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Basic definition and properties of Bessel multipliers[☆]

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Abstract

This paper introduces the concept of Bessel multipliers. These operators are defined by a fixed multiplication pattern, which is inserted between the analysis and synthesis operators. The proposed concept unifies the approach used for Gabor multipliers for arbitrary analysis/synthesis systems, which form Bessel sequences, like wavelet or irregular Gabor frames. The basic properties of this class of operators are investigated. In particular the implications of summability properties of the symbol for the membership of the corresponding operators in certain operator classes are specified. As a special case the multipliers for Riesz bases are examined and it is shown that multipliers in this case can be easily composed and inverted. Finally the continuous dependence of a Bessel multiplier on the parameters (i.e., the involved sequences and the symbol in use) is verified, using a special measure of similarity of sequences.

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1. Introduction

The applications of signal processing algorithms like adaptive or time-variant filters are numerous [18]. If the STFT, the *Short Time Fourier Transformation* [10] is used in its sampled version, the Gabor transform, one possibility to construct a time variant filter is the usage of *Gabor multipliers*. These operators are a current topic of research [6,8]. For them the Gabor transform is used to calculate time frequency coefficients, which are multiplied with a fixed

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time-frequency mask and then the result is synthesized. These operators have been already used for quite some time implicitly in engineering applications and recently have been used in signal-processing applications as time-variant filters called *Gabor filters* [14]. Recent applications can be found for example in the field of system identification [13].

If another way of calculating these coefficients is chosen or if another synthesis is used, many modifications can still be implemented as multipliers. For example it seems quite natural to define *wavelet multipliers*. Also as irregular Gabor frames get more and more attention [12], Gabor multipliers on irregular sets can be investigated [1]. As the sampling set, in this irregular case, does not form a lattice, there is no group structure to work with. Therefore it is quite natural for this case to generalize even more and look at multipliers with frames without any further structure.

All these special types of sequences are used in a lot of applications. They have the big advantage, that it is possible to interpret the analysis coefficients. This would also make the formulation of a concept of a multiplier for other analysis/synthesis systems very profitable, like e.g. gammatone filter banks [11], which are mainly used for analysis based on the auditory system. In [15] a gammatone filter bank was used for analysis and synthesis, for the sound separation part a neuronal network creates a mask for these coefficients. This complies with the definition of a multiplier presented here.

Therefore for Bessel sequences the investigation of operators $\mathbf{M} = \sum_k m_k \langle f, \psi_k \rangle \phi_k$, where the analysis coefficients, $\langle f, \psi_k \rangle$, are multiplied by a fixed *symbol* (m_k) before resynthesis (with ϕ_k), is very natural and useful. These are the *Bessel multipliers* investigated in this paper. As stated above there are numerous applications of this kind of operators. It is the goal of this paper to set the mathematical basis to unify the approach to them for all possible analysis/synthesis sequences, that form a Bessel sequence.

2. Main results

We will introduce the concept of Bessel multipliers and will study their basic properties for the first time in an article. An important result is dealing with the connection of the symbol, the fixed multiplication pattern, to the operator. Most notably if the symbol is in the sequence spaces l^∞ , c_0 , l^2 or l^1 , respectively, then the multiplier is a bounded, compact, trace class or Hilbert–Schmidt operator, respectively. We will also prove that for Riesz bases the Bessel multipliers behave ‘nicely,’ most importantly that the mapping of the symbol to the operator is an injective one. The last result states that the Bessel multiplier depends continuously on the symbol and on the involved Bessel sequences (in a special sense). For this result the investigation of the perturbation of Bessel sequences is important. This topic is given some thought right after the introduction.

This article is organized as follows: Section 3 will first fix some notations and review basic facts in some detail. In Section 4 we are going to present results on the perturbation of Bessel sequences, frames and Riesz bases needed in Section 8. Section 5 will give the basic definition and preliminary results for Bessel multipliers. In Section 6 we will look at the influence the symbol has on the operator and investigate further properties. Section 7 deals with multipliers for Riesz bases and shows that in this case they behave ‘nicely’ in many ways. In Section 8 the influence of “small” changes of the parameters on the operator is examined. The paper is finished with Section 9, Perspectives.

This article is based on parts of [1]. Some straightforward and easy proofs can be found there and are not given here.

3. Notation and preliminaries

In this section basic notation and preliminary result are collected. Let \mathcal{H} denote a separable Hilbert space. The inner product will be denoted by $\langle \cdot, \cdot \rangle$ and will be linear in the first coordinate. Let $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ be the set of all bounded operators from \mathcal{H}_1 to \mathcal{H}_2 . With the *operator norm*, $\|O\|_{\text{Op}} = \sup_{\|x\|_{\mathcal{H}_1} \leq 1} \{\|O(x)\|_{\mathcal{H}_2}\}$, this set forms a Banach algebra. By O^* we denote the *adjoint operator*. For more details on Hilbert space respectively operator theory see [5].

Recall that an operator $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is called *compact*, if $\overline{T(B_1)}$ is compact with B_1 being the unit ball. We know that T is compact, if and only if there exists a sequence $T_n \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ with finite rank, such that $\|T_n - T\|_{\text{Op}} \rightarrow 0$ for $n \rightarrow \infty$. Special classes of compact operators we are using are the trace class (respectively Hilbert–Schmidt class (\mathcal{HS})) operators, which are operators, where the singular values are summable (respectively square summable) with the respective norms $\|\cdot\|_{\text{trace}}$ and $\|\cdot\|_{\mathcal{HS}}$. For details see [16,17] or [1]. We will be using the following special operator:

Definition 3.1. Let $f \in \mathcal{H}_1$, $g \in \mathcal{H}_2$ then define the (*inner*) *tensor product* as the operator from \mathcal{H}_2 to \mathcal{H}_1 by $(f \otimes_i \bar{g})(h) = \langle h, g \rangle f$.

For this operator we know [16] that it is a bounded linear operator from \mathcal{H}_2 to \mathcal{H}_1 with $\|f \otimes_i \bar{g}\|_{\text{Op}} = \|f\|_{\mathcal{H}_1} \cdot \|g\|_{\mathcal{H}_2}$. The last equality is also true for $\|\cdot\|_{\text{trace}}$ and $\|\cdot\|_{\mathcal{HS}}$.

3.1. Frames and bases

For more details on this topic see e.g. [4] or [3].

Definition 3.2. A sequence (ψ_k) is called a *frame* for the Hilbert space \mathcal{H} , if constants $A, B > 0$ exist, such that $A \cdot \|f\|_{\mathcal{H}}^2 \leq \sum_k |\langle f, \psi_k \rangle|^2 \leq B \cdot \|f\|_{\mathcal{H}}^2 \forall f \in \mathcal{H}$. A is a *lower*, B an *upper frame bound*. If the bounds can be chosen such that $A = B$ the frame is called *tight*. A sequence is called *Bessel sequence* if the right inequality above is fulfilled.

The index set will be omitted in the following, if no distinction is necessary. The optimal bounds A_{opt} , B_{opt} are the biggest A and smallest B that fulfill the corresponding inequality.

Lemma 3.3. Let (ψ_k) be a Bessel sequence for \mathcal{H} . Then $\|\psi_k\|_{\mathcal{H}} \leq \sqrt{B}$.

Definition 3.4. For a Bessel sequence (ψ_k) let

- $C_{(\psi_k)}: \mathcal{H} \rightarrow l^2(K)$ be the *analysis operator* $C_{(\psi_k)}(f) = (\langle f, \psi_k \rangle)_k$,
- $D_{(\psi_k)}: l^2(K) \rightarrow \mathcal{H}$ be the *synthesis operator* $D_{(\psi_k)}((c_k)) = \sum_k c_k \cdot \psi_k$ and
- $S_{(\psi_k)}: \mathcal{H} \rightarrow \mathcal{H}$ be the (*associated*) *frame operator* $S_{(\psi_k)}(f) = \sum_k \langle f, \psi_k \rangle \cdot \psi_k$.

If there is no chance of confusion, we will omit the index, so e.g. write C instead of $C_{(\psi_k)}$. These operators have the following properties:

Proposition 3.5.

- (1) Let (ψ_k) be a Bessel sequence. Then C and D are adjoint to each other, $D = C^*$ and so $\|D\|_{\text{op}} = \|C\|_{\text{op}} \leq \sqrt{B}$. The series $\sum_k c_k \cdot \psi_k$ converges unconditionally.
- (2) Let (ψ_k) be a frame. C is a bounded, injective operator with closed range.
- (3) Let (ψ_k) be a frame. $S = C^*C = DD^*$ is a positive invertible operator satisfying $AI_{\mathcal{H}} \leq S \leq BI_{\mathcal{H}}$ and $B^{-1}I_{\mathcal{H}} \leq S^{-1} \leq A^{-1}I_{\mathcal{H}}$.

If we have a frame in \mathcal{H} , we can find an expansion of every member of \mathcal{H} with this frame:

Theorem 3.6. Let (ψ_k) be a frame for \mathcal{H} with bounds $A, B > 0$. Then $(\tilde{g}_k) = (S^{-1}\psi_k)$ is a frame with bounds $B^{-1}, A^{-1} > 0$, the so-called canonical dual frame. Every $f \in \mathcal{H}$ has expansions $f = \sum_{k \in K} \langle f, \tilde{\psi}_k \rangle \psi_k$ and $f = \sum_{k \in K} \langle f, \psi_k \rangle \tilde{\psi}_k$ where both sums converge unconditionally in \mathcal{H} .

Definition 3.7. For two sequences $\{\psi_k\}$ and $\{\phi_k\}$ in \mathcal{H} we call G_{ψ_k, ϕ_k} given by $(G_{\psi_k, \phi_k})_{jm} = \langle \phi_m, \psi_j \rangle$, $j, m \in K$, the cross-Gram matrix. If $(\psi_k) = (\phi_k)$ we call this matrix the Gram matrix G_{ψ_k} .

We can look at the operator induced by the Gram matrix, defined for $c \in l^2$ formally as $(G_{\psi_k, \phi_k}c)_j = \sum_k c_k \langle \phi_k, \psi_j \rangle$. For two Bessel sequences it is well defined and bounded with

$$\|G_{\psi_k, \phi_k}\|_{\text{op}} \leq \|C_{(\psi_k)}\|_{\text{op}} \|D_{(\phi_k)}\|_{\text{op}} \leq B.$$

Definition 3.8. Let (ψ_k) be a complete sequence. If there exist constants $A, B > 0$ such that the inequalities

$$A\|c\|_2^2 \leq \left\| \sum_{k \in K} c_k \psi_k \right\|_{\mathcal{H}}^2 \leq B\|c\|_2^2$$

hold for all $c \in l^2$ the sequence (ψ_k) is called a *Riesz basis*. A sequence (ψ_k) that is a Riesz basis only for $\overline{\text{span}}(\psi_k)$ is called a *Riesz sequence*.

Every subfamily of a Riesz basis is a Riesz sequence. By the lower and upper bounds for a Riesz sequence we will denote the Riesz bounds on the closed span of the elements. It is evident that Riesz bases are frames and the Riesz bounds coincide with the frame bounds. See Christensen [4].

Theorem 3.9. Let (ψ_k) be a frame for \mathcal{H} . Then the following conditions are equivalent:

- (1) (ψ_k) is a Riesz basis for \mathcal{H} .
- (2) The coefficients $(c_k) \in l^2$ for the series expansion with (ψ_k) are unique, i.e., the synthesis operator D is injective.
- (3) The analysis operator C is surjective.
- (4) There exists sequence, which is biorthogonal to (ψ_k) .
- (5) (ψ_k) and (\tilde{g}_k) are biorthogonal.
- (6) (ψ_k) is a basis.

We also need the following estimation of the norm of the elements:

Corollary 3.10. *Let (ψ_k) be a Riesz basis with bounds A and B . Then*

$$\sqrt{A} \leq \|\psi_k\|_{\mathcal{H}} \leq \sqrt{B} \quad \forall k \in K.$$

4. Perturbation of Bessel sequences

For the perturbation results of Bessel multipliers, Section 8, we need some special results on the perturbation of Bessel sequences. The standard question of perturbation theory is whether the Bessel, frame or Riesz properties of a sequence are shared with ‘similar’ sequences. A well-known result is the following [4]: Let $(\phi_k)_{k=1}^{\infty}$ be a frame for \mathcal{H} with bounds A, B . Let $(\psi_k)_{k=1}^{\infty}$ be a sequence in \mathcal{H} . If there exist $\lambda, \mu \geq 0$ such that $\lambda + \frac{\mu}{\sqrt{A}} < 1$ and

$$\left\| \sum_k c_k (\phi_k - \psi_k) \right\|_{\mathcal{H}} \leq \lambda \left\| \sum_k c_k \phi_k \right\|_{\mathcal{H}} + \mu \|c\|_{l^2}$$

for all finite scalar sequences c (we denote $c \in c_{00}$), then (ψ_k) is a frame with lower bound $A(1 - (\lambda + \mu/\sqrt{A}))^2$ and upper bound $B(1 + \lambda + \mu/\sqrt{B})^2$. Moreover if (ϕ_k) is a Riesz basis or Riesz sequence, (ψ_k) is, too. This result can be easily formulated for Bessel sequences using only these parts of the proofs in [4, Theorem 15.1.1], which apply for Bessel sequences:

Corollary 4.1. *Let $(\phi_k)_{k=1}^{\infty}$ be a Bessel sequence for \mathcal{H} . Let $(\psi_k)_{k=1}^{\infty}$ be a sequence in \mathcal{H} . If there exist $\lambda, \mu \geq 0$ such that*

$$\left\| \sum_k c_k (\phi_k - \psi_k) \right\|_{\mathcal{H}} \leq \lambda \left\| \sum_k c_k \phi_k \right\|_{\mathcal{H}} + \mu \|c\|_{l^2}$$

for all $(c_k) \in c_{00}$, then (ψ_k) is a Bessel sequence with bound $B \cdot (1 + \lambda + \mu/\sqrt{B})^2$.

We can specialize and rephrase these results as needed in the later sections. For that let us denote the normed vector space of finite sequences in l^2 by $c_{00}^2 = (c_{00}, \|\cdot\|_2)$.

Proposition 4.2. *Let (ϕ_k) be a Bessel sequence, frame, Riesz sequence or Riesz basis for \mathcal{H} . Let (ψ_k) be a sequence in \mathcal{H} . If there exists μ such that*

$$\|D_{(\phi_k)} - D_{(\psi_k)}\|_{c_{00}^2 \rightarrow \mathcal{H}} \leq \mu < \sqrt{A}$$

then (ψ_k) shares this property with upper bound $B(1 + \mu/\sqrt{B})^2$ and, if applicable, lower bound $A(1 - \mu/\sqrt{A})^2$ and $\|D_{(\phi_k)} - D_{(\psi_k)}\|_{l^2 \rightarrow \mathcal{H}} \leq \mu$.

Proof. For every $c \in c_{00}$ we get

$$\|(D_{(\phi_k)} - D_{(\psi_k)})c\|_{\mathcal{H}} \leq \|D_{(\phi_k)} - D_{(\psi_k)}\|_{\text{Op}} \|c\|_2 \leq \mu \|c\|_2.$$

This is just the condition in the perturbation result mentioned above with $\lambda = 0$ and $\mu < \sqrt{A}$, so that $\lambda + \mu/\sqrt{A} < 1$. Because (ψ_k) is a Bessel sequence, we know that $D_{(\psi_k)} : l^2 \rightarrow \mathcal{H}$ is well defined. Because c_{00}^2 is dense in l^2 we get

$$\|D_{(\phi_k)} - D_{(\psi_k)}\|_{l^2 \rightarrow \mathcal{H}} = \|D_{(\phi_k)} - D_{(\psi_k)}\|_{c_{00}^2 \rightarrow \mathcal{H}} \leq \mu. \quad \square$$

Corollary 4.3. Let (ϕ_k) be a Bessel sequence, frame, Riesz sequence respectively Riesz basis and $(\psi_k^{(n)})$ sequences with

$$\|D_{(\psi_k^{(n)})} - D_{(\phi_k)}\|_{c_{00}^2 \rightarrow \mathcal{H}} \rightarrow 0$$

for $n \rightarrow \infty$. Then there exists N such that $(\psi_k^{(n)})$ are Bessel sequences, frames, Riesz sequences respectively Riesz bases for all $n \geq N$. For the optimal upper frame bounds we get $B_{\text{opt}}^{(n)} \rightarrow B_{\text{opt}}$. And $\|D_{(\psi_k^{(n)})} - D_{(\phi_k)}\|_{l^2 \rightarrow \mathcal{H}} \rightarrow 0$ for $n \rightarrow \infty$.

Proof. The first property is a direct consequence from Proposition 4.2.

To show the second part we note that for all $\varepsilon > 0$ there is N such that for all $n \geq N$,

$$\|D_{(\psi_k^{(n)})}\|_{\text{Op}} \leq \|D_{(\phi_k)}\|_{\text{Op}} + \|D_{(\phi_k)} - D_{(\psi_k^{(n)})}\|_{\text{Op}} \leq B + \varepsilon. \quad \square$$

In Section 8 we are going to investigate the similarity of different frames. Using Proposition 4.2 and Corollary 4.3 the following definition makes sense:

Definition 4.4. Let $\mathfrak{B}_{\text{es}}(\mathcal{H})$ be the set of all Bessel sequences in \mathcal{H} with the index set K . We define the *Bessel norm* on this set: For a sequence $(\psi_k) \in \mathfrak{B}_{\text{es}}(\mathcal{H})$ we define the norm $\|(\psi_k)\|_{\mathfrak{B}_{\text{es}}} = \|D_{(\psi_k)}\|_{\text{Op}}$.

It can be easily shown, that $\|\cdot\|_{\mathfrak{B}_{\text{es}}}$ is well defined and induces a norm. As shown above it is sufficient to use the operator norm on c_{00}^2 . In typical perturbation results like Corollary 4.1, for arbitrary sequences it is investigated, if they share the property with another ‘similar’ sequence, which is a Bessel sequence, frame or Riesz basis. In these cases we cannot use the above norm, as we cannot assume at first, that we start out with a Bessel sequence. We are going to define other measures of similarity of sequences, with the property, that if those are small for two Bessel sequences also the Bessel norm is small.

A simple way to measure the similarity of two frames would be in a uniform sense, using $\sup_k \|\psi_k - \phi_k\|_{\mathcal{H}}$, but this is not a good measure in general for frames since it makes an orthonormal basis similar to every bounded sequence of vectors. Other similarity measures are more useful, as defined below and motivated in the next result.

Corollary 4.5. Let (ψ_k) be a Bessel sequence, frame, Riesz sequence respectively a Riesz basis. Let (ϕ_k) be a sequence with $\sum_k \|\psi_k - \phi_k\|_{\mathcal{H}}^2 < A$ (respectively $\sum_k \|\psi_k - \phi_k\|_{\mathcal{H}} < A$), then (ϕ_k) is a Bessel sequences, frame, Riesz sequence or Riesz basis.

Let $(\phi_k^{(l)} \mid k \in K)$ be sequences such that for all ε there exists $N(\varepsilon)$ with $\sum_k \|\psi_k - \phi_k^{(l)}\|_{\mathcal{H}}^2 < \varepsilon^2$ (respectively $\sum_k \|\psi_k - \phi_k^{(l)}\|_{\mathcal{H}} < \varepsilon$) for all $l \geq N(\varepsilon)$. Then for $\varepsilon < \sqrt{A}$ and for all $l \geq N_0$ the sequences $(\phi_k^{(l)})$ are Bessel sequences, frames, Riesz sequences respectively Riesz bases with the optimal upper frame bound $B_{\text{opt}}^{(l)} \rightarrow B_{\text{opt}}$. Furthermore $\|C_{(\phi_k^{(l)})} - C_{(\psi_k)}\|_{\text{Op}} < \varepsilon$, $\|D_{(\phi_k^{(l)})} - D_{(\psi_k)}\|_{\text{Op}} < \varepsilon$ and for $\varepsilon \leq 1$ $\|S_{(\phi_k^{(l)})} - S_{(\psi_k)}\|_{\text{Op}} < \varepsilon \cdot (\sqrt{B+1} \cdot \sqrt{B})$.

Proof. Let $c \in c_{00}$, then

$$\|D_{(\phi_k)}c - D_{(\psi_k)}c\|_{\mathcal{H}} = \left\| \sum_k c_k(\phi_k - \psi_k) \right\|_{\mathcal{H}} \leq \sum_k |c_k| \|\psi_k - \phi_k\|_{\mathcal{H}}$$

$$\leq \sqrt{\sum_k |c_k|^2} \sqrt{\sum_k \|\psi_k - \phi_k\|_{\mathcal{H}}^2}$$

$$\implies \|D_{(\phi_k)} - D_{(\psi_k)}\|_{\text{Op}} \leq \sqrt{\sum_k \|\psi_k - \phi_k\|_{\mathcal{H}}^2}.$$

So in the first case $\|D_{(\phi_k)} - D_{(\psi_k)}\|_{\text{Op}} < \sqrt{A}$ and therefore (ϕ_k) forms a Bessel sequence, frame, Riesz sequence or Riesz basis.

In the second case we get $\|D_{\phi_k^{(l)}} - D_{(\psi_k)}\|_{\text{Op}} < \varepsilon$ for $l \geq N(\varepsilon)$. With Corollary 4.3 the result for the bounds is proved,

$$\begin{aligned} \|C_{(\phi_k^{(l)})} f - C_{(\psi_k)}\|_{\text{Op}} &= \|D_{(\phi_k^{(l)})}^* - D_{(\psi_k)}^*\|_{\text{Op}} = \|D_{(\phi_k^{(l)})} - D_{(\psi_k)}\|_{\text{Op}} < \varepsilon, \\ \|S_{(\phi_k^{(l)})} - S_{(\psi_k)}\|_{\text{Op}} &= \|D_{(\phi_k^{(l)})} \circ C_{(\phi_k^{(l)})} - D_{(\psi_k)} \circ C_{(\psi_k)}\|_{\text{Op}} \\ &= \|D_{(\phi_k^{(l)})} \circ C_{(\phi_k^{(l)})} - D_{(\phi_k^{(l)})} \circ C_{(\psi_k)} + D_{(\phi_k^{(l)})} \circ C_{(\psi_k)} - D_{(\psi_k)} \circ C_{(\psi_k)}\|_{\text{Op}} \\ &\leq \|D_{(\phi_k^{(l)})}\|_{\text{Op}} \|C_{(\phi_k^{(l)})} - C_{(\psi_k)}\|_{\text{Op}} + \|D_{(\phi_k^{(l)})} - D_{(\psi_k)}\|_{\text{Op}} \|C_{(\psi_k)}\|_{\text{Op}} \\ &\leq \sqrt{B+1} \cdot \varepsilon + \varepsilon \cdot \sqrt{B} = \varepsilon \cdot (\sqrt{B+1} + \sqrt{B}), \end{aligned}$$

which follows from Corollary 4.3, as there is $N(1)$ such that for all $l \geq N(1)$, $\|D_{(\phi_k^{(l)})}\|_{\text{Op}} \leq \sqrt{B+1}$.

For all sequences C we know $\|c\|_1 \geq \|c\|_2$. Therefore the corresponding l^1 property above is also true. \square

With these similarity measures in general neither a norm nor a metric is defined on the set of Bessel sequences. Nevertheless it is useful to use these ‘similarity measures’ as seen in the last two corollaries.

Definition 4.6. Let $(\psi_k)_{k \in K}$ and $(\psi_k^{(l)})_{k \in K}$ be a sequence of elements for all $l \in \mathbb{N}$. The sequences $(\psi_k^{(l)})$ are said to *converge to (ψ_k) in an l^p sense*, denoted by $(\psi_k^{(l)}) \xrightarrow{l^p} (\psi_k)$, if $\forall \varepsilon > 0$ there exists $N > 0$ such that $(\sum_k \|\psi_k^{(l)} - \psi_k\|_{\mathcal{H}}^p)^{1/p} < \varepsilon$ for all $l \geq N$.

The convergence in an l^∞ sense clearly coincides with uniform convergence, which is not valuable for our purposes, as seen above. The convergence in an l^1 and l^2 sense is very useful, in contrast, see Section 8.

5. Bessel multipliers

In [16] R. Schatten provides a detailed study of ideals of compact operators using their singular value decomposition. He investigates the operators of the form $\sum \lambda_i \varphi_i \otimes \psi_i$ where (φ_i) and (ψ_i) are orthonormal families. We are interested in similar operators where the families are Bessel sequences.

Definition 5.1. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, let $(\psi_k)_{k \in K} \subseteq \mathcal{H}_1$ and $(\phi_k)_{k \in K} \subseteq \mathcal{H}_2$ be Bessel sequences. Fix $m \in l^\infty(K)$. Define the operator $\mathbf{M}_{m, (\phi_k), (\psi_k)} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$, the *Bessel multiplier* for the Bessel sequences (ψ_k) and (ϕ_k) , as the operator

$$\mathbf{M}_{m,(\phi_k),(\psi_k)}(f) = \sum_k m_k \langle f, \psi_k \rangle \phi_k.$$

The sequence m is called the *symbol* of \mathbf{M} . For frames we will call the resulting Bessel multiplier a *frame multiplier*, for Riesz sequence a *Riesz multiplier*.

Let us denote $\mathbf{M}_{m,(\psi_k)} = \mathbf{M}_{m,(\psi_k),(\psi_k)}$. Furthermore let us simplify the notation, if there is no chance of confusion, using \mathbf{M}_m or even \mathbf{M} . The definition of a multiplier can also be expressed in the following way:

$$\mathbf{M}_{m,(\phi_k),(\psi_k)} = D_{(\phi_k)}(m \cdot C_{(\psi_k)}) = \sum_k m_k \cdot \phi_k \otimes_i \psi_k.$$

Definition 5.2. For fixed Bessel sequences (ψ_k) and (ϕ_k) , let σ be the relation which assigns the corresponding symbol to a multiplier, $\sigma(\mathbf{M}_{m,(\phi_k),(\psi_k)}) = m$.

This relation σ does not have to be a well-defined function. This is only the case if the operators $\phi_k \otimes_i \psi_k$ have a basis property, cf. Section 6.1.

In [16] the multipliers for orthonormal sequences were investigated and many ‘nice’ properties were shown. A powerful property is the ‘symbolic calculus’ for orthonormal sequences as follows:

$$\mathbf{M}_{m^{(1)},(e_k)} \circ \mathbf{M}_{m^{(2)},(e_k)} = \mathbf{M}_{m^{(1)} \cdot m^{(2)},(e_k)}.$$

In the general Bessel sequence case this is not true anymore.

Corollary 5.3. For two multipliers $\mathbf{M}_{m^{(1)},(\phi_k),(\psi_k)}$ and $\mathbf{M}_{m^{(2)},(\zeta_k),(\xi_k)}$ for the Bessel sequences $(\psi_k), (\zeta_k) \subseteq \mathcal{H}_2$, $(\phi_k) \subseteq \mathcal{H}_2$, $(\xi_k) \subseteq \mathcal{H}_1$ and

$$\mathbf{M}_{m^{(1)},(\phi_k),(\psi_k)} = \sum_k m_k^{(1)} \langle f, \psi_k \rangle \phi_k \quad \text{and} \quad \mathbf{M}_{m^{(2)},(\zeta_k),(\xi_k)} = \sum_l m_l^{(2)} \langle f, \xi_l \rangle \zeta_l$$

the combination is

$$\begin{aligned} (\mathbf{M}_{m^{(1)},(\phi_k),(\psi_k)} \circ \mathbf{M}_{m^{(2)},(\zeta_k),(\xi_k)})(f) &= \sum_k \sum_l m_k^{(1)} m_l^{(2)} \langle f, \xi_l \rangle \langle \zeta_l, \psi_k \rangle \phi_k \\ &= (D_{(\phi_k)} \mathcal{M}_{m^{(1)}} G_{\psi_k, \zeta_k} \mathcal{M}_{m^{(2)}} C_{(\xi_k)})(f). \end{aligned}$$

Thus in the general Bessel sequence case no exact symbolic calculus can be assumed, i.e., the combination of symbols does not correspond to the combination of the operators. See Section 7 for more details on this. In general the product of two frame multipliers is not even a frame multiplier any more.

5.1. The multiplier as an operator from l^2 to l^2

As a preparatory step we will look at this kind of operators on l^2 . Use the symbol \mathcal{M}_m for the mapping $\mathcal{M}_m: l^2 \rightarrow l^2$ and $m \in l^p$ (for $p > 0$) given by the pointwise multiplication $\mathcal{M}_m((c_k)) = (m_k \cdot c_k)$. So a Bessel multiplier \mathbf{M}_m can be written as

$$\mathbf{M}_m = D \circ \mathcal{M}_m \circ C.$$

As preparation for one of the main results, Theorem 6.1, we show:

Lemma 5.4.

- (1) Let $m \in l^\infty$. The operator $\mathcal{M}_m: l^2 \rightarrow l^2$ is bounded with $\|\mathcal{M}_m\|_{\text{Op}} = \|m\|_\infty$.
- (2) $\mathcal{M}_m^* = \mathcal{M}_{\bar{m}}$.
- (3) Let $m \in l^1$. The operator $\mathcal{M}_m: l^2 \rightarrow l^2$ is trace class with $\|\mathcal{M}_m\|_{\text{trace}} = \|m\|_1$.
- (4) Let $m \in l^2$. The operator $\mathcal{M}_m: l^2 \rightarrow l^2$ is a Hilbert–Schmidt (\mathcal{HS}) operator with $\|\mathcal{M}_m\|_{\mathcal{HS}} = \|m\|_2$.
- (5) Let $m \in c_0$. Then there exist finite sequences $m_N = (m_0, \dots, m_N, 0, 0, \dots)$ with $\mathcal{M}_{m_N} \rightarrow \mathcal{M}_m$ as operators in l^2 . Therefore \mathcal{M}_m is compact.

Proof. (1) Clearly $\|m \cdot c\|_2 \leq \|m\|_\infty \|c\|_2 \Rightarrow \|\mathcal{M}_m\|_{\text{Op}} \leq \|m\|_\infty$. On the other hand, $\mathcal{M}_m \delta_i = m_i \Rightarrow \|\mathcal{M}_m\|_{\text{Op}} \geq \|m\|_\infty$.

$$(2) \langle \mathcal{M}_m c, d \rangle_{l^2} = \sum_k m_k c_k \cdot \bar{d}_k = \sum_k c_k \cdot \overline{m_k d_k} = \langle c, \mathcal{M}_{\bar{m}} d \rangle_{l^2}.$$

(3) $[\mathcal{M}_m] = \sqrt{\mathcal{M}_m^* \mathcal{M}_m} = \sqrt{\mathcal{M}_{\bar{m}} \mathcal{M}_m} = \mathcal{M}_{|m|}$ and so using properties of the trace norm [17] $\|\mathcal{M}_m\|_{\text{trace}} = \sum_i \langle [\mathcal{M}_m] \delta_i, \delta_i \rangle = \|m\|_1$.

$$(4) \|\mathcal{M}_m\|_{\mathcal{HS}}^2 = \sum_i \|\mathcal{M}_m \delta_i\|_2^2 = \|m\|_2^2.$$

(5) For $c \in l^2$ $\|m_N \cdot c - m \cdot c\|_2 \leq \|m_N - m\|_\infty \cdot \|c\|_2$ and so $\|\mathcal{M}_{m_N} - \mathcal{M}_m\|_{\text{Op}} \rightarrow 0$. \square

6. Properties of multipliers

Equivalent results as proved in [8] for Gabor multiplier can be shown for Bessel multipliers.

Theorem 6.1. Let $\mathbf{M} = \mathbf{M}_{m, (\phi_k), (\psi_k)}$ be a Bessel multiplier for the Bessel sequences $(\psi_k) \subseteq \mathcal{H}_1$ and $(\phi_k) \subseteq \mathcal{H}_2$ with the bounds B and B' . Then

- (1) If $m \in l^\infty$, \mathbf{M} is a well-defined bounded operator with $\|\mathbf{M}\|_{\text{Op}} \leq \sqrt{B'} \sqrt{B} \cdot \|m\|_\infty$. Furthermore the sum $\sum_k m_k \langle f, \psi_k \rangle \phi_k$ converges unconditionally for all $f \in \mathcal{H}_1$.
- (2) $(\mathbf{M}_{m, (\phi_k), (\psi_k)})^* = \mathbf{M}_{\bar{m}, (\psi_k), (\phi_k)}$. Therefore if m is real-valued and $(\phi_k) = (\psi_k)$, \mathbf{M} is self-adjoint.
- (3) If $m \in c_0$, \mathbf{M} is a compact operator.
- (4) If $m \in l^1$, \mathbf{M} is a trace class operator with $\|\mathbf{M}\|_{\text{trace}} \leq \sqrt{B'} \sqrt{B} \|m\|_1$. And $\text{tr}(\mathbf{M}) = \sum_k m_k \langle \phi_k, \psi_k \rangle$.
- (5) If $m \in l^2$, \mathbf{M} is a Hilbert–Schmidt operator with $\|\mathbf{M}\|_{\mathcal{HS}} \leq \sqrt{B'} \sqrt{B} \|m\|_2$.

Proof. (1)

$$\|\mathbf{M}\|_{\text{Op}} = \|C \circ \mathcal{M}_m \circ D\|_{\text{Op}} \leq \|C\|_{\text{Op}} \cdot \|m\|_\infty \cdot \|D\|_{\text{Op}} \leq \sqrt{B} \|m\|_\infty \sqrt{B'}.$$

As (ϕ_k) is a Bessel sequence, $\sum c_k \phi_k$ converges unconditionally for all $(c_k) \in l^2$, in particular for $(m_k \cdot \langle f, \psi_k \rangle)$.

(2) $\mathbf{M} = C_{(\psi_k)} \circ \mathcal{M}_m \circ D_{(\phi_k)} = C_{(\psi_k)} \circ \mathcal{M}_m \circ C_{(\phi_k)}^*$, so with Lemma 5.4 $\mathbf{M}^* = C_{(\phi_k)} \circ \mathcal{M}_m^* \circ C_{(\psi_k)}^* = C_{(\phi_k)} \circ \mathcal{M}_{\bar{m}} \circ D_{(\psi_k)}$.

(3) Let m_N be the finite sequences from Lemma 5.4, then

$$\begin{aligned} \|\mathbf{M}_{m_N} - \mathbf{M}_m\|_{\text{Op}} &= \|D \mathcal{M}_{m_N} C - D \mathcal{M}_m C\|_{\text{Op}} = \|D(\mathcal{M}_{m_N} - \mathcal{M}_m)C\|_{\text{Op}} \\ &\leq \|D\|_{\text{Op}} \|\mathcal{M}_{m_N} - \mathcal{M}_m\|_{\text{Op}} \|C\|_{\text{Op}} \leq \sqrt{B'} \cdot \varepsilon \sqrt{B}. \end{aligned}$$

For every $\varepsilon' = \varepsilon/\sqrt{B \cdot B'}$, there is a N_ε such that $\|\mathcal{M}_{m_N} - \mathcal{M}_m\|_{\text{Op}} < \varepsilon'$ and therefore $\|\mathbf{M}_{m_N} - \mathbf{M}_m\|_{\text{Op}} < \varepsilon$ for all $N \geq N_\varepsilon$. \mathbf{M}_{m_N} is a finite sum of rank one operators and so has finite rank. This means that \mathbf{M}_m is a limit of finite-rank operators and therefore compact.

(4) $\mathbf{M}(f) = \sum_k \langle f, \psi_k \rangle (m_k \cdot \phi_k)$, so according to the definition of trace class operators [17] we have to show that $\|\mathbf{M}\|_{\text{trace}} = \sum_k \|\psi_k\|_{\mathcal{H}} \cdot \|m_k \phi_k\|_{\mathcal{H}} < \infty$,

$$\begin{aligned} \|\mathbf{M}\|_{\text{trace}} &= \sum_k \|\psi_k\|_{\mathcal{H}} \cdot \|m_k \phi_k\|_{\mathcal{H}} = \sum_k \|\psi_k\|_{\mathcal{H}} |m_k| \|\phi_k\|_{\mathcal{H}} \leq \sqrt{B} \cdot \sqrt{B'} \cdot \|m\|_1 \\ \implies \text{tr}(\mathbf{M}) &= \sum_k \langle m_k \cdot \phi_k, \psi_k \rangle = \sum_k m_k \langle \phi_k, \psi_k \rangle. \end{aligned}$$

(5) The operator $\mathcal{M}_m: l^2 \rightarrow l^2$ is in \mathcal{HS} due to Lemma 5.4 with bound $\|\mathcal{M}_m\|_{\mathcal{HS}} = \|m\|_2$. Using the properties of \mathcal{HS} operators we get $\|D\mathcal{M}_m C\|_{\mathcal{HS}} \leq \|D\|_{\text{Op}} \|m\|_2 \|C\|_{\text{Op}} \leq \sqrt{B} \sqrt{B'} \|m\|_2$. \square

For Riesz and orthonormal bases we can show, see Proposition 7.2, that if the multiplier is well defined, then the symbol must be in l^∞ . This is not true for general Bessel sequences, as can be seen, when using the following frame: Let $(e_i \mid i \in \mathbb{N})$ be an ONB for \mathcal{H} . Let $\psi_{p,q} = \frac{1}{p} \cdot e_q$. Then $(\psi_{p,q} \mid (p,q) \in \mathbb{N}^2)$ is a tight frame as

$$\begin{aligned} \sum_{p,q} |\langle f, \psi_{p,q} \rangle|^2 &= \sum_{p,q} \left| \left\langle f, \frac{1}{p} \cdot e_q \right\rangle \right|^2 = \sum_p \frac{1}{|p|^2} \sum_q |\langle f, e_q \rangle|^2 = \sum_p \frac{1}{|p|^2} \|f\|_{\mathcal{H}}^2 \\ &= \|f\|_{\mathcal{H}}^2 \cdot \frac{\pi^2}{6}. \end{aligned}$$

Define a symbol m by $m_{p,q} = p^2$. Then

$$\mathbf{M}_{m,(\psi_{p,q})}(f) = \sum_{p,q} p^2 \left\langle f, \frac{1}{p} \cdot e_q \right\rangle \frac{1}{p} \cdot e_q = \sum_{p,q} \langle f, e_q \rangle \cdot e_q = f.$$

So the operator $\mathbf{M}_{m_{k,l},(\psi_{k,l})} = \text{Id}$ is bounded although the symbol is not.

6.1. From symbol to operator

When is the operator uniquely defined by the symbol? When is the relation σ a function? This question is equivalent to the question of whether the sequence of operators $(\phi_h \otimes_i \psi_k)$ forms a Riesz sequence, as the rank one operators $\psi_k \otimes_i \bar{f}_k$ form a Bessel sequence in \mathcal{HS} . This follows directly from the following result, as every subsequence of a Bessel sequence is a Bessel sequence again [4].

Proposition 6.2. *Let $(\psi_k \mid k \in K)$ and $(\phi_k \mid k \in K)$ be Bessel sequences in \mathcal{H}_1 respectively \mathcal{H}_2 with bounds B_1 and B_2 . The rank one operators $(\psi_k \otimes_i \bar{\phi}_l)$ with $(k,l) \in K \times K$ form a Bessel sequence in $\mathcal{HS}(\mathcal{H}_2, \mathcal{H}_1)$ with bounds $B_1 \cdot B_2$.*

Proof. Let $O \in \mathcal{HS}(\mathcal{H}_2, \mathcal{H}_1)$. Then by properties of the Hilbert–Schmidt inner product,

$$\sum_{k,l} |\langle O, \psi_k \otimes_i \bar{\phi}_l \rangle_{\mathcal{HS}}|^2 = \sum_l \sum_k |\langle O \phi_l, \psi_k \rangle|^2 \leq B_1 \sum_l \|O \phi_l\|_{\mathcal{H}_1}^2.$$

Let now $(e_i \mid i \in I)$ be any ONB of \mathcal{H}_2 , then

$$\begin{aligned} \sum_l \|O\phi_l\|_{\mathcal{H}}^2 &= \sum_l \sum_i |\langle O\phi_l, e_i \rangle|^2 = \sum_i \sum_l |\langle \phi_l, O^*e_i \rangle|^2 \leq B_2 \cdot \sum_i \|O^*e_i\|_{\mathcal{H}_2}^2 \\ &= B_2 \cdot \|O^*\|_{\mathcal{H}_S}^2 = B_2 \cdot \|O\|_{\mathcal{H}_S}^2 \\ \Rightarrow \sum_{k,l} |\langle O, \psi_k \otimes_i \bar{f}_l \rangle|^2 &\leq B_1 B_2 \|O\|_{\mathcal{H}_S}^2. \quad \square \end{aligned}$$

7. Riesz multipliers

For Riesz sequences the family $(\psi_k \otimes_i \bar{\phi}_k)$ is certainly a Riesz sequence in \mathcal{H}_S , following Proposition 6.2 and the fact that

$$\langle \psi_k \otimes_i \bar{\phi}_k, \tilde{g}_l \otimes_i \bar{\phi}_l \rangle = \langle \psi_k, \tilde{g}_l \rangle \cdot \langle \bar{\phi}_l, \phi_k \rangle = \delta_{k,l} \cdot \delta_{k,l}.$$

In this case for $m \in l^2$ the function $m \mapsto \mathbf{M}_m$ is injective as the multiplier is just the synthesis operator of the sequence $(\psi_k \otimes_i \bar{\phi}_k)$ applied on m . We can state a more general property:

Lemma 7.1. *Let $(\psi_k) \subseteq \mathcal{H}_1$ be a Bessel sequence with no zero elements, and $(\phi_k) \subseteq \mathcal{H}_2$ a Riesz sequence. Then the mapping $m \mapsto \mathbf{M}_{m, \phi_k, \psi_k}$ is injective from l^∞ into $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$.*

Proof. Suppose $\mathbf{M}_{m, (\phi_k), (\psi_k)} = \mathbf{M}_{m', (\phi_k), (\psi_k)} \Rightarrow \sum_k m_k \langle f, \psi_k \rangle \phi_k = \sum_k m'_k \langle f, \psi_k \rangle \phi_k$ for all f . As ϕ_k is a Riesz basis for its span $\Rightarrow m_k \langle f, \psi_k \rangle = m'_k \langle f, \psi_k \rangle$ for all f, k . For any $k \in K$ we know $\psi_k \neq 0$. So there exists f such that $\langle f, \psi_k \rangle \neq 0$. Therefore $m_k = m'_k$ for all k . And so $(m_k) = (m'_k)$. \square

So if the conditions in Lemma 7.1 are fulfilled, the Bessel sequence $(\psi_k \otimes_i \bar{f}_k)$ is a Riesz sequence in \mathcal{H}_S .

For Riesz bases the multiplier is bounded if and only if the symbol is bounded:

Proposition 7.2. *Let (ψ_k) be a Riesz basis with bounds A, B and (ϕ_k) be one with bounds A', B' . Then*

$$\sqrt{AA'} \|m\|_\infty \leq \|\mathbf{M}_{m, (\phi_k), (\psi_k)}\|_{\text{Op}} \leq \sqrt{BB'} \|m\|_\infty.$$

Particularly $\mathbf{M}_{m, (\phi_k), (\psi_k)}$ is bounded if and only if m is bounded.

Proof. Theorem 6.1 gives us the upper bound. For the lower bound let k_0 be arbitrary, then $\mathbf{M}_{m, (\phi_k), (\psi_k)}(\tilde{\psi}_{k_0}) = \sum_k m_k \langle \tilde{\psi}_{k_0}, \psi_k \rangle \phi_k = \sum_k m_k \delta_{k_0, k} \phi_k = m_{k_0} \phi_{k_0}$. Therefore

$$\begin{aligned} \|\mathbf{M}_{m, (\phi_k), (\psi_k)}\|_{\text{Op}} &= \sup_{f \in \mathcal{H}} \left\{ \frac{\|\mathbf{M}_{m, (\phi_k), (\psi_k)}(f)\|_{\mathcal{H}}}{\|f\|_{\mathcal{H}}} \right\} \geq \frac{\|\mathbf{M}_{m, (\phi_k), (\psi_k)}(\tilde{\psi}_{k_0})\|_{\mathcal{H}}}{\|\tilde{\psi}_{k_0}\|_{\mathcal{H}}} \\ &= \frac{\|m_{k_0} \phi_{k_0}\|_{\mathcal{H}}}{\|\tilde{\psi}_{k_0}\|_{\mathcal{H}}} \geq \frac{|m_{k_0}| \sqrt{A'}}{1/\sqrt{A}} \geq \sqrt{A'A} |m_{k_0}|, \end{aligned}$$

using Corollary 3.10 and the properties of the dual frame. \square

For an orthonormal sequence (ϵ_k) the combination of multipliers $\mathbf{M} = \mathbf{M}_{m,(\epsilon_k)} \cdot \mathbf{M}_{m',(\epsilon_k)}$ is just the multiplier with symbol $\sigma(\mathbf{M}) = m \cdot m'$. This is true for all biorthogonal sequences in the following way:

Corollary 7.3. *Let (ψ_k) , (ϕ_k) , (ζ_k) and (ξ_k) be Bessel sequences, such that (ϕ_k) and (ψ_k) are biorthogonal to each other, then*

$$(\mathbf{M}_{m^{(1)},(\xi_k),(\psi_k)} \circ \mathbf{M}_{m^{(2)},(\phi_k),(\zeta_k)})(f) = \mathbf{M}_{m^{(1)} \cdot m^{(2)},(\xi_k),(\zeta_k)}.$$

So for Riesz sequences we get that the symbol of the combination of multipliers is the multiplication of the symbols. The reverse of this is also true as stated in Corollary 7.5. For this result we first show the following property:

Proposition 7.4. *Let (ψ_k) and (ϕ_k) be Bessel sequences in \mathcal{H} with the same index set K . If $\forall m^{(1)}, m^{(2)} \in c_{00}$,*

$$\mathbf{M}_{m^{(1)},(\psi_k),(\phi_k)} \circ \mathbf{M}_{m^{(2)},(\psi_k),(\phi_k)} = \mathbf{M}_{m^{(1)} \cdot m^{(2)},(\psi_k),(\phi_k)}$$

then for all pairs $(k, l) \in K \times K$ either $\phi_l = 0$, $\psi_k = 0$ or $\langle \psi_k, \phi_l \rangle = \delta_{k,l}$.

Proof. Choose k_0, k_1 in the index set. Let $m^{(1)} = \delta_{k_0}$ and $m^{(2)} = \delta_{k_1}$.

$$\mathbf{M}_{m^{(1)},(\psi_k),(\phi_k)} \circ \mathbf{M}_{m^{(2)},(\psi_k),(\phi_k)} = \mathbf{M}_{m^{(1)} \cdot m^{(2)},(\psi_k),(\phi_k)}$$

is in this case equivalent via Lemma 5.3 to

$$\langle f, \phi_{k_1} \rangle \langle \psi_{k_1}, \phi_{k_0} \rangle \cdot \psi_{k_0} = \delta_{k_0, k_1} \langle f, \phi_{k_1} \rangle \psi_{k_0} \quad \forall f \in \mathcal{H}.$$

Let $k_1 \neq k_0$ then this means that we obtain $\langle f, \phi_{k_1} \rangle \langle \psi_{k_1}, \phi_{k_0} \rangle \cdot \psi_{k_0} = 0$. So either $\psi_{k_0} = 0$ or $\langle f, \phi_{k_1} \rangle = 0$ for all f and so $\phi_{k_1} = 0$, or $\langle \psi_{k_1}, \phi_{k_0} \rangle = 0$.

Let $k_1 = k_0$. $\langle f, \phi_{k_1} \rangle (\langle \psi_{k_1}, \phi_{k_0} \rangle - 1) \psi_{k_0} = 0$. Either $\psi_{k_0} = 0$ or $\langle f, \phi_{k_1} \rangle = 0$ for all f and so $\phi_{k_1} = 0$ or $\langle \psi_{k_1}, \phi_{k_0} \rangle = 1$. \square

This means we have found a way to classify Riesz bases by multipliers:

Corollary 7.5. *Let (ψ_k) and (ϕ_k) be Bessel sequences with $\psi_k \neq 0$ and $\phi_k \neq 0$ for all $k \in K$. If and only if $\sigma(\mathbf{M}_{m^{(1)},(\phi_k),(\psi_k)} \circ \mathbf{M}_{m^{(2)},(\phi_k),(\psi_k)}) = \sigma(\mathbf{M}_{m^{(1)},(\phi_k),(\psi_k)}) \cdot \sigma(\mathbf{M}_{m^{(2)},(\phi_k),(\psi_k)})$ for all multipliers $\mathbf{M}_{m^{(1)},(\phi_k),(\psi_k)}$, $\mathbf{M}_{m^{(2)},(\phi_k),(\psi_k)}$ with $m^{(1)}, m^{(2)}$ finite, then these frames are biorthogonal to each other and therefore have to be Riesz bases.*

The commutation of multipliers involving Riesz sequences behaves also very canonically:

Corollary 7.6. *Let (ψ_k) be a Riesz sequence, then*

$$\mathbf{M}_{m^{(1)},(\tilde{\psi}_k),(\psi_k)} \circ \mathbf{M}_{m^{(2)},(\tilde{\psi}_k),(\psi_k)} = \mathbf{M}_{m^{(2)},(\tilde{\psi}_k),(\psi_k)} \circ \mathbf{M}_{m^{(1)},(\tilde{\psi}_k),(\psi_k)}.$$

Finally we can ask, when a Riesz multiplier is invertible, or more precisely when it is the inverse of another multiplier. Let us call a sequence (m_k) for which $0 < \inf |m_k| \leq \sup |m_k| < \infty$ a *semi-normalized* sequence.

Proposition 7.7. *Let (ψ_k) and (ϕ_k) be Riesz bases and let the symbol m be semi-normalized. Then $\mathbf{M}_{m_k, (\phi_k), (\psi_k)}^{-1} = \mathbf{M}_{\frac{1}{m_k}, (\tilde{\psi}_k), (\tilde{\phi}_k)}$.*

Proof. If (m_k) is semi-normalized, $(\frac{1}{m_k})$ is, too. Therefore $(\frac{1}{m_k}) \in l^\infty$. Corollary 5.3 tells us that

$$\begin{aligned} (\mathbf{M}_{m, (\phi_k), (\psi_k)} \circ \mathbf{M}_{\frac{1}{m}, (\tilde{\psi}_k), (\tilde{\phi}_k)})(f) &= \sum_k \sum_l m_k \frac{1}{m_l} \langle f, \tilde{\phi}_l \rangle \langle \tilde{\psi}_l, \psi_k \rangle \phi_k \\ &= \sum_k \sum_l m_k \frac{1}{m_l} \langle f, \tilde{\phi}_l \rangle \delta_{l,k} \phi_k = \sum_k m_k \frac{1}{m_k} \langle f, \tilde{\phi}_k \rangle \phi_k = f. \end{aligned}$$

With the commutativity shown in Corollary 7.6 we can finish the proof. \square

8. Changing the ingredients

A Bessel multiplier clearly depends on the chosen symbol, analysis and synthesis sequence. A natural question arises: What happens if these items are changed? Are the frame multipliers similar to each other if the symbol or the frames are similar to each other (in the right similarity sense)?

Theorem 8.1. *Let \mathbf{M} be a multiplier for the Bessel sequences (ψ_k) and (ϕ_k) with Bessel bounds B_1 and B_2 , respectively. Then the operator \mathbf{M} depends continuously on m , (ψ_k) and (ϕ_k) , in the following sense: Let $(\psi_k^{(l)})$ and $(\phi_k^{(l)})$ be sequences indexed by $l \in \mathbb{N}$.*

- (1) (a) *Let $m^{(l)} \rightarrow m$ in l^∞ then $\|\mathbf{M}_{m^{(l)}, (\psi_k), (\phi_k)} - \mathbf{M}_{m, (\psi_k), (\phi_k)}\|_{\text{Op}} \rightarrow 0$.*
 (b) *Let $m^{(l)} \rightarrow m$ in l^2 then $\|\mathbf{M}_{m^{(l)}, (\psi_k), (\phi_k)} - \mathbf{M}_{m, (\psi_k), (\phi_k)}\|_{\mathcal{HS}} \rightarrow 0$.*
 (c) *Let $m^{(l)} \rightarrow m$ in l^1 then $\|\mathbf{M}_{m^{(l)}, (\psi_k), (\phi_k)} - \mathbf{M}_{m, (\psi_k), (\phi_k)}\|_{\text{trace}} \rightarrow 0$.*
- (2) (a) *Let $m \in l^1$ and let the sequences $(\psi_k^{(l)})$ be Bessel sequences converging uniformly to (ψ_k) . Then for $l \rightarrow \infty$ $\|\mathbf{M}_{m, (\psi_k^{(l)}), (\phi_k)} - \mathbf{M}_{m, (\psi_k), (\phi_k)}\|_{\text{trace}} \rightarrow 0$.*
 (b) *Let $m \in l^2$ and let the sequences $(\psi_k^{(l)})$ converge to (ψ_k) in an l^2 sense. Then for $l \rightarrow \infty$ $\|\mathbf{M}_{m, (\psi_k^{(l)}), (\phi_k)} - \mathbf{M}_{m, (\psi_k), (\phi_k)}\|_{\mathcal{HS}} \rightarrow 0$.*
 (c) *Let $m \in l^\infty$ and let the sequences $(\psi_k^{(l)})$ converge to (ψ_k) in an l^1 sense. Then for $l \rightarrow \infty$ $\|\mathbf{M}_{m, (\psi_k^{(l)}), (\phi_k)} - \mathbf{M}_{m, (\psi_k), (\phi_k)}\|_{\text{Op}} \rightarrow 0$.*
- (3) *For Bessel sequences $(\phi_k^{(l)})$ converging to (ϕ_k) , corresponding properties as in (2) apply.*
- (4) (a) *Let $m^{(l)} \rightarrow m$ in l^1 , $(\psi_k^{(l)})$ and $(\phi_k^{(l)})$ be Bessel sequences with bounds $B_1^{(l)}$ and $B_2^{(l)}$, such that there exists \mathbf{B}_1 and \mathbf{B}_2 with $B_1^{(l)} \leq \mathbf{B}_1$ and $B_2^{(l)} \leq \mathbf{B}_2$. Let the sequences $(\psi_k^{(l)})$ and $(\phi_k^{(l)})$ converge uniformly to (ψ_k) respectively (ϕ_k) . Then for $l \rightarrow \infty$*

$$\|\mathbf{M}_{m^{(l)}, (\psi_k^{(l)}), (\phi_k^{(l)})} - \mathbf{M}_{m, (\psi_k), (\phi_k)}\|_{\text{trace}} \rightarrow 0.$$

- (b) *Let $m^{(l)} \rightarrow m$ in l^2 and let the sequences $(\psi_k^{(l)})$ respectively $(\phi_k^{(l)})$ converge to (ψ_k) respectively (ϕ_k) in an l^2 sense. Then for $l \rightarrow \infty$ $\|\mathbf{M}_{m, (\psi_k^{(l)}), (\phi_k)} - \mathbf{M}_{m, (\psi_k), (\phi_k)}\|_{\mathcal{HS}} \rightarrow 0$.*
- (c) *Let $m^{(l)} \rightarrow m$ in l^∞ and let the sequences $(\psi_k^{(l)})$ respectively $(\phi_k^{(l)})$ converge to (ψ_k) respectively (ϕ_k) in an l^1 sense. Then for $l \rightarrow \infty$ $\|\mathbf{M}_{m, (\psi_k^{(l)}), (\phi_k)} - \mathbf{M}_{m, \psi_k, \phi_k}\|_{\text{Op}} \rightarrow 0$.*

Proof. (1) For a sequence of symbols this is a direct result of Theorem 6.1 and

$$\|\mathbf{M}_{m^{(l)},(\psi_k),(\phi_k)} - \mathbf{M}_{m,(\psi_k),(\phi_k)}\|_{\mathcal{HS}} = \|\mathbf{M}_{(m^{(l)}-m),(\psi_k),(\phi_k)}\|_{\mathcal{HS}} \leq \|m^{(l)} - m\|_2 \sqrt{BB'}.$$

The result for the operator and infinity norm respectively trace and l^1 norms can be proved in an analogue way.

(2) For points (b) and (c) we know from Corollary 4.5 that the sequences are Bessel sequences. For all the norms (Op, \mathcal{HS} , trace) $\|\psi_k \otimes_i \phi_k\| = \|\psi_k\|_{\mathcal{H}} \|\phi_k\|_{\mathcal{H}}$ and so

$$\begin{aligned} & \left\| \sum m_k \psi_k^{(l)} \otimes_i \phi_k - \sum m_k \psi_k \otimes_i \phi_k \right\| \\ &= \left\| \sum m_k (\psi_k^{(l)} - \psi_k) \otimes_i \phi_k \right\| \leq \sum_k |m_k| \|\psi_k^{(l)} - \psi_k\|_{\mathcal{H}} \sqrt{B'} = (*) \end{aligned}$$

case (a): $(*) \leq \sqrt{B'} (\sum_k |m_k|) \sup_l \{\|\psi_k^{(l)} - \psi_k\|_{\mathcal{H}}\} \leq \sqrt{B'} \|m\|_1 \varepsilon$,

case (b): $(*) \leq \sqrt{B'} \sqrt{\sum_k |m_k|^2} \sqrt{\sum \|\psi_k^{(l)} - \psi_k\|_{\mathcal{H}}^2} \leq \sqrt{B'} \|m\|_2 \varepsilon$,

case (c): $(*) \leq \sqrt{B'} \|m\|_{\infty} \sum \|\psi_k^{(l)} - \psi_k\|_{\mathcal{H}} \leq \sqrt{B'} \|m\|_{\infty} \varepsilon$.

(3) Use a corresponding argumentation as in (2).

(4) For points (b) and (c) Corollary 4.3 states that $(\psi_k^{(l)})$ and $(\phi_k^{(l)})$ are Bessel sequences and there are common Bessel bounds \mathbf{B}_1 and \mathbf{B}_2 for $l \geq N_1$. So using the results above we get

$$\begin{aligned} & \|\mathbf{M}_{m^{(l)},(\psi_k^{(l)}),(\phi_k^{(l)})} - \mathbf{M}_{m,(\psi_k),(\phi_k)}\| \\ & \leq \|\mathbf{M}_{m^{(l)},(\psi_k^{(l)}),(\phi_k^{(l)})} - \mathbf{M}_{m,(\psi_k^{(l)}),(\phi_k^{(l)})}\| \\ & \quad + \|\mathbf{M}_{m,(\psi_k^{(l)}),(\phi_k^{(l)})} - \mathbf{M}_{m,(\psi_k),(\phi_k^{(l)})}\| + \|\mathbf{M}_{m,(\psi_k),(\phi_k^{(l)})} - \mathbf{M}_{m,(\psi_k),(\phi_k)}\| \\ & \leq \varepsilon \sqrt{\mathbf{B}\mathbf{B}'} + \|m\| \varepsilon \sqrt{\mathbf{B}'} + \|m\| \sqrt{B} \varepsilon = \varepsilon \cdot (\sqrt{\mathbf{B}\mathbf{B}'} + \|m\| (\sqrt{\mathbf{B}'} + \sqrt{B})) \end{aligned}$$

for l bigger than the maximum N needed for the convergence conditions. This is true for all pairs or norms (Op, ∞), (\mathcal{HS} , l^2) and (trace, l^1). \square

9. Perspectives

For the future many questions are still open. For example it seems very likely, that for symbols $m \in l^p$ the multiplier lies in the p -Schatten operator class. Connected to that an investigation of the singular values of these operators might be worthwhile. The combination of two multipliers are connected to the Gram matrix, see Corollary 5.3. It will be interesting to apply the results for the decay properties of the Gram matrix in [9] to this topic. An interesting step away from the unstructured frame will be to investigate frame multipliers for structured frames, found e.g. in [7]. The topic of frame multipliers is closely related to the notion of *weighted frames* as introduced in [2]. It can be easily proved that for a positive, semi-normalized symbol the multiplier corresponds to the frame operator of a weighted frame. This connection should be investigated further and the theory of frame multiplier should be applied to the further context of the paper [2], computational issues for wavelets on the sphere.

Applications of these objects already exist. It seems that acoustics is a very interesting field for that. Frame multipliers there are not only used as (irregular or regular) Gabor multipliers like in [1] or [13], but also as a multipliers for a gammatone filter bank in [15]. In the engineering

literature frame multipliers for regular Gabor frames are known as *Gabor filters* [14]. Also first ideas are investigated to use this concept with wavelets to apply in the context of evaluations of noise barriers. The importance of the theoretical results in these and other application should be investigated further.

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