β = γ = 0, that is to say, the stress momentum is lost in rotation of the particles, the macrostructure plays an important role as it usually increases the load capacity and stabilizes the flows, this sort micropolar fluid is less prone to instability than that of a classical fluid (refer to [13, 24]). Some polymeric fluids and fluids containing certain additives in narrow films may be represented by this mathematical model (see Eringen [12, Section 1, Section 6]). Moreover, Experiments with the fluids contain extremely small amount of polymeric addition additives indicate that the skin friction near a rigid body in such fluids are considerably lower (up to 30–50%) than the same fluids without additives (see [22]).

Because of their mathematical and physical importance, there is a large literature on the mathematical theory of micropolar fluid flows. The Cauchy problem of (1.1) has been studied by many authors and a lot of good results have been obtained (a complete literature in this direction is beyond the scope of this paper). To go directly to the main points of the present paper, in what follows we only review some known results which are closely related to our main result. Galdi and Rionero [14], Łukaszewicz [19] (and references therein) proved the global existence of weak solutions of micropolar flows (1.1) with the methods of Ladyzhenskaya [17] and Temam [25], Chen and Price [7], Rojas-Medar et al. [3, 8, 21, 23] investigated the local existence and uniqueness of strong solutions to the micropolar flows (or magneto-micropolar flows) by some different methods.

However, like the 3D Navier–Stokes equations, the problem of global regularity or finite time singularity for strong solutions of the 3D micropolar fluid with large initial data is still a challenging open problem. In this paper, we are concerned with the global regularity problem of the 2D micropolar fluid. For the 2D micropolar fluid, we assume that the velocity component in the x_3-direction is zero and the axes of rotation of particles are parallel to the x_3-axis. That is,

\[ \mathbf{v}(x, t) = (v_1(x, t), v_2(x, t), 0), \quad \mathbf{w}(x, t) = (0, 0, w_3(x, t)). \]

Substituting v, w of the above form into the system (1.1), we obtain

\[
\begin{align*}
\partial_t v - (v + \kappa) \Delta v - 2\kappa \nabla \times w + \nabla \pi + v \cdot \nabla v &= 0, \\
\partial_t w - \gamma \Delta w + 4\kappa w - 2\kappa \nabla \times v + v \cdot \nabla w &= 0, \\
\nabla \cdot v &= 0.
\end{align*}
\]

(1.2)

Here and in what follows,

\[ \nabla \times v = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}, \quad \nabla \times w = \left( \frac{\partial w}{\partial x_2}, -\frac{\partial w}{\partial x_1} \right). \]

For 2D micropolar fluid (1.2) with full viscosity (i.e., all the viscous coefficients are positive), the global well-posedness of smooth solution has been obtained by Łukaszewicz [19]. Based on the decay estimates of the linearized equations, a more explicit result has been recently obtained by Dong and Chen [10]. The purpose of this paper is to study the global regularity of smooth solutions of 2D micropolar fluid flows with zero angular viscosity γ = 0. That is, we will consider the following system

\[
\begin{align*}
\partial_t v - (v + \kappa) \Delta v - 2\kappa \nabla \times w + \nabla \pi + v \cdot \nabla v &= 0, \\
\partial_t w + 4\kappa w - 2\kappa \nabla \times v + v \cdot \nabla w &= 0, \\
\nabla \cdot v &= 0.
\end{align*}
\]

(1.3)
associated with the initial conditions

\[ v|_{t=0} = v_0, \quad w|_{t=0} = w_0. \] (1.4)

Our interest is partially motivated by global well-posedness problem of the 2D Boussinesq equations with partial viscosity,

\[
\begin{aligned}
\partial_t v - \nu \Delta v - \theta e_2 + \nabla \pi + v \cdot \nabla v &= 0, \\
\partial_t \theta - \kappa \Delta \theta + v \cdot \nabla \theta &= 0, \\
\nabla \cdot v &= 0,
\end{aligned}
\] (1.5)

and the 2D magnetohydrodynamic (MHD) equations with partial viscosity,

\[
\begin{aligned}
\partial_t v - \nu \Delta v + v \cdot \nabla v - b \cdot \nabla b + \nabla \pi &= 0, \\
\partial_t b - \kappa \Delta b + v \cdot \nabla b - b \cdot \nabla v &= 0, \\
\nabla \cdot v &= 0, \\
\nabla \cdot b &= 0.
\end{aligned}
\] (1.6)

Here partial viscosity means that one of the viscous coefficients \( \nu \) and \( \kappa \) is taken to zero. Chae [5] and Hou and Li [15] independently established the global regularity of 2D Boussinesq equations (1.5) with zero thermal diffusivity \( \kappa = 0 \) or zero kinetic viscosity \( \nu = 0 \). However, the global existence of smooth solutions of 2D MHD equations (1.6) with zero magnetic diffusivity \( \kappa = 0 \) or zero kinetic viscosity \( \nu = 0 \) is still open although partial results are recently obtained by Cao and Wu [4]. One may also refer to [9,18,20] for the global well-posedness problem of 2D polymeric fluids.

Since the second equation of (1.3) is a transport equation, we need to obtain a uniform bound of \( \| \nabla v \|_{L^1_t L^\infty} \) in order to propagate the global regularity of the initial data. From the first equation, we at least require \( w \in L^1_t L^\infty \) in order to obtain \( \nabla v \in L^1_t L^\infty \). Compared with the 2D Boussinesq equations (1.5) with \( \kappa = 0 \), the global regularity also requires the bound of \( \| \theta \|_{L^\infty} \), which can be directly obtained by maximum principle, i.e.,

\[ \| \theta(t) \|_{L^\infty} \leq \| \theta_0 \|_{L^\infty}. \]

For the 2D micropolar fluid flows (1.3), however, maximum principle allows to obtain

\[ \| w(t) \|_{L^\infty} \leq \| w_0 \|_{L^\infty} + 2\kappa \int_0^t \| \nabla \times v(\tau) \|_{L^\infty} d\tau. \] (1.7)

So, this is a recursive argument. Fortunately, we find a new quantity \( Z = \nabla \times v - \frac{2\kappa}{\nu+\kappa} w \) which has the following elegant structure

\[
\partial_t Z - (\nu + \kappa) \Delta Z + v \cdot \nabla Z = \left( \frac{8\kappa^2}{\nu + \kappa} - \frac{8\kappa^3}{(\nu + \kappa)^2} \right) w - \frac{4\kappa^2}{\nu + \kappa} Z,
\]

from which we can obtain

\[ \| Z(t) \|_{L^\infty} \leq \| \Omega_0 - w_0 \|_{L^\infty} + C \int_0^t \| w(\tau) \|_{L^\infty} d\tau, \]

which combined with (1.7) gives the \( L^\infty \) bounds of \( \nabla \times v \) and \( w \). Then the global existence of smooth solutions can be deduced from a Beale–Kato–Majda type blow-up criterion. Now our main result reads:
Theorem 1.1. Suppose $\nu > 0, \kappa > 0$, $(v_0, w_0) \in H^s(\mathbb{R}^2)$, $s > 2$ and $\nabla \cdot v_0 = 0$. Then (1.3)–(1.4) has a unique global solution $(v, w)$ such that for any $T > 0$,

$$
\begin{align*}
v & \in C([0, \infty); H^s(\mathbb{R}^2)) \cap L^2(0, T; H^{s+1}(\mathbb{R}^2)), \\
w & \in C([0, \infty); H^s(\mathbb{R}^2)).
\end{align*}
$$

Remark 1.2. In the case of periodic domain, similar result also holds. However, our method cannot work for the case of bounded domain. The main reason is that there is no boundary condition on the vorticity if we impose the Dirichlet boundary condition on the velocity.

2. Preliminaries

Let us firstly recall some basic facts about the Littlewood–Paley decomposition. One may check [6] for more details. Choose two nonnegative radial functions $\chi, \varphi \in \mathcal{S}(\mathbb{R}^2)$ supported respectively in $B = \{ \xi \in \mathbb{R}^2, |\xi| \leq \frac{3}{4} \}$ and $C = \{ \xi \in \mathbb{R}^2, \frac{3}{4} \leq |\xi| \leq \frac{2}{3} \}$ such that for any $\xi \in \mathbb{R}^2$,

$$
\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j} \xi) = 1. \tag{2.1}
$$

Let $h = \mathcal{F}^{-1}\varphi$ and $\tilde{h} = \mathcal{F}^{-1}\chi$, the frequency localization operator $\Delta_j$ and $S_j$ are defined by

$$
\begin{align*}
\Delta_j f & = \varphi(2^{-j} D) f = 2^{2j} \int_{\mathbb{R}^2} h(2^j y) f(x - y) \, dy, \quad \text{for } j \geq 0, \\
S_j f & = \chi(2^{-j} D) f = \sum_{-1 \leq k \leq j-1} \Delta_k f = 2^{2j} \int_{\mathbb{R}^2} \tilde{h}(2^j y) f(x - y) \, dy, \quad \text{and} \\
\Delta_{-1} f & = S_0 f, \quad \Delta_j f = 0 \quad \text{for } j \leq -2.
\end{align*}
$$

With our choice of $\varphi$, one can easily verify that

$$
\begin{align*}
\Delta_j \Delta_k f & = 0 \quad \text{if } |j - k| \geq 2 \quad \text{and} \\
\Delta_j (S_{k-1} \Delta_k f) & = 0 \quad \text{if } |j - k| \geq 5. \tag{2.2}
\end{align*}
$$

For any $f \in \mathcal{S}'(\mathbb{R}^2)$, we have by (2.1) that

$$
f = S_0(f) + \sum_{j \geq 0} \Delta_j f, \tag{2.3}
$$

which is called the Littlewood–Paley decomposition. The norm of Sobolev space $H^s(\mathbb{R}^2)$ can be characterized in terms of $\Delta_j$,

$$
\|f\|_{H^s} = \left\| S_0(f) \right\|_{L^2} + \left( \sum_{j \geq 0} 2^{2js} \| \Delta_j f \|_{L^2}^2 \right)^{1/2}. \tag{2.4}
$$

In the sequel, we will use Bony’s decomposition from [2] that

$$
uv = T_u v + T_v u + R(u, v), \tag{2.5}
$$
\[ Tu v = \sum_j S_{j-1} u \Delta_j v, \quad R(u, v) = \sum_{|j'-j| \leq 1} \Delta_j u \Delta_{j'} v, \]

and we also denote

\[ T'_u v = T_u v + R(u, v). \]

Let us conclude this section by some useful lemmas.

**Lemma 2.1.** (See [6].) Let \( 1 \leq p \leq q \leq \infty \). Assume that \( f \in L^p(\mathbb{R}^2) \), then there hold

\[ \text{supp} \hat{f} \subset \{ |\xi| \leq C_2^j \} \Rightarrow \| \partial^\alpha f \|_{L^q} \leq C_2^j |\alpha| + 2j (1/p - 1/q) \| f \|_{L^p}, \]

\[ \text{supp} \hat{f} \subset \{ 1/C_2^j \leq |\xi| \leq C_2^j \} \Rightarrow \| f \|_{L^p} \leq C_2^{-j} \| \partial^\alpha f \|_{L^p}. \]

Here the constant \( C \) is independent of \( f \) and \( j \).

**Lemma 2.2.** Let \( s > -1 \). Assume that \( f \in H^s(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \) and \( \nabla v \in H^s(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \) with \( \nabla \cdot v = 0 \). Then there holds

\[ \| [\Delta_j, v] \cdot \nabla f \|_{L^2} \leq C c_j 2^{-js} (\| \nabla v \|_{L^\infty} \| f \|_{H^s} + \| \nabla v \|_{H^s} \| f \|_{L^\infty}). \] (2.6)

Here \( \{ c_j \} \) is a sequence satisfying \( \| \{ c_j \} \|_{l^2} \leq 1 \).

**Proof.** Using the Bony's decomposition (2.5), we write

\[ \Delta_j (v \cdot \nabla f) = \Delta_j (T_{v^i} \partial_i f) + \Delta_j (T_{\partial_i f} v^i) + \Delta_j R(v^i, \partial_i f), \]

\[ v \cdot \nabla \Delta_j f = T_{v^i} \partial_i \Delta_j f + T'_{\partial_i} \Delta_j f v^i. \]

Then we have

\[ [\Delta_j, v] \cdot \nabla f = \Delta_j (T_{v^i} \partial_i f) + \Delta_j (T_{\partial_i f} v^i) + \Delta_j R(v^i, \partial_i f) - T'_{\partial_i} \Delta_j f v^i. \]

Due to (2.2), we have

\[ \Delta_j (T_{\partial_i f} v^i) = \sum_{|j'-j| \leq 4} \Delta_j (S_{j'-1} \partial_i f \Delta_{j'} v^i). \]

This gives by Lemma 2.1 that

\[ \| \Delta_j (T_{\partial_i f} v^i) \|_{L^2} \leq \sum_{|j'-j| \leq 4} \| \nabla S_{j'-1} f \|_{L^\infty} \| \Delta_{j'} v \|_{L^2} \]

\[ \leq C \sum_{|j'-j| \leq 4} 2^{j'} \| f \|_{L^\infty} \| \Delta_{j'} v \|_{L^2} \]

\[ \leq C c_j 2^{-js} \| f \|_{L^\infty} \| \nabla v \|_{H^s}. \] (2.7)
Due to $\nabla \cdot v = 0$, we have

$$\Delta_j R(v^i, \partial_i f) = \sum_{j', j'' \geq j - 3; |j' - j''| \leq 1} \partial_i \Delta_j (\Delta_{j'} v^{i} \Delta_{j''} f),$$

from which and Lemma 2.1, it follows that

$$\| \Delta_j R(w^i, \partial_i u) \|_{L^2} \leq C 2^j \sum_{j', j'' \geq j - 3; |j' - j''| \leq 1} \| \Delta_{j'} f \|_{L^\infty} \| \Delta_{j''} v \|_{L^2}$$

$$\leq C 2^{-js} \| f \|_{L^\infty} \sum_{j' \geq j - 2} 2^{(s+1)(j-j')} 2^{j's} \| \Delta_{j'} \nabla v \|_{L^2}$$

$$\leq C c j 2^{-js} \| f \|_{L^\infty} \| \nabla v \|_{H^s}. \quad (2.8)$$

In view of the definition of $T'_{\partial_i \Delta_j f} v^i$,

$$T'_{\partial_i \Delta_j f} v^i = \sum_{j' \geq j - 2} S_{j' + 2} \Delta_j \partial_i f \Delta_{j'} v^i,$$

thus by Lemma 2.1, we get

$$\| T'_{\partial_i \Delta_j f} v^i \|_{L^2} \leq C 2^j \| f \|_{L^\infty} \sum_{j' \geq j - 2} \| \Delta_{j'} v \|_{L^2}$$

$$\leq C 2^{-js} \| f \|_{L^\infty} \sum_{j' \geq j - 2} 2^{(s+1)(j-j')} 2^{j's} \| \Delta_{j'} \nabla v \|_{L^2}$$

$$\leq C c j 2^{-js} \| f \|_{L^\infty} \| \nabla v \|_{H^s}. \quad (2.9)$$

Now, we turn to estimate $[T_{vi}, \Delta_j] \partial_i f$. In view of the definition of $\Delta_j$, we write

$$[T_{vi}, \Delta_j] \partial_i f = \sum_{|j' - j| \leq 4} [S_{j' - 1} v^i, \Delta_j] \partial_i f$$

$$= \sum_{|j' - j| \leq 4} 2^{2j} \int_{\mathbb{R}^2} h(2^j (x - y)) (S_{j' - 1} v^i(x) - S_{j' - 1} v^i(y)) \partial_i \Delta_{j'} f(y) \, dy$$

$$= \sum_{|j' - j| \leq 4} 2^{2j} \int_{\mathbb{R}^2} \int_{0}^{1} y \cdot \nabla S_{j' - 1} v^i(x - \tau y) \, d\tau \, h(2^j y) \Delta_{j'} f(x - y) \, dy,$$

from which and the Minkowski inequality, we deduce that

$$\| [T_{vi}, \Delta_j] \partial_i f \|_{L^2} \leq \sum_{|j' - j| \leq 4} \| \nabla S_{j' - 1} v \|_{L^\infty} \| f \|_{H^s}$$

$$\leq C c j 2^{-js} \| \nabla v \|_{L^\infty} \| f \|_{H^s}. \quad (2.10)$$

Summing up (2.7)–(2.10), we obtain (2.6). \qed
Lemma 2.3. (See [16].) Assume that \( f \in H^s(\mathbb{R}^2), s > 1. \) Then there holds
\[
\|f\|_{L^\infty} \leq C \left( 1 + \|f\|_{BMO} \right) \ln(e + \|f\|_{H^s}).
\]

3. Local smooth solution and blow-up criterion

In this section, we will prove the local well-posedness of (1.3)-(1.4) and give a Beale, Kato and Majda [1] type blow-up criterion for thus obtained solution. More precisely,

**Theorem 3.1.** Suppose \( \nu > 0, \kappa > 0, (\nu_0, w_0) \in H^s(\mathbb{R}^2), s > 2 \) and \( \nabla \cdot \nu_0 = 0. \) Then there exists \( T > 0 \) such that (1.3)-(1.4) has a unique solution \( (\nu, w) \) satisfying
\[
\nu \in C([0, T); H^s(\mathbb{R}^2)) \cap L^2(0, T; H^{s+1}(\mathbb{R}^2)),
\]
\[
w \in C([0, T); H^s(\mathbb{R}^2)).
\]
Furthermore, if \( T^* \) is the maximal existence time of the solution, we have the following necessary condition for blow up
\[
T^* < \infty \quad \Rightarrow \quad \int_0^{T^*} \|\nabla \times \nu(t)\|_{L^\infty} \, dt = +\infty. \tag{3.1}
\]

**Proof of Theorem 3.1.** We will use the energy method, and the proof is divided into three steps.

**Step 1. Existence.** We first prove the existence of local smooth solution by the classic Friedrichs method which consists of an approximation of (1.3)-(1.4) by a cut-off in the frequency space. Denote \( J_n f = \mathcal{F}^{-1}(\chi_{B(0,n)}(\xi) \hat{f}(\xi)) \) for \( n \in \mathbb{N} \) and consider the approximate system of (1.3)-(1.4),
\[
\begin{align*}
\partial_t \nu_n &- (\nu + \kappa) J_n \Delta \nu_n = 2\kappa J_n P \nabla \times w_n - J_n P(J_n \nu_n \cdot \nabla J_n \nu_n), \\
\partial_t w_n + 4\kappa J_n \nu_n = 2\kappa J_n \nabla \times \nu_n - J_n (J_n \nu_n \cdot \nabla J_n w_n), \\
v_n|_{t=0} = J_n \nu_0, \quad w_n|_{t=0} = J_n w_0,
\end{align*} \tag{3.2}
\]
where \( P \) is the projection mapping \( L^2 \) onto the subspace \( \{ \nu \in L^2(\mathbb{R}^2); \nabla \cdot \nu = 0 \}. \) This is an ODE system on \( L^2 \) and the classic Cauchy–Lipschitz theorem ensures that there exists a unique solution which is continuous in time \( [0, T_0] \) with value in \( L^2. \) Furthermore, thanks to \( J_n^2 = J_n, \) we claim that \( J_n(\nu_n, \omega_n) \) is also a solution of (3.2), so the uniqueness implies that \( J_n(\nu_n, \omega_n) = (\nu_n, \omega_n). \) Thus \( (\nu_n, \omega_n) \) is also a solution of the following system
\[
\begin{align*}
\partial_t \nu_n &- (\nu + \kappa) \Delta \nu_n = 2\kappa P \nabla \times w_n - J_n P(\nu_n \cdot \nabla \nu_n), \\
\partial_t w_n + 4\kappa \nu_n = 2\kappa \nabla \times \nu_n - J_n (\nu_n \cdot \nabla w_n), \\
v_n|_{t=0} = J_n \nu_0, \quad w_n|_{t=0} = J_n w_0.
\end{align*} \tag{3.3}
\]

It is easy to verify that the solution \( (\nu_n, w_n) \) of (3.3) satisfies
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left( \|\nu_n(t)\|_{L^2}^2 + \|w_n(t)\|_{L^2}^2 + (\nu + \kappa) \|\nabla \nu_n(t)\|_{L^2}^2 + 4\kappa \|w_n(t)\|_{L^2}^2 \right) \\
= \int_{\mathbb{R}^2} \left( 2\kappa (\nabla \times w_n) \cdot \nu_n + 2\kappa (\nabla \times \nu_n) \cdot w_n \right) \, dx \\
\leq 4\kappa \|\nabla \nu_n\|_{L^2} \|w_n(t)\|_{L^2} \leq \kappa \|\nabla \nu_n(t)\|_{L^2}^2 + 4\kappa \|w_n(t)\|_{L^2}^2.
\end{align*}
\]
where we have used the fact that
\[
\int_{\mathbb{R}^2} J_n(v_n \cdot \nabla v_n) \cdot v_n \, dx = 0, \quad \int_{\mathbb{R}^2} J_n(v_n \cdot \nabla w_n) \cdot w_n \, dx = 0.
\]

Integrating in time to obtain
\[
\left\| v_n(t) \right\|_{L^2}^2 + \left\| w_n(t) \right\|_{L^2}^2 + 2\nu \int_0^t \left\| \nabla v_n(\tau) \right\|_{L^2}^2 \, d\tau \leq \left\| v_0 \right\|_{L^2}^2 + \left\| w_0 \right\|_{L^2}^2,
\]
(3.4)

which ensures that \( T_n = +\infty \).

Next we present the uniform estimate for the approximate solutions \((v_n, w_n)\) in \( H^s \). Taking the operator \(\Delta_j\) for \( j \geq 0 \) to both sides of (3.3), we obtain
\[
\begin{align*}
\partial_t \Delta_j v_n - (\nu + \kappa) \Delta \Delta_j v_n &= 2\kappa P \nabla \times \Delta_j w_n - J_n P \Delta_j (v_n \cdot \nabla v_n), \\
\partial_t \Delta_j w_n + 4\kappa \Delta_j w_n &= 2\kappa \nabla \times \Delta_j v_n - J_n \Delta_j (v_n \cdot \nabla w_n), \\
\Delta_j v_n|_{t=0} &= J_n \Delta_j v_0, \quad \Delta_j w_n|_{t=0} = J_n \Delta_j w_0.
\end{align*}
\]

The standard energy method implies that
\[
\frac{d}{dt} \left( \left\| \Delta_j v_n \right\|_{L^2}^2 + \left\| \Delta_j w_n \right\|_{L^2}^2 \right) + 2\nu \left\| \nabla \Delta_j v_n \right\|_{L^2}^2 \leq I + II.
\]
(3.5)

Here we used the following estimate due to integration by parts and the Hölder inequality,
\[
2\kappa \left| \int_{\mathbb{R}^2} (\nabla \times \Delta_j w_n) \cdot \Delta_j v_n \, dx + \int_{\mathbb{R}^2} (\nabla \times \Delta_j v_n) \cdot \Delta_j w_n \, dx \right|
\leq 4\kappa \int_{\mathbb{R}^2} |\Delta_j w_n| \left| \nabla \Delta_j v_n \right| \, dx
\leq \kappa \left\| \nabla \Delta_j v_n \right\|_{L^2}^2 + 4\kappa \left\| \Delta_j w_n \right\|_{L^2}^2.
\]

For \( I \), thanks to \( \nabla \cdot v_n = 0 \), we have
\[
I = -2 \int_{\mathbb{R}^2} \left( \Delta_j (v_n \cdot \nabla v_n) - v_n \cdot \nabla \Delta_j v_n \right) \cdot \Delta_j v_n \, dx
= -2 \int_{\mathbb{R}^2} \left[ \Delta_j, v_n \right] \cdot \nabla v_n \cdot \Delta_j v_n \, dx
\]
which together with Lemma 2.2 yields

\[
|I| \leq C \left\| \Delta_j \cdot v_n \right\|_{L^2} \left\| \Delta_j v_n \right\|_{L^2} \\
\leq C c_j 2^{-2js} \left\| \nabla v_n \right\|_{L^\infty} \left\| v_n \right\|_{H^s} \left\| v_n \right\|_{H^s}.
\]  

(3.6)

Similarly,

\[
II = -2 \int_{\mathbb{R}^2} \left( \Delta_j (v_n \cdot \nabla w_n) - v_n \cdot \nabla \Delta_j w_n \right) \cdot \Delta_j w_n \, dx \\
= -2 \int_{\mathbb{R}^2} [\Delta_j, v_n] \cdot \nabla w_n \cdot \Delta_j w_n \, dx,
\]

thus we get

\[
|II| \leq C \left\| [\Delta_j, v_n] \cdot \nabla w_n \right\|_{L^2} \left\| \Delta_j w_n \right\|_{L^2} \\
\leq C c_j 2^{-2js} \left( \left\| \nabla v_n \right\|_{H^s} \left\| w_n \right\|_{L^\infty} + \left\| \nabla v_n \right\|_{L^\infty} \left\| w_n \right\|_{H^s} \right) \left\| w_n \right\|_{H^s}.
\]  

(3.7)

Plugging (3.6) and (3.7) into (3.5) gives

\[
\frac{d}{dt} \left( \left\| \Delta_j v_n \right\|_{L^2}^2 + \left\| \Delta_j w_n \right\|_{L^2}^2 \right) + 2\nu \left\| \nabla \Delta_j v_n \right\|_{L^2}^2 \\
\leq C c_j 2^{-2js} \left( \left\| \nabla v_n \right\|_{L^\infty} \left\| v_n \right\|_{H^s}^2 + \left\| w_n \right\|_{L^\infty} \left\| w_n \right\|_{H^s} \left\| \nabla v_n \right\|_{H^s} \right),
\]

from which, (3.4) and Young's inequality, it follows that

\[
\frac{d}{dt} \left( \left\| v_n(t) \right\|_{H^s}^2 + \left\| w_n(t) \right\|_{H^s}^2 \right) + \nu \left\| \nabla v_n(t) \right\|_{H^s}^2 \\
\leq C \left( \left\| w_n \right\|_{L^2}^2 + \left\| \nabla v_n \right\|_{L^\infty} \right) \left( \left\| w_n \right\|_{H^s}^2 + \left\| v_n \right\|_{H^s}^2 \right),
\]

which together with Gronwall's inequality yields that

\[
\left( \left\| v_n(t) \right\|_{H^s}^2 + \left\| w_n(t) \right\|_{H^s}^2 \right) + \nu \int_0^t \left\| \nabla v_n(\tau) \right\|_{H^s}^2 \, d\tau \\
\leq \left( \left\| v_0 \right\|_{H^s}^2 + \left\| w_0 \right\|_{H^s}^2 \right) \exp \left\{ C \int_0^t \left( \left\| w_n(\tau) \right\|_{L^\infty}^2 + \left\| \nabla v_n(\tau) \right\|_{L^\infty} \right) \, d\tau \right\}.
\]  

(3.8)

Denote

\[
E_s(v, w, t) = \left\| v(t) \right\|_{H^s}^2 + \left\| w(t) \right\|_{H^s}^2,
\]
\[
F(v, w, t) = \nu \int_0^t \left\| \nabla v(\tau) \right\|_{H^s}^2 \, d\tau.
\]
and define $T_n$ as

$$T_n = \sup\{ t \mid \forall t' \leq t, \ E_s(v_n, w_n, t') + F(v_n, w_n, t') \leq 2E_s(v_0, w_0) \}.$$ 

From Sobolev embedding inequality, we infer that for $0 \leq t < T_n$,

$$C \int_0^t (\| w_n(\tau) \|_{L^\infty}^2 + \| \nabla v_n(\tau) \|_{L^\infty}^2) d\tau \leq C(t^{1/2} + t(1 + E_s(v_0, w_0))).$$

(3.9)

Choosing $T > 0$ such that

$$e^{C(T^{1/2} + T(1 + E_s(v_0, w_0)))} \leq \frac{3}{2},$$

Then $T_n \geq T$. Otherwise, we have by (3.8) and (3.9) that

$$E_s(v_n, w_n, t) + F(v_n, w_n, t) \leq \frac{3}{2} E_s(v_0, w_0), \quad \forall n \in \mathbb{N}, \ t \in [0, T_n),$$

which contradicts with the definition of $T_n$. Thus there holds for any $t \in [0, T],$

$$\| v_n(t) \|_{H^s}^2 + \| w_n(t) \|_{H^s}^2 + \nu \int_0^t \| \nabla v_n(\tau) \|_{H^s}^2 d\tau \leq 2(\| v_0 \|_{H^s} + \| w_0 \|_{H^s}).$$

(3.10)

Thus based on the estimate (3.10), a standard compactness argument ensures the existence of the solution $(v, w)$ of the system (1.3)–(1.4) on the interval $[0, T)$. Here we omit the details.

**Step 2. Uniqueness.** Let $(v_1, w_1)$ and $(v_2, w_2)$ be two solutions of (1.3)–(1.4). We denote $V = v_1 - v_2$, $W = w_1 - w_2$. Then there holds

$$\begin{cases}
\partial_t V - (\nu + \kappa) \Delta V - 2\kappa \nabla \times W + \nabla \pi + V \cdot \nabla v_1 + v_2 \cdot \nabla V = 0, \\
\partial_t W + 4\kappa W - 2\kappa \nabla \times V + V \cdot \nabla w_1 + v_2 \cdot \nabla W = 0.
\end{cases}$$

Using the standard $L^2$ energy estimate, it follows that

$$\frac{d}{dt}(\| V \|_{L^2}^2 + \| W \|_{L^2}^2) + 2\nu \| \nabla V \|_{L^2}^2 = -2 \int_{\mathbb{R}^2} (V \cdot \nabla v_1) \cdot V \ dx - 2 \int_{\mathbb{R}^2} (V \cdot \nabla w_1) W \ dx$$

$$\leq 2\| \nabla v_1 \|_{L^\infty} \| V \|_{L^2}^2 + 2\| \nabla v_1 \|_{L^2} \| W \|_{L^\infty} \| V \|_{L^2}$$

$$\leq C\left( \| v_1 \|_{H^s} + \| w_1 \|_{H^s} \right)(\| V \|_{L^2}^2 + \| W \|_{L^2}^2),$$

where we have used

$$\int_{\mathbb{R}^2} (v_2 \cdot \nabla V) \cdot V \ dx = 0, \quad \int_{\mathbb{R}^2} (v_2 \cdot \nabla W) W \ dx = 0.$$
and

\[ 2\kappa \left| \int_{\mathbb{R}^2} (\nabla \times W) \cdot V \, dx + \int_{\mathbb{R}^2} (\nabla \times V) \cdot W \, dx \right| \leq 4\kappa \left| W \right| \left| \nabla V \right| \, dx \leq 4\kappa \| W \|_2^2 + \kappa \left( \nabla V \right) \|_2^2. \]

Thus from Gronwall’s inequality, it follows that

\[ \| V(t) \|_2^2 + \| W(t) \|_2^2 \leq \left( \| V_0 \|_2^2 + \| W_0 \|_2^2 \right) \exp \left\{ C \int_0^t \left( \left\| v_1(\tau) \right\|_{H^s} + \left\| w_1(\tau) \right\|_{H^s} \right) \, d\tau \right\}, \]

which implies the uniqueness of the solution.

**Step 3. Blow-up criterion.** Exactly as in the proof of (3.8), we have

\[ \left( \left\| v(t) \right\|_{H^s}^2 + \left\| w(t) \right\|_{H^s}^2 \right) \leq \left( \left\| v_0 \right\|_{H^s}^2 + \left\| w_0 \right\|_{H^s}^2 \right) \exp \left\{ C \int_0^t \left( \left\| w(\tau) \right\|_{L^\infty}^2 + \left\| \nabla v(\tau) \right\|_{L^\infty} \right) \, d\tau \right\}, \]

which gives

\[ \ln(e + \| v(t) \|_{H^s}^2 + \| w(t) \|_{H^s}^2) \leq \ln(e + \| v_0 \|_{H^s}^2 + \| w_0 \|_{H^s}^2) + C \int_0^t \left( \left\| w(\tau) \right\|_{L^\infty}^2 + \left\| \nabla v(\tau) \right\|_{L^\infty} \right) \, d\tau. \] (3.11)

From the second equation of (1.3), it is easy to deduce that (see also the estimate of \( w \) in Section 4)

\[ \| w(t) \|_{L^\infty} \leq \| w_0 \|_{L^\infty} + C \int_0^t \left\| \nabla \times v(\tau) \right\|_{L^\infty} \, d\tau, \]

and using Lemma 2.3 to obtain

\[ \int_0^t \left\| \nabla v(\tau) \right\|_{L^\infty} \, d\tau \leq C \int_0^t \left( 1 + \left\| \nabla v(\tau) \right\|_{BMO} \right) \ln(e + \| v(\tau) \|_{H^s}) \, d\tau \]

\[ \leq C \int_0^t \left( 1 + \left\| \nabla \times v(\tau) \right\|_{L^\infty} \right) \ln(e + \| v(\tau) \|_{H^s}) \, d\tau, \]

where we used in the last inequality the fact that \( \nabla v = T(\nabla \times v) \) with \( T \) a singular integral operator (Biot–Savart law) and

\[ \left\| T(\nabla \times v) \right\|_{BMO} \leq C \left\| \nabla \times v \right\|_{L^\infty}. \]
Substituting them into (3.11) yields that
\[
\ln(e + \|v(t)\|_{H^s}^2 + \|w(t)\|_{H^s}^2) \leq \ln(e + \|v_0\|_{H^s}^2 + \|w_0\|_{H^s}^2)
+ C t \left(\|w_0\|_{L^\infty} + \int_0^t \|\nabla \times v(\tau)\|_{L^\infty} d\tau\right)^2
+ C \int_0^t (1 + \|\nabla \times v(\tau)\|_{L^\infty}) \ln(e + \|v(\tau)\|_{H^s}) d\tau,
\]
which together with Gronwall’s inequality implies (3.1). This completes the proof of Theorem 3.1. \(\square\)

4. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Thanks to Theorem 3.1, there exists a unique strong solution \((v, w)\) to the system (1.3)–(1.4) on a maximal time interval \([0, T^*)\). In what follows, we prove \(T^* = +\infty\). With the aid of the blow-up criterion, it suffices to show that if \(T^* < +\infty\), then
\[
\int_0^{T^*} \|\nabla \times v(t)\|_{L^\infty} dt < +\infty.
\tag{4.1}
\]
Denote the vorticity of velocity \(v\) by \(\Omega = \nabla \times v\) and it is easy to check that \(\nabla \times \nabla \times w = -\Delta w\).
Taking the rot operator to the first equation of (1.3) yields that
\[
\partial_t \Omega - (\nu + \kappa) \Delta \Omega + 2\kappa \Delta w + v \cdot \nabla \Omega = 0,
\tag{4.2}
\]
subtracting \(2\kappa \nu + \kappa \times (1.3)\) from (4.2) and denoting \(Z(x, t) = \Omega - \frac{2\kappa}{\nu + \kappa} w\) give
\[
\partial_t Z - (\nu + \kappa) \Delta Z + v \cdot \nabla Z = \left(\frac{8\kappa^2}{\nu + \kappa} - \frac{8\kappa^3}{(\nu + \kappa)^2}\right) w - \frac{4\kappa^2}{\nu + \kappa} Z.
\tag{4.3}
\]
Multiplying (4.3) by \(|Z|^{p-2} Z\) with \(p \geq 2\) and integrating over \(\mathbb{R}^2\), we obtain
\[
\frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}^2} |Z(t)|^p dx + \frac{(p - 2)(\nu + \kappa)}{2} \int_{\mathbb{R}^2} |\nabla Z|^2 |Z|^{p-2} dx
= \int_{\mathbb{R}^2} \left(\frac{8\kappa^2}{\nu + \kappa} - \frac{8\kappa^3}{(\nu + \kappa)^2}\right) w|Z|^{p-2} Z dx - \frac{4\kappa^2}{\nu + \kappa} \int_{\mathbb{R}^2} |Z|^p dx
\leq C \|w\|_{L^p} \|Z\|_{L^p}^{p-1},
\]
where we have used the equality due to the divergence free of velocity \(v\),
\[
\int_{\mathbb{R}^2} v \cdot \nabla Z |Z|^{p-2} Z dx = 0.
\]
Integrating in time yields

\[ \| Z(t) \|_{L^p} \leq \| \Omega_0 - w_0 \|_{L^p} + C \int_0^t \| w(\tau) \|_{L^p} \, d\tau, \quad (4.4) \]

for any \( 2 \leq p \leq \infty \).

Similarly, from the second equation of (1.3), it follows that

\[ \frac{1}{p} \frac{d}{dt} \| w \|_{L^p}^p + 4\kappa \| w \|_{L^p}^p = 2\kappa \int_{\mathbb{R}^2} |w|^{p-2} w \, dx. \quad (4.5) \]

Notice that

\[ \int_{\mathbb{R}^2} |w|^{p-2} w \, dx = \int_{\mathbb{R}^2} \left( \Omega - \frac{2\kappa}{v + \kappa} w \right) |w|^{p-2} w \, dx + \frac{2\kappa}{v + \kappa} \int_{\mathbb{R}^2} w |w|^{p-2} w \, dx \]

\[ \leq \| Z \|_{L^p} \| w \|_{L^p}^{p-1} + C \| w \|_{L^p}^p. \]

Inserting it into (4.5) and integrating in time yield

\[ \| w(t) \|_{L^p} + \int_0^t \| w(\tau) \|_{L^p} \, d\tau \leq e^{Ct} \left( \| w_0 \|_{L^p} + \int_0^t \| Z(\tau) \|_{L^p} \, d\tau \right). \]

for any \( 2 \leq p \leq \infty \). Especially for \( p = \infty \),

\[ \| w(t) \|_{L^\infty} \leq e^{Ct} \left( \| w_0 \|_{L^\infty} + \int_0^t \| Z(\tau) \|_{L^\infty} \, d\tau \right). \]

which combined with (4.4) gives

\[ \| w(t) \|_{L^\infty} \leq e^{Ct} \left( \| w_0 \|_{L^\infty} + \| \Omega_0 - w_0 \|_{L^\infty} + \int_0^t \| w(\tau) \|_{L^\infty} \, d\tau \right). \]

Then Gronwall’s inequality ensures that

\[ \| w(t) \|_{L^\infty} \leq C(t, \| w_0 \|_{L^\infty}, \| \Omega_0 \|_{L^\infty}) \leq C(t, \| w_0 \|_{H^s}, \| v_0 \|_{H^s}). \]

With this, we infer from (4.4) that

\[ \| \Omega(t) \|_{L^\infty} \leq \| Z(t) \|_{L^\infty} + C \| w(t) \|_{L^\infty} \leq C(t, \| w_0 \|_{H^s}, \| v_0 \|_{H^s}), \]

for any \( t \in [0, T^*) \). Thus, the inequality (4.1) holds.
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References