Spectrum and principal functions of the non-self-adjoint Sturm–Liouville operators with a singular potential

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Abstract

We consider the operator $L$ generated in $L^2(\mathbb{R}^+)$ by the differential expression

$$l(y) = -y'' + \left[ v^2 - \frac{1}{x^2} + q(x) \right] y, \quad x \in \mathbb{R}^+ := (0, \infty)$$

and the boundary condition

$$\lim_{x \to 0} x^{-\nu-\frac{1}{2}} y(x) = 1,$$

where $q$ is a complex valued function and $\nu$ is a complex number with $\text{Re} \, \nu > 0$. In this work we investigate the eigenvalues and the spectral singularities of $L$. We also obtain the properties of the principal functions corresponding to the spectral singularities of $L$.

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1. Introduction

Let $T$ be a non-self-adjoint, closed operator in a Hilbert space $H$. We will denote the continuous spectrum and the set of all eigenvalues of $T$ by $\sigma_c(T)$ and $\sigma_d(T)$, respectively. Let us assume that $\sigma_c(T) \neq \phi$.

**Definition 1.1.** If $\lambda = \lambda_0$ is a pole of the resolvent of $T$ and $\lambda_0 \in \sigma_c(T)$, but $\lambda_0 \notin \sigma_d(T)$, then $\lambda_0$ is called a spectral singularity of $T$.

Let us consider the non-self-adjoint operator $L_0$ generated in $L^2(\mathbb{R}_+)$ by the differential expression

$$l_0(y) = -y'' + q(x)y, \quad x \in \mathbb{R}_+, \quad (1.1)$$

and the boundary condition $y(0) = 0$, where $q$ is a complex valued function. The spectrum and spectral expansion of $L_0$ were investigated by Naimark [23]. He proved that the spectrum of $L_0$ is composed of continuous spectrum, eigenvalues and spectral singularities. He showed that spectral singularities are on the continuous spectrum and are the poles of the resolvent’s kernel, which are not eigenvalues.

Lyance investigated the effect of the spectral singularities on the spectral expansion in terms of the principal functions of $L_0$ [20]. He also showed that the spectral singularities play an important role in the spectral analysis of $L_0$.

The spectral analysis of the non-self-adjoint operator $L_1$ generated in $L^2(\mathbb{R}_+)$ by (1.1) and the boundary condition

$$\int_0^{\infty} K(x) y(x) dx + \alpha y'(0) - \beta y(0) = 0,$$

in which $K \in L^2(\mathbb{R}_+)$ is a complex valued function and $\alpha, \beta \in \mathbb{C}$, was investigated in detail by Krall [13–17]. In [13] he obtained the adjoint $L_1^\ast$ of the operator $L_1$. Note that $L_1^\ast$ deserves special interest, since it is not a purely differential operator. The eigenfunction expansions of $L_1$ and $L_1^\ast$ were investigated in [14].

In [11] the results of Naimark were extended to the three dimensional Schrödinger operators.

The Laurent expansion of the resolvents of the abstract non-self-adjoint operators in the neighborhood of the spectral singularities was studied in [12].

Using the boundary uniqueness theorems of analytic functions, the structure of the eigenvalues and the spectral singularities of a quadratic pencil of Schrödinger, Klein–Gordon, discrete Dirac and discrete Schrödinger operators was investigated in [1–8,18,19]. By regularization of a divergent integral, the effect of the spectral singularities in the spectral expansion of a quadratic pencil of Schrödinger operators was obtained in [3]. In [18] and [19] the spectral expansion of the discrete Dirac and Schrödinger operators with spectral singularities was derived using the generalized spectral function (in the sense of Marchenko [22]) and the analytical properties of the Weyl function.

Let $L$ denote the operator generated in $L^2(\mathbb{R}_+)$ by the differential expression

$$l(y) = -y'' + \left[\frac{v^2 - 1}{4} + q(x)\right] y, \quad x \in \mathbb{R}_+,$$

and the boundary condition

$$\lim_{x \to 0} x^{-\frac{1}{2}} y(x) = 1,$$
where \( q \) is a complex valued function and \( \nu \) is a complex number with \( \text{Re} \, \nu > 0 \). In this work we investigate the spectrum and the spectral singularities of \( L \). Moreover we obtain the properties of the principal functions corresponding to the spectral singularities of \( L \).

2. The Jost solution and Jost function

We consider the equation

\[
- \frac{d^2 y}{dx^2} + \left[ \nu^2 - \frac{1}{4} + q(x) \right] y = k^2 y, \quad x \in \mathbb{R}_+
\]

related to the operator \( L \).

Now we will assume that the complex valued function \( q \) is almost everywhere continuous in \( \mathbb{R}_+ \) and satisfies the following [9, Chapter 3]:

\[
\int_a^\infty |q(x)| \, dx < \infty, \quad \int_0^{a'} x |q(x)| \, dx < \infty, \quad (a, a' > 0).
\]

Let \( \varphi(x, k, \nu) \) and \( f(x, k, \nu) \) denote the solutions of (2.1) satisfying the conditions

\[
\lim_{x \to 0} x^{-\nu - \frac{1}{2}} y(x) = 1,
\]

and

\[
\lim_{x \to \infty} e^{-ikx} y(x) = 1,
\]

respectively. The solution \( f(x, k, \nu) \) is called the Jost solution of (2.1). Note that, under the condition (2.2), the solution \( \varphi(x, k, \nu) \) is an entire function of \( k \) and the Jost solution is an analytic function of \( k \) in \( \mathbb{C}_+ := \{ k : k \in \mathbb{C}, \text{Im} \, k > 0 \} \) and continuous in \( \overline{\mathbb{C}}_+ = \{ k : k \in \mathbb{C}, \text{Im} \, \lambda \geq 0 \} \) [9, Chapter 4]. Moreover the Jost solution also satisfies

\[
f(x, k, \nu) = e^{ikx} [1 + o(1)], \quad k \in \overline{\mathbb{C}}_+, \text{Re} \, \nu > 0, x \to \infty.
\]

Let us consider the function

\[
f_0(x, k, \nu) = \sqrt{\frac{1}{2\pi}} k x e^{-\frac{1}{2}i\pi(\nu + \frac{1}{2})} H_\nu^2(kx),
\]

where \( H_\nu^2(kx) \) is the Hankel function of the second kind. It is obvious that the function \( f_0(x, k, \nu) \) is the solution of the equation

\[
- \frac{d^2 y}{dx^2} + \nu^2 - \frac{1}{4} y = k^2 y.
\]

Under the condition (2.2) the Jost solution has the representation

\[
f(x, k, \nu) = f_0(x, k, \nu) + \int_x^\infty K^{(\nu)}(x, t) f_0(t, k, \nu) \, dt,
\]

where the kernel \( K^{(\nu)}(x, t) \) may be expressed in terms of \( q \) [21, Chapter 5].

We will denote the Wronskian of the solutions \( f(x, k, \nu) \) and \( \varphi(x, k, \nu) \) by \( f_\nu(k) \), i.e.,

\[
f_\nu(k) = W[f(x, k, \nu), \varphi(x, k, \nu)], \quad k \in \overline{\mathbb{C}}_+, \text{Re} \, \nu > 0.
\]
The function $f_\nu$ is called the Jost function of $L$. Under the condition (2.1) the Jost function is analytic with respect to $k$ in $\mathbb{C}_+$ and continuous in $\overline{\mathbb{C}}_+$, and

$$f_\nu(k) = 1 + o(1), \quad k \in \overline{\mathbb{C}}_+, \quad \text{Re } \nu > 0, \quad |k| \to \infty,$$

holds [9, Chapter 5].

### 3. Eigenvalues and spectral singularities of $L$

Let $G(x, t, k, \nu)$ be the Green function of $L$, i.e.,

$$G(x, t, k, \nu) = \begin{cases} \varphi(t, k, \nu)f(x, k, \nu), & 0 < t < x \\ \varphi(x, k, \nu)f(t, k, \nu), & x \leq t < \infty \end{cases} \quad (3.1)$$

We will denote the set of eigenvalues and spectral singularities of $L$ by $\sigma_d(L)$ and $\sigma_{ss}(L)$, respectively. From (2.3), (2.6) and (3.1) we get the following:

**Theorem 3.1.**

$$\sigma_d(L) = \{ \lambda : \lambda = k^2, k \in \mathbb{C}_+, f_\nu(k) = 0 \}, \quad (3.2)$$

$$\sigma_{ss}(L) = \{ \lambda : \lambda = k^2, k \in \mathbb{R}^*, f_\nu(k) = 0 \}, \quad (3.3)$$

where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

From (3.2) and (3.3) we obtain that to investigate the structure of the eigenvalues and the spectral singularities of $L$, we need to discuss the structure of the zeros of the function $f_\nu$ in $\overline{\mathbb{C}}_+$.

**Theorem 3.2. Under the condition (2.2)**

(i) The set $\sigma_d(L)$ is bounded, has at most a countable number of elements and its limit points can lie only in a bounded subinterval of $\mathbb{R}_+$.

(ii) The set $\sigma_{ss}(L)$ is bounded and $\mu[\sigma_{ss}(L)] = 0$, where $\mu$ denotes the Lebesgue measure on the real axis.

**Proof.** (2.7) shows the boundedness of the zeros of $f_\nu$ in $\overline{\mathbb{C}}_+$. Consequently, it follows from (3.2) and (3.3) that the sets $\sigma_d(L)$ and $\sigma_{ss}(L)$ are bounded. From the analyticity of $f_\nu$ in $\mathbb{C}_+$, we get that the set $\sigma_d(L)$ has at most a countable number of elements. Using boundary uniqueness theorems of analytic functions we obtain that the limit points of $\sigma_d(L)$ can lie in $\mathbb{R}_+$ and $\mu[\sigma_{ss}(L)] = 0$ [10].

Now let us assume that, for some $\varepsilon > 0$,

$$\int_0^\infty x|q(x)|e^{\varepsilon x}dx < \infty \quad (3.4)$$

holds. Under the condition (3.4), the solution $f(x, k, \nu)$ has an analytic continuation in terms of $k$, from the real axis to the half-plane $\text{Im } k > -\frac{\varepsilon}{2}$ [9, Chapter 4]. Moreover the kernel of the Jost solution satisfies

$$|K^{(\nu)}(x, t)| \leq Ce^{-\frac{\varepsilon}{2}(x+t)}, \quad (3.5)$$

where $C > 0$ is a constant.
**Definition 3.1.** The multiplicity of zero of the function $f_v$ in $\mathbb{C}_+$ is called the multiplicity of the corresponding eigenvalue or spectral singularity of $L$.

**Theorem 3.3.** Under the condition (3.4) the operator $L$ has a finite number of eigenvalues and spectral singularities and each of them is of finite multiplicity.

**Proof.** It follows from (2.6) that the condition (3.4) guarantees the analytic continuation of $f_v$ from the real axis to the lower half-plane $\text{Im } k > -\frac{\varepsilon}{2}$, since the limits of its zeros in $\mathbb{C}_+$ cannot lie in $\mathbb{R}$, using Theorem 3.2 we have the finiteness of zeros of $f_v$ in $\mathbb{C}_+$. Moreover all zeros $f_v$ in $\mathbb{C}_+$ have a finite multiplicity. We get the proof of the theorem by (3.2) and (3.3).

4. **Principal functions**

In this section we assume that (3.4) holds. Let $\lambda_1 = k_1^2, \ldots, \lambda_\alpha = k_\alpha^2$ and $\lambda_{\alpha+1} = k_{\alpha+1}^2, \ldots, \lambda_n = k_n^2$ denote the eigenvalues and the spectral singularities of $L$ with multiplicities $m_1, \ldots, m_\alpha$ and $m_{\alpha+1}, \ldots, m_n$, respectively. It is obvious that

$$\left\{ \frac{d^j}{dk^j} W[f(x, k, v), \varphi(x, k, v)] \right\}_{k=k_p} = \left\{ \frac{d^j}{dk^j} f_v(k) \right\}_{k=k_p} = 0, \quad (4.1)$$

for $j = 0, 1, \ldots, m_p - 1, p = 1, \ldots, \alpha, \alpha + 1, \ldots, n$.

**Theorem 4.1.** The formula

$$\left\{ \frac{\partial^j}{\partial k^j} \varphi(x, k, v) \right\}_{k=k_p} = \sum_{\beta=0}^{j} \binom{j}{\beta} a_{j-\beta} \left\{ \frac{\partial^\beta}{\partial k^\beta} f(x, k, v) \right\}_{k=k_p}, \quad (4.2)$$

for $j = 0, 1, \ldots, m_p - 1, p = 1, \ldots, \alpha, \alpha + 1, \ldots, n$, holds, where the constants $a_0, a_1, \ldots, a_j$ depend on $k_p$.

**Proof.** We will proceed by mathematical induction. Let $j = 0$. From (4.1) we get

$$\varphi(x, k_p, v) = a_0(k_p) f(x, k_p, v),$$

where $a_0(k_p) \neq 0$. Let us assume that for $1 \leq j_0 \leq m_p - 2$ (4.2) holds; i.e.,

$$\left\{ \frac{\partial^{j_0}}{\partial k^{j_0}} \varphi(x, k, v) \right\}_{k=k_p} = \sum_{\beta=0}^{j_0} \binom{j_0}{\beta} a_{j_0-\beta} \left\{ \frac{\partial^\beta}{\partial k^\beta} f(x, k, v) \right\}_{k=k_p}. \quad (4.3)$$

Now we will prove that (4.2) holds for $j_0 + 1$. If $y(x, k, v)$ is a solution of (2.1), then $\frac{\partial^{j+1}}{\partial k^{j+1}} y(x, k, v)$ satisfies

$$\begin{align*}
-\frac{d^2}{dx^2} + \frac{v^2 - 1}{x^2} + q(x) - k^2 \frac{\partial^j}{\partial k^j} y(x, k, v) & = 2kj \frac{\partial^{j-1}}{\partial k^{j-1}} y(x, k, v) + j(j-1) \frac{\partial^{j-2}}{\partial k^{j-2}} y(x, k, v),
\end{align*} \quad (4.4)$$
Writing (4.4) for \( \varphi(x, k_p, v) \) and \( f(x, k_p, v) \), and using (4.3), we find
\[
\left[ -\frac{d^2}{dx^2} + \frac{v^2 - \frac{1}{4}}{x^2} + q(x) - k^2 \right] F_{j_0+1}(x, k_p, v) = 0,
\]
where
\[
F_{j_0+1}(x, k_p, v) = \left\{ \frac{\partial^{j_0+1}}{\partial k^{j_0+1}} \varphi(x, k, v) \right\}_{k=k_p}
- \sum_{\beta=0}^{j_0+1} \left( j_0 + 1 \right) a_{j_0+1-\beta}(k_p) \left\{ \frac{\partial^{\beta}}{\partial k^\beta} f(x, k, v) \right\}_{k=k_p}.
\]

From (4.1) we have
\[
W[f(x, k_p, v), F_{j_0+1}(x, k_p, v)] = \left\{ \frac{\partial^{j_0+1}}{\partial k^{j_0+1}} W[f(x, k, v), \varphi(x, k, v)] \right\}_{k=k_p} = 0.
\]

Hence there exists a constant \( a_{j_0+1}(k_p) \) such that
\[
F_{j_0+1}(x, k_p, v) = a_{j_0+1}(k_p) f(x, k_p, v).
\]

This shows that (4.2) holds for \( j = j_0 + 1 \).

Using the notation
\[
A_{j-\beta}(k_p) = \frac{a_{j-\beta}(k_p)}{(j-\beta)!},
\]
we can write (4.2) as
\[
\frac{1}{j!} \left\{ \frac{\partial^j}{\partial k^j} \varphi(x, k, v) \right\}_{k=k_p} = \sum_{\beta=0}^{j} A_{j-\beta}(k_p) \frac{1}{\beta!} \left\{ \frac{\partial^{\beta}}{\partial k^\beta} f(x, k, v) \right\}_{k=k_p}, \quad (4.5)
\]
\( j = 0, 1, \ldots, m_p - 1, \ p = 1, \ldots, \alpha, \alpha + 1, \ldots, n. \)

**Definition 4.1.** Let \( \lambda = \lambda_0 \) be an eigenvalue of \( L \). If the functions
\[
y_0(x, \lambda_0, v), y_1(x, \lambda_0, v), \ldots, y_s(x, \lambda_0, v)
\]
satisfy the equations
\[
l(y_0) - \lambda_0 y_0 = 0, \quad l(y_j) - \lambda_0 y_j - y_{j-1} = 0, \quad j = 1, 2, \ldots, s,
\]
then \( y_0(x, \lambda_0, v) \) is called the eigenfunction corresponding to the eigenvalue \( \lambda = \lambda_0 \) of \( L \). The functions \( y_1(x, \lambda_0, v), \ldots, y_s(x, \lambda_0, v) \) are called the associated functions corresponding to \( \lambda = \lambda_0 \). The eigenfunctions and associated functions corresponding to \( \lambda = \lambda_0 \) are called the principal functions of the eigenvalue \( \lambda = \lambda_0 \).

The principal functions of the spectral singularities of \( L \) are defined similarly.

Let us introduce the functions
\[
\Phi_j(x, k_p, v) = \frac{1}{j!} \left\{ \frac{\partial^j}{\partial k^j} \varphi(x, k, v) \right\}_{k=k_p}, \quad (4.6)
\]
\( j = 0, 1, \ldots, m_p - 1, \ p = 1, \ldots, \alpha, \alpha + 1, \ldots, n. \)
It follows from (4.5) that

\[ \Phi_j(x, k_p, v) = \sum_{\beta=0}^{j} A_{j-\beta}(k_p) \frac{1}{\beta!} \left\{ \frac{\partial^\beta}{\partial k_p^\beta} f(x, k, v) \right\}_{k=k_p}, \tag{4.7} \]

\( j = 0, 1, \ldots, m_p - 1, p = 1, \ldots, \alpha, \alpha + 1, \ldots, n. \)

From (4.4), (4.6) and (4.7) we get

\[ l[\Phi_0(x, k_p, v)] - k_p^2 \Phi_0(x, k_p, v) = 0 \]
\[ l[\Phi_j(x, k_p, v)] - k_p^2 \Phi_j(x, k_p, v) - \Phi_{j-1}(x, k_p, v) = 0, \]

\( j = 0, 1, \ldots, m_p - 1, p = 1, \ldots, \alpha, \alpha + 1, \ldots, n. \)

Consequently the functions \( \Phi_j(x, k_p, v), j = 0, 1, \ldots, m_p - 1, p = 1, \ldots, \alpha \) and \( \Phi_j(x, k_p, v), j = 0, 1, \ldots, m_p - 1, p = \alpha + 1, \ldots, n, \) are the principal functions corresponding to the eigenvalues and the spectral singularities of \( L, \) respectively.

**Theorem 4.2.**

\[ \Phi_j(., k_p, v) \in L^2(\mathbb{R}^+), \quad j = 0, 1, \ldots, m_p - 1, p = 1, \ldots, \alpha, \tag{4.8} \]
\[ \Phi_j(., k_p, v) \not\in L^2(\mathbb{R}^+), \quad j = 0, 1, \ldots, m_p - 1, p = \alpha + 1, \ldots, n. \tag{4.9} \]

**Proof.** From (2.4), (2.5) and (3.5) we obtain

\[ \left\{ \frac{\partial^j}{\partial k_p^j} f(x, k, v) \right\}_{k=k_p} \leq C x^j e^{-\lambda p x} + C \int_0^\infty t^j e^{-\tau x} e^{-t \lambda p x} \, dt, \tag{4.10} \]

\( j = 0, 1, \ldots, m_p - 1, p = 1, \ldots, \alpha, \alpha + 1, \ldots, n, \) where \( C > 0 \) is a constant. Since \( \text{Im} \, k_p > 0 \) for the eigenvalues \( \lambda p = k_p^2, p = 1, \ldots, \alpha, \) of \( L, \) (4.10) implies that

\[ \left\{ \frac{\partial^j}{\partial k_p^j} f(., k, v) \right\}_{k=k_p} \in L^2(\mathbb{R}^+), \quad j = 0, 1, \ldots, m_p - 1, p = 1, \ldots, \alpha. \tag{4.11} \]

Using (4.7) and (4.11), we arrive at (4.8).

If we consider (4.10) for the principal functions corresponding to the spectral singularities \( \lambda p = k_p^2, p = 1, \ldots, \alpha, \alpha + 1, \ldots, n, \) of \( L \) and consider that \( \text{Im} \, k_p = 0 \) for the spectral singularities, then we have (4.9), by (4.7) and (4.10).

Let us introduce the Hilbert spaces

\[ H_m = \left\{ f : \int_0^\infty (1 + x)^2 m |f(x)|^2 \, dx < \infty \right\}, \quad m = 0, 1, \ldots, \]
\[ H_{-m} = \left\{ g : \int_0^\infty (1 + x)^{-2m} |g(x)|^2 \, dx < \infty \right\}, \quad m = 0, 1, \ldots, \]

with

\[ \|f\|^2_m = \int_0^\infty (1 + x)^2 m |f(x)|^2 \, dx; \quad \|g\|^2_{-m} = \int_0^\infty (1 + x)^{-2m} |g(x)|^2 \, dx, \]

respectively. It is obvious that \( H_0 = L^2(\mathbb{R}^+) \) and

\[ H_{m+1} \subsetneq H_m \subsetneq L^2(\mathbb{R}^+) \subsetneq H_{-m} \subsetneq H_{-(m+1)}. \tag{4.12} \]
Also $H_{-m}$ is isomorphic to the dual of $H_m$.

**Lemma 4.1.**

\[ \Phi_j(., k_p, \nu) \in H_{-(j+1)}, \quad j = 0, 1, \ldots, m_p - 1, \quad p = \alpha + 1, \ldots, n. \]

**Proof.** It follows from (4.10) that

\[ \left\{ \frac{\partial^j}{\partial k^j} f (., k, \nu) \right\}_{k=k_p} \in H_{-(j+1)}, \quad j = 0, 1, \ldots, m_p - 1, \quad p = \alpha + 1, \ldots, n. \]

The proof of the lemma may easily be obtained from (4.7).

Let

\[ H_- := H_{-m_0}, \]

where

\[ m_0 = \max\{m_{\alpha+1}, m_{\alpha+2}, \ldots, m_n\}. \]

Now Lemma 4.1 and (4.12) yield the following immediately:

**Theorem 4.3.**

\[ \Phi_j(., k_p, \nu) \in H_-, \quad j = 0, 1, \ldots, m_p - 1, \quad p = \alpha + 1, \ldots, n. \]

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**References**


