JOURNAL OF DIFFERENTIAL EQUATIONS 71, 288–314 (1988)

Stochastic Oscillators

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Received November 4, 1986

1. STOCHASTIC OSCILLATORS: STATEMENT OF PROBLEM AND RESULTS

A stochastic oscillator is described mathematically as the solution of an appropriate ordinary differential equation, which is driven by an external disturbance of white noise. Accordingly, such solutions are stochastic processes. We investigate these stochastic oscillations, and the statistical distribution of their zeros, particularly, the first zero of the oscillation.

Our studies deal with the scalar *stochastic oscillator* (the differential equation and its solution $x(t)$):

$$\ddot{x} + k(x, \dot{x}, t) = h\dot{w}(t) \qquad \text{on} \quad t \geqslant 0,$$

where $k(x, y, t)$ is a suitably smooth and bounded real function, and $h > 0$ is a positive parameter. As usual, $\dot{w}(t)$ represents white noise; that is, $w(t)$ on $0 \leqslant t < \infty$ is a Brownian motion (or Wiener process).

The corresponding first-order differential system in the (x, y)-plane $\mathbb{R}^2$ is

$$\begin{aligned} dx &= y\,dt \\ dy &= -k(x, y, t)\,dt + h\,dw \qquad \text{on} \quad t \geqslant 0. \end{aligned}$$

From each initial point $x(0) = x_0$, $y(0) = y_0$ in $\mathbb{R}^2$ there exists a unique solution $x(t)$, $y(t)$ on $0 \leqslant t < \infty$, defined as a stochastic process, according to the theory of Itô integration. In particular, the real stochastic function $x(t)$ is a stochastic oscillator (although $x(t)$ is not generally a martingale or

* The first-mentioned author acknowledges research support from NSF Grant DMS 84-13129 and the SERC, and the second author from NSF Grant DMS 86-03437.

0022-0396/88 $3.00

a Markov process), and we shall be interested in the zeros of the C^1 function $x(t)$ on $0 < t < \infty$.

The linear stochastic oscillator

$$\ddot{x} + kx = h\dot{w}(t) \qquad \text{on} \quad t \geqslant 0,$$

for positive constants k and h, will be of primary interest for our investigations. However, in Section 2 we demonstrate how the nonlinear stochastic oscillator can be reduced to the linear case, at least for the analysis of properties holding almost surely (a.s.), for instance, the property that all zeros of $x(t)$ are simple.

In Section 3 we demonstrate that the linear stochastic oscillator $x(t)$ has infinitely many zeros (a.s.). Furthermore we obtain explicit upper and lower estimates for the expected values of these zeros, with emphasis on the first positive zero of $x(t)$. In Section 4 we offer some comments and conjectures concerning the stochastic winding angle around the origin in the phase-plane.

2. Reduction of Scalar Nonlinear Stochastic Oscillator to the Linear Oscillator $\ddot{x} + x = h\dot{w}(t)$ on $t \geqslant 0$

Consider a real scalar stochastic oscillator

$$\dot{x} + k(x, \dot{x}, t) = h\dot{w}(t) \qquad \text{on} \quad t \geqslant 0.$$

Here $k(x, y, t)$ is a real function on the domain $(x, y) \in \mathbb{R}^2$ and $t \geqslant 0$, where we assume

(i) $k(x, y, t)$, $(\partial k/\partial x)(x, y, t)$, $(\partial k/\partial y)(x, y, t)$ are continuous, and

(ii) $|k(x, y, t)| \leqslant \gamma(t)(1 + |x| + |y|)$ for some continuous bound $\gamma(t)$ on $0 \leqslant t < \infty$.

The parameter $h >$ is a positive constant, and the stochastic perturbation is scalar white noise $\dot{w}(t)$.

The corresponding stochastic differential system is

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} y \\ -k(x, y, t) \end{bmatrix} dt + \begin{bmatrix} 0 & 0 \\ 0 & h \end{bmatrix} \begin{bmatrix} dw_1 \\ dw_2 \end{bmatrix},$$

where $w_1(t)$ and $w_2(t) \equiv w(t)$ are independent Brownian motions on $0 \leqslant t < \infty$; that is, $w(t)$ on $t \geqslant 0$ satisfies the usual axioms: $w(0) = 0$, and independent increments $[w(t) - w(s)]$, normally distributed with mean zero, variance $|t - s|$. In full notation we designate the stochastic process $w(t)$ defined on the Wiener probability space $(\Omega, \mathscr{F}, P)$ where

$\Omega = C_0[0, \infty)$ and $\mathscr{F}$ is the family of σ-fields adapted to $w(t)$; that is, the sample points of $C_0[0, \infty)$ are real continuous functions $w(t)$ on $0 \leqslant t < \infty$ with $w(0) = 0$, and such a typical sample is denoted by $w(\cdot)$ or $\omega = w(\cdot)$; see [4]. Of course, the 2-vector Wiener process $\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]$ is defined on $\Omega_2 = \Omega \times \Omega$ so the components are stochastically independent.

Under these circumstances there exists a unique solution (in the sense of Itô, and without regard to the particular model of Brownian motion) $(x(t), y(t))$ on $0 \leqslant t < \infty$, through a prescribed initial point $x(0) = x_0$, $y(0) = y_0$ in the plane $\mathbb{R}^2$. Of course, $(x(t), y(t))$ is a stochastic process, in fact $x(t)$ is a stochastic process on Ω (with probability induced by the Wiener measure on the sample functions $w(t)$), and the sample functions $x(t)$ almost surely lie in class C^1 on $0 \leqslant t < \infty$.

We shall show that this curve $(x(t), y(t))$ in $\mathbb{R}^2$ is also a solution of the linear stochastic differential equation

$$\ddot{x} + x = h\dot{\hat{w}}(t) \qquad \text{on} \quad t \geqslant 0,$$

or the Itô differential system

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix} dt + \begin{bmatrix} 0 & 0 \\ 0 & h \end{bmatrix} \begin{bmatrix} d\hat{w}_1 \\ d\hat{w}_2 \end{bmatrix},$$

but for a different choice of the sample function $\hat{w}_2(t) \equiv w(t)$ from the Wiener probability space $\Omega = C_0[0, \infty)$. In fact, using techniques of Cameron–Martin–Girsanov [4], we shall show that all such scalar stochastic oscillators have precisely the same (i.e., stochastically equivalent) solution curves $(x(t), y(t))$ in $\mathbb{R}^2$, as for the trivial oscillator:

$$\ddot{x} = h\dot{B}(t) \qquad \text{on} \quad t \geqslant 0$$

or the first-order system

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix} dt + \begin{bmatrix} 0 & 0 \\ 0 & h \end{bmatrix} \begin{bmatrix} dB_1 \\ dB_2 \end{bmatrix},$$

where $B_1(t)$ and $B_2(t) \equiv B(t)$ are independent Brownian motions on the product space $\Omega_2 = \Omega \times \Omega$, as usual. The main conclusion is that properties of the solutions, that hold almost surely, are not influenced by the particular choice of the coefficient $k(x, \dot{x}, t)$. In order to apply the required transformation of Cameron–Martin–Girsanov we need a certain a priori estimate for the expectation (over Ω) of exponentials of $B(t)$ and $\int_0^t B(s)\, ds$.

LEMMA. *Let $k(x, y, t)$ be a real function continuous for all $(x, y) \in \mathbb{R}^2$ and $0 \leqslant t < \infty$, and assume*

$$|k(x, y, t)| \leqslant \gamma(t)(1 + |x| + |y|) \qquad \textit{for continuous } \gamma(t).$$

Let

$$x(t)=x_0+y_0t+h\int_0^t B(s)\,ds, \qquad y(t)=y_0+hB(t),$$

for fixed $(x_0, y_0)\in\mathbb{R}^2$ *and* $0\leqslant t<\infty$.

Define $\varphi(t)=k(x(t), y(t), t)$, *and for the indicated exponential we assert the following bound: for each positive* $T>0$ *there exist positive constants* $\mu=\mu(T)$ *and* $C=C(T)$ *such that the expectation*

$$\mathbb{E}\exp[\mu|\varphi(t)|^2]\leqslant C \qquad on \quad 0\leqslant\ \leqslant t\leqslant T.$$

This lemma follows from a straightforward calculation using the customary inequalities of Schwarz and Jenson [7].

THEOREM 1. *Let* $(x(t), y(t))$ *on* $0\leqslant t<\infty$ *be the unique solution, initiating at* $x(0)=x_0$, $y(0)=y_0$ *in* $\mathbb{R}^2$, *for the trivial stochastic oscillator*:

$$\begin{bmatrix} dx\\ dy\end{bmatrix}=\begin{bmatrix} y\\ 0\end{bmatrix}dt+\begin{bmatrix} 0 & 0\\ 0 & h\end{bmatrix}\begin{bmatrix} dB_1\\ dB_2\end{bmatrix} \qquad on \quad t\geqslant 0.$$

Then $(x(t), y(t))$ *is also the unique solution, initiating at* $(x_0, y_0)\in\mathbb{R}^2$, *for the nonlinear stochastic oscillator*:

$$\begin{bmatrix} dx\\ dy\end{bmatrix}=\begin{bmatrix} y\\ -k(x, y, t)\end{bmatrix}dt+\begin{bmatrix} 0 & 0\\ 0 & h\end{bmatrix}\begin{bmatrix} dw_1\\ dw_2\end{bmatrix} \qquad on \quad t\geqslant 0.$$

Here $\mathbf{B}=(B_1, B_2)$ *and* $\mathbf{w}=(w_1, w_2)$ *are each Brownian motions on* $\Omega\times\Omega$, *with respect to two probability measures* $\tilde{P}$ *and* P. *Moreover for each finite duration* $0\leqslant t\leqslant T$, *and* σ-*field correspondingly restricted,* $\tilde{P}$ *and* P *have the same null sets.*

Remark. The conclusion of the theorem implies that a subset of sample curves of the stochastic process $(x(t), y(t))$ on $0\leqslant t\leqslant T$ has $\tilde{P}$-probability zero, with reference to the trivial oscillator

$$\ddot{x}=h\dot{B}(t)$$

if and only if it also has P-probability zero, with reference to the nonlinear oscillator

$$\ddot{x}+k(x, \dot{x}, t)=h\dot{w}(t).$$

Here the function $k(x, y, t)$ is assumed to satisfy the standing hypotheses (i) and (ii) listed at the beginning of this section, and $h>0$ is a constant.

Proof of Theorem 1. Write $\xi(t)=\left[\begin{smallmatrix} x(t) \\ y(t) \end{smallmatrix}\right]$ as the solution of the Itô integral equation

$$\xi(t)=\xi(0)+\int_0^t b(s)\,ds+\int_0^t \sigma(s)\,d\mathbf{B}$$

where $b(s)=\left[\begin{smallmatrix} y(s) \\ 0 \end{smallmatrix}\right]$ and $\sigma(s)=\left[\begin{smallmatrix} 0 & 0 \\ 0 & h \end{smallmatrix}\right]$; that is, the solution of the trivial stochastic oscillator is

$$x(t)=x_0+y_0 t+h\int_0^t B(s)\,ds, \qquad y(t)=y_0+hB(t)$$

so

$$b(s)=\begin{bmatrix} y_0+hB(s) \\ 0 \end{bmatrix}$$

where we write $\mathbf{B}=(B_1, B_2)$ and define $B(s)\equiv B_2(s)$, as before.

The proof of the theorem consists in verifying the hypotheses of the Cameron–Martin–Girsanov theorem [4], and then interpreting its conclusions for the current Theorem 1.

We define a bijection of $\Omega\times\Omega=C_0[0,\infty)\times C_0[0,\infty)$ onto itself by the map

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\to\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}=\begin{bmatrix} B_1(t) \\ B_2(t)-\int_0^t \varphi(s)\,ds \end{bmatrix}$$

where $\varphi(t)=(-1/h)\,k(x(t), y(t), t)$; that is,

$$\varphi(t)=\frac{-1}{h}\,k\left(x_0+y_0 t+h\int_0^t B(s)\,ds,\, y_0+hB(t),\, t\right).$$

The sufficient condition that guarantees that $\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]$ is again a Brownian motion on $\Omega\times\Omega$, relative to the same family of σ-fields, but with respect to a probability measure, possibly different, but with the same null sets (non-anticipating $T<\infty$) is

> for $T>0$ there exist positive constants $\mu=\mu(T)$ and $C=C(T)$ for which $\mathbb{E}\exp[\mu|\varphi(t)|^2]\leqslant C$ on $0\leqslant t\leqslant T$.

Thus the lemma assures us that

$$B(t)\to w(t)=B(t)-\int_0^t \varphi(s)\,ds$$

(and trivially $B_1 \to w_1 \equiv B_1$), defines the required bijection on $\Omega \times \Omega$, and $\mathbf{w} = \left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]$ is also a Brownian motion. In fact, $w(t) \equiv w_2(t)$ is a scalar Brownian motion on Ω.

Then the Cameron–Martin–Girsanov theorem asserts that $\xi(t) = \left[\begin{smallmatrix} x(t) \\ y(t) \end{smallmatrix}\right]$ also satisfies the Itô integral equation:

$$\xi(t) = \xi(0) + \int_0^t \tilde{b}(s)\, ds + \int_0^t \sigma(s)\, d\mathbf{w}$$

where

$$\tilde{b}(s) = b(s) + \sigma(s) \begin{bmatrix} 0 \\ \varphi(s) \end{bmatrix} = \begin{bmatrix} y(s) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ h\varphi(s) \end{bmatrix}$$

so

$$\tilde{b}(s) = \begin{bmatrix} y(s) \\ -k(x(s), y(s), s) \end{bmatrix}.$$

This means that $\xi(t)$ is the unique solution, initiating at $\xi(0) = \left[\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right]$, of the stochastic differential equation

$$\begin{aligned} dx &= y\, dt \\ dy &= -k(x, y, t)\, dt + h\, dw \qquad \text{on} \quad t \geqslant 0. \end{aligned}$$

Thus the theorem is proved. ▮

In order to indicate the force of the reduction theorem we apply these methods to the study of the simplicity of the zeros of a stochastic oscillator

$$\ddot{x} + k(x, \dot{x}, t) = h\dot{w}(t) \qquad \text{on} \quad t \geqslant 0;$$

that is, we show that, almost surely,

$$x(t)^2 + \dot{x}(t)^2 > 0 \qquad \text{for all} \quad t > 0.$$

By Theorem 1 it is sufficient to demonstrate the result for the trivial oscillator

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix} dt + \begin{bmatrix} 0 & 0 \\ 0 & h \end{bmatrix} \begin{bmatrix} dB_1 \\ dB_2 \end{bmatrix},$$

with solution

$$x(t) = x_0 + y_0 t + h \int_0^t B(s)\, ds, \qquad y(t) = y_0 + hB(t).$$

In order to measure the deviation of $(x(t), y(t))$ from the origin $(0, 0)$ of $\mathbb{R}^2$, we utilize a positive real function $V(x, y)$ with a pole at the origin; that is, $1/V$ plays a role like that of a polar distance.

LEMMA. *There exists a real function $V(x, y)$ in class C^∞ on its domain $\mathbb{R}^2 - (0, 0)$ where it satisfies*

(1) $V(x, y) > 0$,

(2) $\lim_{|x| + |y| \to 0} V(x, y) = +\infty$,

and along the stochastic process

$$x(t) = x_0 + y_0 t + h \int_0^t B(s)\, ds, \; y(t) = y_0 + hB(t) \qquad \textit{for} \quad t > 0,$$

the Itô differential of $V(x(t), y(t))$ reduces to

$$dV(x(t), y(t)) = h \frac{\partial V}{\partial y} dB(t) \qquad \textit{when} \quad x(t)^2 + y(t)^2 > 0.$$

Hence, the differential operator of the stochastic process, namely,

$$L = y \frac{\partial}{\partial x} + \frac{h^2}{2} \frac{\partial^2}{\partial y^2}$$

annihilates $V(x, y)$; in other words,

(3) $LV(x, y) \equiv 0$ *when* $(x, y) \neq (0, 0)$ *in* $\mathbb{R}^2$.

Proof. The proof consists in displaying an explicit formula for $V(x, y)$, namely,

$$V(x, y) = \int_0^\infty \frac{\sqrt{3}}{\pi t^2} \exp\left(\frac{-2}{h^2 t^3} [t^2 y^2 + 3txy + 3x^2]\right) dt.$$

We omit the routine calculations verifying the conditions in the lemma; see [7].

Remark. It is of interest to indicate our motivation for seeking $V(x, y)$ in the form

$$V(x, y) = \int_0^\infty p(t, x, y)\, dt,$$

and for defining

$$p(t, x, t) = \frac{\sqrt{3}}{\pi t^2} \exp\left(\frac{-2}{h^2 t^3} [t^2 y^2 + 3txy + 3x^2]\right).$$

We start with the Gaussian process

$$\begin{bmatrix} -\int_0^t B(s)\,ds \\ B(t) \end{bmatrix},$$

which can be considered as the solution of the trivial oscillator initiating at the origin, but with the time t reversed to $-t$. Here $B(t)$ is again scalar Brownian motion, and we have set $h = 1$ for clarity of exposition in this remark.

This process has the covariance matrix

$$Q(t) = \begin{bmatrix} t^3/3 & -t^2/2 \\ -t^2/2 & t \end{bmatrix}$$

and the generator

$$L^* = -y\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial^2}{\partial y^2},$$

which is the formal adjoint of

$$L = y\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial^2}{\partial y^2}.$$

Moreover the appropriate probability transition density is

$$\mathbf{P}(t, \xi, \zeta) = \frac{\sqrt{3}}{\pi t^2}\exp\left(\frac{-1}{2}\langle(\xi - \zeta), Q^{-1}(t)(\xi - \zeta)\rangle\right)$$

where $\xi = (x, y)$ and $\zeta = (u, v)$ are points in $\mathbb{R}^2$.

In this situation we consider

$$p(t, w, y) \equiv \mathbf{p}(t. \xi, \mathbf{0}) = \frac{\sqrt{3}}{\pi t^2}\exp\left(\frac{-2}{t^3}[t^2 y + 3txy + 3x^2]\right)$$

which satisfies the forward parabolic equation for L^*, and hence we are led to our desired result:

$$Lp(t, w, y) = \frac{\partial p}{\partial t}, \qquad \text{and} \qquad LV = \int_0^\infty \frac{\partial p}{\partial t}\,dt = p\Big]_0^\infty = 0.$$

We are now able to present the theorem concerning the simplicity of the zeros for the stochastic oscillator

$$\ddot{x} + k(x, \dot{x}, t) = h\dot{w}(t) \qquad \text{on} \quad t \geqslant 0,$$

where $k(x, y, t)$ satisfies the standing hypothesis listed at the beginning of this section, and the constant parameter $h > 0$.

THEOREM 2. *Let* $(x(t), y(t))$ *on* $0 \leqslant t < \infty$ *be the unique solution, initiating at* $(x_0, y_0) \in \mathbb{R}^2 - (0, 0)$, *for the nonlinear stochastic oscillator*

$$\ddot{x} + k(x, \dot{x}, t) = h\dot{w}(t) \qquad \text{on} \quad t \geqslant 0,$$

or the Itô differential system for $(x, y) \in \mathbb{R}^2$:

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} y \\ -k(x, y, t) \end{bmatrix} dt + \begin{bmatrix} 0 & 0 \\ 0 & h \end{bmatrix} \begin{bmatrix} dw_1 \\ dw_2 \end{bmatrix},$$

as earlier. Then almost surely

$$x(t)^2 + y(t)^2 > 0 \qquad \textit{for all} \quad 0 \leqslant t < \infty.$$

Proof. By use of the reduction Theorem 1, we shall show that we need only consider the trivial stochastic oscillator

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix} dt + \begin{bmatrix} 0 & 0 \\ 0 & h \end{bmatrix} \begin{bmatrix} dB_1 \\ dB_2 \end{bmatrix}$$

or $\ddot{x} = h\dot{B}(t)$ on $t \geqslant 0$. Using familiar estimates on $B(t)$ and $\int_0^t B(s)\, ds$, we investigate the hitting times for $(x(t), y(t))$ into small disks centered at the origin. In this way we prove (see [7]) that with probability one (for the appropriate B-measures),

$$x(t) = x_0 + y_0 t + h \int_0^t B(s)\, ds \qquad \text{and} \qquad y(t) = y_0 + hB(t)$$

do not vanish simultaneously on any finite interval $0 \leqslant t \leqslant T$.

Take the function

$$V(x, y) = \int_0^\infty p(t, x, y)\, dt$$

with

$$p(t, x, y) = \frac{\sqrt{3}}{\pi t^2} \exp\left(\frac{-2}{h^2 t^3} [t^2 y^2 + 3txy + 3x^2]\right),$$

as in the lemma.

The paths $\xi(t) = \left[\begin{smallmatrix} x(t) \\ y(t) \end{smallmatrix}\right]$ are each continuous so, since $x_0^2 + y_0^2 > 0$, there is a first time $\tau > 0$ when each sample path hits the origin, unless $\tau = +\infty$.

Thus $\tau(\omega)$ is a random variable defined on the Wiener probability space $(\Omega, \mathscr{F}, P)$. We shall show that the random variable τ satisfies $\mathbb{P}[\tau < T] = 0$, for each preassigned positive T, and thus conclude that $\tau \equiv +\infty$, a.s., that is, the Wiener probability $\mathbb{P}[\tau = +\infty] = 1$.

Certainly $V(x(t), y(t)) < \infty$ on $0 \leqslant t < \tau$. Define the random time τ_n as the first time when $|\xi(t_n)| = 1/n$, for each $n = 1, 2, 3, \ldots$. Clearly $\tau_n < \tau$ (except when both are $+\infty$) and $\tau_n < \tau_{n+1}$ with $\lim_{n \to \infty} \tau_n = \tau$. From Itô's formula it follows directly that

$$\mathbb{E}[V(\xi(T \wedge \tau_n))] = V(\xi_0) < \infty.$$

Let $n \to \infty$ and use Fatou's lemma to obtain

$$\mathbb{E}[V(\xi(T \wedge \tau))] \leqslant V(\xi_0) < \infty.$$

But integrating over the Wiener space $(\Omega, \mathscr{F}, P)$, we find

$$\mathbb{E}[V(\xi(T \wedge \tau)] = \int_{[\tau \geqslant T]} V(\xi(T))\, dP + \int_{[\tau < T]} V(\xi(\tau))\, dP$$

so

$$\int_{[\tau < T]} V(\xi(\tau))\, dP \leqslant \mathbb{E}[V(\xi(T \wedge \tau))] \leqslant V(\xi_0) < \infty.$$

However, $V(\xi(\tau)) = +\infty$ and so the Wiener probability measure of the set $[\tau < T]$ must be zero. Therefore the Wiener probability $\mathbb{P}[\tau < T] = 0$, for each given $T > 0$. Thus we conclude that $\tau \equiv +\infty$, almost surely.

Now refer to the reduction of Theorem 1 for a prescribed finite $T > 0$. Almost surely (relative to the probability measure appropriate for the Brownian motion $w(t)$) we have

$$x(t)^2 + y(t)^2 > 0 \qquad \text{on} \quad 0 \leqslant t \leqslant T.$$

But $T > 0$ is arbitrary, and furthermore the algebra of null sets (for the duration $0 \leqslant t \leqslant T$) increases with T. We therefore conclude that, with w-probability of one

$$x(t)^2 + y(t)^2 > 0 \qquad \text{for all} \quad 0 \leqslant t < \infty. \quad \blacksquare$$

3. The Scalar Linear Stochastic Oscillator

Consider the scalar linear oscillator

$$\ddot{x} + kx = h\dot{w}(t) \qquad \text{on} \quad t \geqslant 0,$$

where k and h are positive constants, and $w(t)$ is scalar Brownian motion or Wiener process on the probability space $(\Omega = C_0[0, \infty), \mathscr{F}, P)$, as usual. The Itô stochastic differential equation

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} y \\ -kx \end{bmatrix} dt + \begin{bmatrix} 0 & 0 \\ 0 & h \end{bmatrix} \begin{bmatrix} dw_1 \\ dw_2 \end{bmatrix},$$

with $w_1(t)$, $w_2(t) \equiv w(t)$ independent scalar Brownian motions, has a unique solution, from $(x_0, y_0) \in \mathbb{R}^2$,

$$x(t) = x_0 \cos \sqrt{k}\, t + \frac{y_0}{\sqrt{k}} \sin \sqrt{k}\, t + \frac{h}{\sqrt{k}} \int_0^t \sin \sqrt{k}\, (t-s)\, dw$$

$$y(t) = -\sqrt{k}\, x_0 \sin \sqrt{k}\, t + y_0 \cos \sqrt{k}\, t + h \int_0^t \cos \sqrt{k}\, (t-s)\, dw.$$

After integration by parts (via Itô calculus), this solution can be written

$$x(t) = x_0 \cos \sqrt{k}\, t + \frac{y_0}{\sqrt{k}} \sin \sqrt{k}\, t + h \int_0^t w(s) \cos \sqrt{k}\, (t-s)\, ds$$

$$y(t) = -\sqrt{k}\, x_0 \sin \sqrt{k}\, t + y_0 \sin \sqrt{k}\, t + hw(t) - \sqrt{k}\, h \int_0^t w(s) \sin \sqrt{k}\, (t-s)\, ds.$$

For simplicity of treatment we shall consider primarily the case $k = 1$, $x_0 = 1$, $y_0 = 0$, that is, the stochastic oscillator

$$\ddot{x} + x = h\dot{w}(t) \qquad \text{on} \quad t \geqslant 0$$

or

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix} dt + \begin{bmatrix} 0 & 0 \\ 0 & h \end{bmatrix} \begin{bmatrix} dw_1 \\ dw_2 \end{bmatrix}.$$

We shall be interested in the zeros of the C^1-stochastic process $x(t)$ on $t \geqslant 0$.

Remark 1. Of course the linear stochastic oscillator

$$\ddot{x} + kx = h\dot{w}(t) \qquad \text{on} \quad t \geqslant 0$$

is a special case of the nonlinear stochastic oscillator

$$\ddot{x} + k(x, \dot{x}, t) = h\dot{w}(t) \qquad \text{on} \quad t \geqslant 0,$$

as discussed in the previous section. Accordingly, many of these general results are valid, for instance, the unique solution $(x(t), y(t))$ on $t \geqslant 0$, from $(x_0, y_0) \neq (0, 0)$, almost surely misses the origin for all $0 \leqslant t < \infty$ (see Theorem 2).

However, certain probabilistic results hold only with some positive probability depending on the parameters k and h, and are particular to the theory of linear stochastic oscillators.

Remark 2. Consider the linear stochastic oscillator

$$dx = y\,dt$$
$$dy = -kx\,dt + h\,dw \qquad \text{on} \quad t \geqslant 0,$$

with initial data $x(0) = x_0 > 0$ and $\dot{x}(0) = y_0$. By appropriate changes of scale we can reduce this study to the case $k = 1$, $x_0 = 1$, and for simplicity we also assume $y_0 = 0$.

THEOREM 3. *Consider the scalar stochastic process $x(t)$ satisfying the linear stochastic oscillator*:

$$\ddot{x} + x = h\dot{w}(t) \qquad \textit{on} \quad t \geqslant 0$$

from $x(0) = 1$, $\dot{x}(0) = 0$, with parameter $h > 0$. Then, almost surely, $x(t)$ has infinitely many zeros, all simple, on each half line $[t_0 < t < \infty)$. Moreover, the expectation for the first zero $\hat{T}$ satisfies

$$\mathbb{E}(\hat{T}) \geqslant 2(\operatorname{arc\,cot} h)[\operatorname{Erf}((\operatorname{arc\,cot} h)^{-1/2})].$$

Proof. Note. In the next theorem we show that $\mathbb{E}(\hat{T}) < \infty$ and find explicit upper bounds for the first two moments of $\hat{T}$. In Theorem 3 we first establish a positive lower estimate for $\mathbb{E}(\hat{T})$, and thereafter analyse the oscillatory behavior of $x(t)$.

The stochastic process $x(t)$ is defined as the solution of the differential equation

$$x(t) = \cos t + h \int_0^t \sin(t-s)\,dw\,(s)$$

or

$$x(t) = \cos t + h \int_0^t w(s) \cos(t-s)\,ds,$$

and hence $x(t) \in C^1[0, \infty)$, almost surely.

Consider the random sample function $w(t)$, from the Wiener space $\Omega = C_0[0, \infty)$ of real continuous functions, satisfying the constraint

$$w(t) > -\gamma \qquad \text{on} \quad 0 \leqslant t \leqslant T < \pi/2, \qquad \text{for fixed} \quad \gamma > 0.$$

Use the Kač–Erdös version of the central limit theorem [1] to obtain the familiar formula concerning the measure of such subsets of Wiener space $(\Omega, \mathscr{F}, P)$ (see [1, 4]),

$$\mathbb{P}[w(t) > -\gamma \qquad \text{on} \quad 0 \leqslant t \leqslant T] = 2\,\text{Erf}(\gamma T^{-1/2}),$$

where the error function is given by

$$\text{Erf}(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-u^2/2}\,du, \qquad \text{so} \quad \text{Erf}(\infty) = \tfrac{1}{2}.$$

Now set $\gamma = 1$ and $T = \text{arc}\cot(h)$, so for instance, $T = \pi/4$ when $h = 1$. Then for such sample functions $w(t) > -1$ on $0 \leqslant t \leqslant T$, and we have

$$\mathbb{P}[w(t) > -1 \quad \text{on} \quad 0 \leqslant t \leqslant \text{arc}\cot(h)] = 2\,\text{Erf}(T^{-1/2}).$$

Further, since $\cos(t-s) > 0$ on $0 \leqslant s \leqslant t < \pi/2$, we find that

$$x(t) > \cos t - h \int_0^t \cos(t-s)\,ds = \cos t - h \sin t > 0,$$

for $0 < t < \text{arc}\cot(h)$. Hence for these sample functions $w(t)$ we find

$$\hat{T} \geqslant \text{arc}\cot(h),$$

and therefore

$$\mathbb{P}[\hat{T} \geqslant \text{arc}\cot(h)] \geqslant 2\,\text{Erf}((\text{arc}\cot h)^{-1/2}).$$

This result yields a lower estimate for the expected value of $\hat{T}$,

$$\mathbb{E}(\hat{T}) \geqslant (\text{arc}\cot h)[2\,\text{Erf}((\text{arc}\cot h)^{-1/2})],$$

as asserted in the theorem.

We next proceed to demonstrate the oscillatory behavior of the stochastic solution

$$x(t) = \cos t - h \cos t \int_0^t \sin s\,dw(s) + h \sin t \int_0^t \cos s\,dw(s).$$

For this purpose we can introduce new Brownian motions $\tilde{w}_1(t)$ and $\tilde{w}_2(t)$ on $0 \leqslant t < \infty$, so that

$$x(t) = \cos t - h \cos t \, \tilde{w}_1(f(t)) + h \sin t \, \tilde{w}_2(g(t)).$$

Here the "changed time rates" are defined by [4]

$$f(t) = \tfrac{1}{2}(t - \tfrac{1}{2} \sin 2t), \qquad g(t) = \tfrac{1}{2}(t + \tfrac{1}{2} \sin 2t).$$

Now consider $x(t)$ at the discrete instants $t = (2m + \frac{1}{2})\,\pi$, for $m = 1, 2, 3, \ldots$, when

$$f((2m + \tfrac{1}{2})\,\pi) = (m + \tfrac{1}{4})\,\pi, \qquad g((2m + \tfrac{1}{2})\,\pi) = (m + \tfrac{1}{4})\,\pi,$$

and when

$$x((2m + \tfrac{1}{2})\,\pi) = h \, \hat{w}_2((m + \tfrac{1}{4})\,\pi).$$

Next define a sequence of independent normal random variables $\{Y_0, Y_1, Y_2, Y_3, \ldots\}$, as

$$\begin{aligned}
Y_0 &= \tilde{w}_2(\pi/4) \\
Y_1 &= \tilde{w}_2((1 + \tfrac{1}{4})\,\pi) - \tilde{w}_2(\pi/4) \\
Y_2 &= \tilde{w}_2((2 + \tfrac{1}{4})\,\pi) - \tilde{w}_2((1 + \tfrac{1}{4})\,\pi) \\
&\;\vdots \\
Y_m &= \tilde{w}_2((m + \tfrac{1}{4})\,\pi) - \tilde{w}_2((m - 1 + \tfrac{1}{4})\,\pi), \qquad \text{etc.}
\end{aligned}$$

Because of the independence of the increments in the Wiener process on disjoint intervals, the sequence $\{Y_m\}$ consists of pairwise independent Gaussian random variables, each with mean zero and variance π, for $m = 1, 2, 3, \ldots$.

This construction was arranged so that the partial sums of the sequence $\{Y_m\}$ are

$$Y_0 + Y_1 + Y_2 + \cdots + Y_m = (1/h)\, x((2m + \tfrac{1}{2})\,\pi) \qquad \text{for} \quad m = 1, 2, 3, \ldots.$$

Familiar theorems on the limits of sums of independent random variables (e.g., law of the iterated logarithm) show that, almost surely, the terms of the sequence $\{x((2m + \frac{1}{2})\,\pi)\}$ have infinitely many switches of sign as $m \to \infty$. Moreover, since the solution curves $x(t)$ are each continuous on $0 \leqslant t < \infty$ (almost surely), then each $x(t)$ must have infinitely many zeros on each right half-line $[t_0 < t < \infty)$.

The simplicity of the zeros of $x(t)$ has already been proved in Theorem 2. ∎

It may be of interest to examine this lower estimate for $\mathbb{E}(\hat{T})$ for particular values of $h > 0$. For example,

$$\mathbb{E}(\hat{T}) \geqslant \frac{\pi}{2} \operatorname{Erf}\left(\left(\frac{\pi}{4}\right)^{-1/2}\right) > 0.57 \quad \text{for} \quad h = 1,$$

and

$$\mathbb{E}(\hat{T}) \geqslant 0.42 \quad \text{for} \quad 0 < h \leqslant 1.$$

But our formula shows only $\mathbb{E}(\hat{T}) \geqslant 0.89$ as $h \to 0$, rather than the correct deterministic value $\hat{T}_d = \pi/2 \doteq 1.57$.

We now demonstrate that the probability distribution

$$\mathbb{P}[\hat{T} > T] \qquad \text{for each positive } T,$$

has finite moments, and we estimate the first two moments $\mathbb{E}(\hat{T})$ and $\mathbb{E}_2(\hat{T})$ from above.

THEOREM 4. *Consider the scalar stochastic process $x(t)$ satisfying the linear stochastic oscillator:*

$$\ddot{x} + x = h\dot{w}(t) \qquad \text{for} \quad t \geqslant 0.$$

from $x(0) = 1$, $\dot{x}(0) = 0$, with parameter $h > 0$. Let $\hat{T} > 0$ be the first zero of $x(t)$ on $0 \leqslant t < \infty$.

Then the probability distribution of the random variable $\hat{T}$ satisfies

$$\mathbb{P}[\hat{T} > T] < 4\, c(h) \exp\left(\frac{-\ln 2}{\pi} T\right), \qquad \text{for each} \quad T > \pi,$$

where the constant $c(h) = \frac{1}{2} - \operatorname{Erf}[(1/h)\sqrt{2/\pi}]$, so $\lim_{h \to 0} c(h) = 0$. In consequence,

$$\mathbb{P}[\hat{T} < \infty] = 1,$$

and every moment of $\hat{T}$ is finite, with the first two moments having the upper bounds:

$$\mathbb{E}(\hat{T}) < \pi(1 + 6\, c(h)) \leqslant 4\pi$$

and

$$\mathbb{E}_2(\hat{T}) < \pi^2(1 + 22\, c(h)) \leqslant 12\pi^2.$$

Proof. The solution of the stochastic differential equation is

$$x(t) = \cos t + h \int_0^t w(s) \cos(t - s)\, ds,$$

and again, as in the previous theorem,

$$x(t) = \cos t - h \cos t\, \tilde{w}_1(f(t)) + h \sin t\, \tilde{w}_2(g(t))$$

where

$$f(t) = \tfrac{1}{2}\,(t - \tfrac{1}{2}\sin 2t), \qquad g(t) = \tfrac{1}{2}\,(t + \tfrac{1}{2}\sin 2t);$$

and $\tilde{w}_1(t)$ and $\tilde{w}_2(t)$ are Brownian motions on $0 \leqslant t < \infty$.

Evaluate $x(t)$ at the discrete instants $t = m\pi$ for $m = 1, 2, 3, \ldots$, to obtain

$$x(m\pi) = (\cos m\pi)[1 - h\, \tilde{w}_1(m\pi/2)].$$

Hence

$$x(m\pi) > 0$$

if and only if

$$\tilde{w}_1(m\pi/2) > 1/h \qquad \text{for} \quad m = 1, 3, 5, \ldots$$
$$\tilde{w}_2(m\pi/2) < 1/h \qquad \text{for} \quad m = 2, 4, 6, \ldots.$$

Using the fundamental formula for the Wiener measure of "an interval in Ω," we have

$$\mathbb{P}[\tilde{w}_1(\pi/2) > 1/h] = [\tfrac{1}{2} - \mathrm{Erf}(1/h)\sqrt{2/\pi}] \equiv c(h).$$

Since $\hat{T} > \pi$ implies that $x(\pi) > 0$, we then obtain an upper bound for this probability

$$\mathbb{P}[\hat{T} > \pi] < c(h).$$

Furthermore, since the probability that a random sample function $\tilde{w}_1(t)$ will decrease over the interval $\pi/2 \leqslant t \leqslant 2\pi/2$ is just $\tfrac{1}{2}$, we note that

$$\mathbb{P}[\tilde{w}_1(\pi/2) > 1/h \qquad \text{and} \qquad \tilde{w}_1(2\pi/2) < 1/h] < c(h) \cdot (\tfrac{1}{2}).$$

Thus

$$\mathbb{P}[\hat{T} > 2\pi] < c(h) \cdot (\tfrac{1}{2}).$$

Continuing this line of argument, we find

$$\mathbb{P}[\hat{T} > m\pi] < c(h) \cdot (1/2^{m-1}) \qquad \text{for} \quad m = 1, 2, 3, \ldots$$

or

$$\mathbb{P}[\hat{T} > m\pi] < c(h)\, 2e^{-m \ln 2}.$$

In order to estimate the probability that $\hat{T}$ exceeds an arbitrary positive number $T > \pi$, we let $[T/\pi]$ be the greatest integer not exceeding T/π, so $[T/\pi] \cdot \pi \leqslant T$. Then

$$\mathbb{P}[\hat{T} > T] \leqslant \mathbb{P}[T > [(T/\pi)\,\pi] < c(h)\, 2e^{-[T/\pi] \ln 2}.$$

and

$$\mathbb{P}[\hat{T} > T] < 2c(h)\, e^{-(\ln 2)(T/\pi - 1)} < 4c(h)\, e^{-(\ln 2)\, T/\pi}.$$

From this upper estimate on the probability distribution for the random variable $\hat{T}$ we easily conclude that

$$\lim \mathbb{P}[\hat{T} > T] = 0 \qquad \text{or} \qquad \mathbb{P}[\hat{T} < \infty] = 1.$$

Also, for each fixed $T > \pi$,

$$\lim_{h \to 0} \mathbb{P}[\hat{T} > T] = 0.$$

Incidentally, this is a rather weak result in view of the computation for the deterministic case $(h = 0)$ where $x(t) = \cos t$ has its first positive zero at $\pi/2$.

Since $\mathbb{P}[\hat{T} > T]$ satisfies a bound of exponential decay, we are assured that each moment of $\hat{T}$ is finite. We next indicate an elementary method for estimating these moments, and give details for the bounds for the expectation of $\hat{T}$ and of $(\hat{T})^2$; that is, we shall give upper bounds for $\mathbb{E}(\hat{T})$ and $\mathbb{E}_2(\hat{T}) = \mathbb{E}(\hat{T}^2)$.

Recall first that

$$\mathbb{P}[\hat{T} > m\pi] < 2c(h)\, 2^{-m} \qquad \text{for} \quad m = 1, 2, 3, \ldots.$$

Then a trivial estimate for the mean of $\hat{T}$ is

$$\mathbb{E}(\hat{T}) \leqslant \mathbb{P}[0 < T \leqslant \pi] \cdot \pi + \mathbb{P}[\pi < \hat{T} \leqslant 2\pi] \cdot 2\pi + \mathbb{P}[2\pi < \hat{T} \leqslant 3\pi] \cdot 3\pi + \cdots$$

so

$$\mathbb{E}(\hat{T}) < \pi \left[1 + 2c(h) \cdot \frac{2}{2} + 2c(h) \cdot \frac{3}{2^2} + \cdots \right]$$

and

$$\mathbb{E}(\hat{T}) < 4\pi c(h) \left[\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \cdots \right] + [\pi - 2\pi c(h)].$$

The sum of the infinite series is easily calculated:

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = q'(1) = 2 \qquad \text{where} \quad q(z) = \sum_{n=1}^{\infty} \frac{z^n}{2^n} = \frac{z}{2-z}.$$

Thus we have the required estimate

$$\mathbb{E}(\hat{T}) < 8\pi c(h) + \pi[1 - 2c(h)] = 6\pi c(h) + \pi.$$

Since $c(h) < \frac{1}{2}$, we have the uniform estimate

$$\mathbb{E}(\hat{T}) < 4\pi \qquad \text{for all} \quad h > 0.$$

In the same way we estimate the second moment of $\hat{T}$:

$$\mathbb{E}_2(\hat{T}) < \pi^2 \left[1 + 2c(h) \frac{2^2}{2} + c(h) \frac{3^2}{2^2} + \cdots \right]$$

so

$$\mathbb{E}_2(\hat{T}) < 4\pi^2 c(h) \left[\frac{1}{2} + \frac{2^2}{2^2} + \frac{3^2}{2^3} + \cdots \right] + [\pi^2 - 2\pi^2 c(h)].$$

Then, using the infinite sum

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = (zq'(z))' \,|_{z=1} = 6,$$

we have the estimate for the second moment

$$\mathbb{E}_2(\hat{T}) < \pi^2[1 + 22c(h)] \leqslant 12\pi^2. \quad \blacksquare$$

We designate the first zero of $x(t)$ by $\hat{T}_1 = \hat{T}$, and the subsequent zeros by $\hat{T}_2, \hat{T}_3, \hat{T}_4, \ldots$. Since the zeros of $x(t)$ are all simple (a.s.), these random variables $\hat{T}_1 < \hat{T}_2 < \hat{T}_3 < \cdots$ are all well defined. In the next corollary we improve our estimate for the expectation of $\hat{T}_1$, and then give similar estimates for the expectations of all the remaining zeros of $x(t)$.

COROLLARY. *Consider the scalar stochastic process $x(t)$ satisfying the linear stochastic oscillator*

$$\ddot{x} + x = h\dot{w}(t) \qquad \text{for} \quad t \geqslant 0,$$

with $x(0) = 1$, $\dot{x}(0) = 0$. Let $\hat{T}_1 < \hat{T}_2 < \hat{T}_3 < \cdots$ be the successive positive zeros of $x(t)$ on $0 < t < \infty$. Then the expectation for the lth zero satisfies

$$\mathbb{E}(\hat{T}_l) \leqslant 2l\pi \qquad \text{for each} \quad l = 1, 2, 3, \ldots.$$

Proof. The calculation in the prior theorem shows that

$$\mathbb{P}[\hat{T}_1 > m\pi] < c(h)/2^{m-1} \qquad \text{for} \quad m = 1, 2, 3, \ldots .$$

But a standard result for any non-negative random variable X asserts [1]

$$\sum_{m=1}^{\infty} \mathbb{P}[X \geqslant m] \leqslant \mathbb{E}(X) \leqslant 1 + \sum_{m=1}^{\infty} \mathbb{P}[X \geqslant m].$$

Let $X = \hat{T}_1/\pi$ to obtain

$$\frac{1}{\pi}\, \mathbb{E}(\hat{T}_1) \leqslant 1 + c(h)[1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots] = 1 + 2c(h),$$

so

$$\mathbb{E}(\hat{T}_1) \leqslant \pi[1 + 2c(h)] \leqslant 2\pi.$$

In order to consider the second zero $\hat{T}_2$ of $x(t)$, we shift the time scale from $t \geqslant 0$ to $s = t - \hat{T}_1$ (where $\hat{T}_1(\omega)$ is constant for each sample path). Then write $z(s) = x(s + \hat{T}_1)$ and note that

$$\frac{d^2 z}{ds^2} + z(s) = h\, \frac{d}{ds}\, B(s), \qquad \text{with} \quad z(0) = 0, \quad \frac{dz}{ds}\,(0) = \dot{x}(\hat{T}_1) \neq 0,$$

where $B(s) = w(s + \hat{T}_1(\omega)) - w(\hat{T}_1(\omega))$ is a new Brownian motion for $s \geqslant 0$. Moreover, the first positive zero $\hat{S}_1$ of $z(s)$ will yield the second zero $\hat{T}_2 = \hat{T}_1 + \hat{S}_1$ for $x(t)$.

In this spirit we proceed as before to write

$$z(s) = \dot{z}(0) \sin s + (h \cos s)\, \tilde{B}_1(f(s)) - (h \sin s)\, \tilde{B}_2(g(s))$$

where $\tilde{B}_1$ and $\tilde{B}_2$ are Brownian motions and

$$f(s) = \tfrac{1}{2}(s - \tfrac{1}{2} \sin 2s), \qquad g(s) = \tfrac{1}{2}(s + \tfrac{1}{2} \sin 2s).$$

We next examine $z(s)$ at the discrete times $s = \pi, 2\pi, \ldots, m\pi, \ldots$ to find

$$z(m\pi) = h(-1)^m\, \tilde{B}_1(m\pi/2)$$

and thus

$$z(m\pi) < 0$$

if and only if

$$\tilde{B}_1(m\pi/2) > 0 \qquad \text{for} \quad m = 1, 3, 5, \ldots$$
$$\tilde{B}_1(m\pi/2) < 0 \qquad \text{for} \quad m = 2, 4, 6, \ldots .$$

These results do not depend on the value for $\dot{z}(0) \neq 0$.

From this calculation we note that

$$z(\pi) < 0 \qquad \text{if and only if} \quad \tilde{B}_1(\pi/2) > 0$$

so

$$\mathbb{P}[z(s) < 0 \text{ on } 0 < s \leqslant m\pi] \leqslant \mathbb{P}[z(\pi) < 0] \leqslant \tfrac{1}{2}.$$

Continue as before to compute

$$\mathbb{P}[z(s) < 0 \text{ on } 0 < s \leqslant m\pi] \leqslant 1/2^m \qquad \text{for each} \quad m = 1, 2, 3, \ldots .$$

Therefore

$$\mathbb{P}[\hat{S}_1 > m\pi] \leqslant 1/2^m \qquad \text{for each} \quad m = 1, 2, 3, \ldots .$$

Now take the random variable $X_1 = \hat{S}_1/\pi$ to compute

$$\mathbb{E}(\hat{S}_1) \leqslant \pi(1 + \tfrac{1}{2} + \tfrac{1}{4} + \cdots) = 2\pi.$$

Therefore

$$\mathbb{E}(\hat{T}_2) = \mathbb{E}(\hat{T}_1) + \mathbb{E}(\hat{S}_1) \leqslant 4\pi,$$

and an elementary repetition of this argument yields the desire result

$$\mathbb{E}(\hat{T}_l) \leqslant 2l\pi \qquad \text{for each} \quad l = 1, 2, 3, \ldots . \quad \blacksquare$$

The next theorem produces a positive lower bound for the probability distribution of $\hat{T}$ $(= \hat{T}_1)$, and the proof involves more difficult calculations on Wiener measure, than for the previous results.

THEOREM 5. *Consider the scalar stochastic process $x(t)$ satisfying the linear stochastic oscillator*

$$\ddot{x} + x = h\dot{w}(t) \qquad \textit{for} \quad t \geqslant 0,$$

from $x(0) = 1$, $\dot{x}(0) = 0$, with parameter $h > 0$. Let $\hat{T} > 0$ be the first positive zero of $x(t)$ on $0 \leqslant t < \infty$.

Then the probability distribution of the random variable $\hat{T}$ satisfies

$$\mathbb{P}[\hat{T} > T] > Ae^{-BT^3}, \qquad \textit{for all suitably large} \quad T > 0,$$

for each fixed $h > 0$. *Further, for* $0 < h < 1$ *and* $T > 2$, *we have explicit estimates with* $A(h) = e^{-1/h^2}$, $B(h) = 2/h^2$.

Proof. The stochastic process $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ solving

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix} dt + \begin{bmatrix} 0 & 0 \\ 0 & h \end{bmatrix} \begin{bmatrix} dw_1 \\ dw_2 \end{bmatrix},$$

from $x(0) = 1$, $y(0) = 0$, can also be understood as the solution of

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix} dt + \begin{bmatrix} 0 & 0 \\ 0 & h \end{bmatrix} \begin{bmatrix} dB_1 \\ dB_2 \end{bmatrix};$$

that is, $x(t) = 1 + h \int_0^t B(s)\, ds$, $y(t) = hB(t)$. Moreover, in the first instance the probability measure on $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ is induced from the measure P on the Brownian motion $\mathbf{w}(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$ on $\Omega \times \Omega$, whereas in the second instance of the trivial oscillator the measure $\tilde{P}$ on $\Omega \times \Omega$ is appropriate for the Brownian motion $\mathbf{B}(t) = \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix}$. The bijection of $\Omega \times \Omega$ onto itself, that connects these Brownian motions, is given by the Cameron–Martin–Girsanov formula

$$\mathbf{w}(t) = \mathbf{B}(t) - \int_0^t \begin{bmatrix} 0 \\ (-1/h)\, k(x(s), y(s), s) \end{bmatrix} ds$$

where $k(x, y, t) = x$ for this linear oscillator.

Since we write $w(t) = w_2(t)$ and $B(t) = B_2(t)$ as scalar Brownian motions on $\Omega = C_0[0, \infty)$, we can restrict attention to Ω whereon

$$w(t) = B(t) + \frac{1}{h} \int_0^t x(s)\, ds.$$

Also the connection between the two measures P and $\tilde{P}$ on Ω is given by [4]

$$dP = \exp\left\{ \frac{-1}{h} \int_0^T k\, dB - \frac{1}{2h^2} \int_0^T k^2\, ds \right\} d\tilde{P}.$$

Of course, $x(t) = 1 + h \int_0^t B(s)\, ds$, $y(t) = hB(t)$, and $k(x(t), y(t), t) = 1 + h \int_0^t B(s)\, ds$ in the case in hand. We seek to estimate the integral over Wiener space Ω,

$$\int_{[x(t) > 0 \text{ on } 0 \leq t \leq T]} dP$$
$$= \int_{[1 + h \int_0^t B(s)\, ds > 0 \text{ on } 0 \leq t \leq T]} \exp\left\{ \frac{-1}{h} \int_0^T k\, dB - \frac{1}{2h^2} \int_0^T k^2\, dt \right\} d\tilde{P}.$$

Thus we seek a positive lower bound for the integral (denoted by ψ)

$$\psi = \int_{[1 + h \int_0^t B(s)\, ds > 0 \text{ on } 0 \leqslant t \leqslant T]} \exp\left\{\frac{-1}{h} \int_0^T \left(1 + h \int_0^t B(s)\, ds\right) dB - \frac{1}{2h^2} \int_0^T \left(1 + h \int_0^t B(s)\, ds\right)^2 dt\right\} d\tilde{P}.$$

The significance of this integral ψ is that it involves only the Brownian motion $B(t)$ on Ω, appropriate for the Wiener space probability measure $\tilde{P}$, but some care must be taken in the evaluation of the Itô stochastic integrals.

First we simplify the estimation of ψ by restricting attention to the subset of Ω

$$\left\{\sup_{0 \leqslant t \leqslant T} |B(t)| < \frac{1}{Th}\right\} \subset \left\{1 + h \int_0^t B(s)\, ds > 0 \text{ on } 0 \leqslant t \leqslant T\right\}.$$

This smaller subset of Ω has the probability $\tilde{P}$ such that

$$\frac{\pi}{4} e^{(\pi^2/8)\, T^3 h^2} \tilde{\mathbb{P}} \left[\sup_{0 \leqslant t \leqslant T} |B(t)| < \frac{1}{Th}\right] \to 1 \qquad \text{as} \quad T \to \infty.$$

This assertion follows from the familiar limit

$$\lim_{T \to \infty} \frac{\pi}{4} e^{\pi^2 T/8} \tilde{\mathbb{P}}[\sup_{0 \leqslant t \leqslant T} |B(t)| < 1] = 1.$$

But for each $\delta > 0$ we know that $\delta B(t/\delta^2)$ is also Brownian motion (for same $\tilde{P}$), and take $\delta = 1/Th$ and then

$$\begin{aligned} \tilde{\mathbb{P}}[\sup_{0 \leqslant t \leqslant T/\delta^2} |B(t)| < 1] &= \tilde{\mathbb{P}}[\sup_{0 \leqslant t \leqslant T} |B(t/\delta^2)| < 1] \\ &= \tilde{\mathbb{P}}[\sup_{0 \leqslant t \leqslant T} (1/\delta)\, |B(t)| < 1] \\ &= \tilde{\mathbb{P}}[\sup_{0 \leqslant t \leqslant T} |B(t)| < \delta]. \end{aligned}$$

Hence

$$\tilde{\mathbb{P}} \left[\sup_{0 \leqslant t \leqslant T} |B(t)| < \frac{1}{Th}\right] = \tilde{\mathbb{P}}[\sup_{0 \leqslant t \leqslant T^3 h^2} |B(t)| < 1]$$

and the prior assertion is verified. If we adopt this restriction on the sample paths $B(t)$, namely, $\sup_{0 \leq t \leq T} |B(t)| < 1/Th$, then we can conclude that

$$\psi > \left\{ \exp \left[\frac{-1}{h} \int_0^T \left(1 + h \int_0^t B(s)\, ds \right) dB - \frac{1}{2h^2} \int_0^T \left(1 + h \int_0^t B(s)\, ds \right)^2 dt \right\} \times \left\{ \frac{4}{\pi} e^{(-\pi^2/8)\, T^3 h^2} \cdot \frac{\pi}{4} e^{(\pi^2/8)\, T^3 h^2} \tilde{\mathbb{P}} \left[\sup_{0 \leq t \leq T} |B(t)| < \frac{1}{Th} \right] \right\},$$

and the second factor can be replaced by

$$\frac{4-\varepsilon}{\pi} e^{(-\pi^2/8)\, T^3 h^2}, \qquad \text{for each preassigned} \quad \varepsilon > 0,$$

and all sufficiently large $T > T(\varepsilon)$.

In order to evaluate the stochastic integrals that appear in the preceding exponential we note the Itô formula for the differential

$$d\left(\int_0^t B(s)\, ds \right) \cdot B(t) = \left(\int_0^t B(s)\, ds \right) dB + B(t)\, (B(t)\, dt).$$

This yields a formula of "integration by parts"

$$\int_0^T \left(\int_0^t B(s)\, ds \right) dB = B(T). \int_0^T B(s)\, ds - \int_0^T B(t)^2\, dt,$$

and so

$$\int_0^T \left(1 + h \int_0^t B(s)\, ds \right) dB = B(T) + hB(T) \int_0^T B(s)\, ds - h \int_0^T B(t)^2\, dt.$$

Once the integrals are deterministic (measure dt on $\mathbb{R}$) then we can use the positivity $(1/Th) - |B(t)| > 0$ to obtain the estimates

$$\int_0^T \left(1 + h \int_0^t B(s)\, ds \right) dB < B(T) + hB(T) \cdot \frac{T}{Th} < \frac{2}{Th},$$

and

$$\int_0^T \left(1 + h \int_0^t B(s)\, ds \right)^2 dt < \int_0^T (1 + ht/Th)^2\, dt = 7T/3.$$

Therefore, using the condition $\sup_{0 \leq t \leq T} |B(t)| < 1/Th$, we compute (for each preassigned $\varepsilon > 0$)

$$\psi > \left\{ \exp\left(\frac{-2}{Th^2} - \frac{7T}{6h^2} \right) \right\} \cdot \left\{ \frac{4-\varepsilon}{\pi} \exp\left(\frac{-\pi^2}{8} T^3 h^2 \right) \right\}$$

when $T \to \infty$. This yields the desired lower bound

$$\psi > \exp\left(\frac{-2}{h^2}\frac{1}{T} - \frac{7}{6h^2}T - \frac{\pi^2}{8}h^2T^3\right) \qquad (\text{when } \varepsilon = 4 - \pi).$$

But the first zero $\hat{T}$ of $x(t)$ must occur after time T, since we have $x(t) = 1 + h\int_0^t B(s)\,ds > 0$ on $0 \leqslant t \leqslant T$. Therefore

$$\mathbb{P}[\hat{T} > T] = \int_{[x(t) > 0 \text{ on } 0 \leqslant t \leqslant T]} dP$$
$$= \psi(T) > \exp\left(\frac{-2}{h^2}\frac{1}{T} - \frac{7}{6h^2}T - \frac{\pi^2}{8}h^2T^3\right).$$

Set $A(h) = e^{-1/h^2}$, $B(h) = +2/h^2$, and then $\mathbb{P}[\hat{T} > T] > A(h)\,e^{-B(h)T^3}$ for $T > 2$, when we take $0 < h < 1$. ∎

Remark. It is of interest to investigate the lower bound for $\mathbb{P}[\hat{T} > T]$ in the case of the nonlinear stochastic oscillator

$$\ddot{x} + k(x, \dot{x}, t) = h\dot{w}(t) \qquad \text{on} \quad t \geqslant 0.$$

as specified in Section 2. For the general case where $k(x, y, t)$ satisfies the standing conditions (i) and (ii), we have not been able to obtain satisfactory results. However, we can follow the argument of Theorem 5 quite directly for the case $k(x, y, t) = k(x, t) + \beta y$, for constant β and $k(x, t) \in C^1$ satisfying the usual linear growth condition

$$|k(x, t)| \leqslant \gamma(t)(1 + |x|), \qquad \text{for continuous } \gamma(t) \quad \text{on} \quad 0 \leqslant t < \infty.$$

Again we obtain, for each fixed $h > 0$, and then large $T \to \infty$, $\mathbb{P}[\hat{T} > T] > A_1 e^{-B_1 T^3}$, for positive constants A_1, B_1.

4. The Stochastic Winding Angle

Once again consider the scalar stochastic oscillator

$$\ddot{x} + k(x, \dot{x}, t) = h\dot{w}(t) \qquad \text{on} \quad t \geqslant 0,$$

or the differential system

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} y \\ -k(x, y, t) \end{bmatrix} dt + \begin{bmatrix} 0 & 0 \\ 0 & h \end{bmatrix} \begin{bmatrix} dw_1 \\ dw_2 \end{bmatrix}$$

as in the opening paragraph of Section 2. Let $(x(t), y(t))$ on $0 \leqslant t < \infty$ be the stochastic solution initiating at $(x_0, y_0) \neq (0, 0)$ in the (x, y)-plane $\mathbb{R}^2$.

From Theorem 2 we conclude that

$$x(t)^2+y(t)^2>0 \qquad \text{for all} \quad 0\leqslant t<\infty \qquad \text{(a.s.)}.$$

In this situation we define the stochastic winding angle $\Omega(t)$ to be the algebraic angle (in radians) from the initial vector (x_0, y_0) along the solution curve to the vector $(x(t), y(t))$, clockwise around the origin in the (x, y)-plane. Of course, $\Omega(t)$ on $0\leqslant t<\infty$ is a scalar stochastic process, depending on the stochastic oscillator and on the initial state (x_0, y_0).

In the case of a linear stochastic oscillator, say,

$$\ddot{x}+x=h\dot{w}(t) \qquad \text{on} \quad t\geqslant 0$$

with $(x_0, y_0)=(1, 0)$ as in Theorem 3, we conclude that

$$\mathbb{P}[\lim_{t\to\infty} \Omega(t)=\infty]=1;$$

compare results in [8]. It seems reasonable to conjecture that this result also holds in many nonlinear cases (say, $k=x^3$).

Furthermore, for the linear stochastic oscillator

$$\ddot{x}+x=h\dot{w}(t) \qquad \text{on} \quad 0\leqslant t<\infty$$

with $x(0)=1$, $\dot{x}(0)=0$ we would conjecture that the expectation

$$\mathbb{E}(\Omega(t))=t \qquad \text{for each} \quad t\geqslant 0.$$

[Note this is different from the triviality arctan $\mathbb{E}(y)/\mathbb{E}(x)=t$.] But it also seems likely that the deviations $|\Omega(t)-t|$ should satisfy

$$\limsup_{t\to\infty} |\Omega(t)-t|=\infty \qquad \text{(a.s.)}.$$

We cannot demonstrate these conjectures, but we can provide a rather weak lower bound for $\Omega(t)$, namely,

$$\liminf_{t\to\infty} \frac{\Omega(t)}{\log t}\geqslant\frac{\pi}{2} \qquad \text{(a.s.)}.$$

The proof of this lower estimate for $\Omega(t)$ rests on the calculations obtained in Theorem 3 where we found

$$(1/h)\, x((2m+\tfrac{1}{2})\pi)=Y_0+Y_1+Y_2+\cdots+Y_m \qquad \text{for} \quad m=0, 1, 2, 3, \ldots,$$

with Y_0, Y_1, Y_2, ... mutually independent Gaussian random variables of mean zero and variance π (except for var $Y_0=\pi/4$).

THEOREM 6. *Consider the linear stochastic oscillator*

$$dx = y\,dt$$
$$dy = -x\,dt + h\,dw \qquad on \quad 0 \leqslant t < \infty \qquad (h > 0),$$

with solution $x(t)$, $y(t)$ *from* $x(0) = 1$, $y(0) = 0$, *as before. Let* $\Omega(t)$ *be the stochastic winding angle of this solution clockwise about the origin. Then*

$$\liminf_{t \to \infty} \frac{\Omega(t)}{\log t} \geqslant \frac{\pi}{2} \qquad \text{(a.s.)}.$$

Proof. Let $Z(T)$ be the number of sign changes of $x(t)$ in the open interval $0 < t < T$, or equally well, the number of zeros of $x(t)$ in $(0, T)$. From the geometry of the dynamics in the (x, y)-plane, we note that the winding angle $\Omega(t)$ satisfies

$$\pi \cdot [Z(t) - 1] + \pi/2 \leqslant \Omega(t) \leqslant \pi \cdot [Z(t) + 1], \qquad \text{for all} \quad t > \hat{T}.$$

Let N_m be the number of sign changes in the finite sequence $S_0 = Y_0$, $S_1 = Y_0 + Y_1, \ldots, S_m = Y_0 + Y_1 + \cdots + Y_m$, for each positive integer $m \geqslant 1$. Then trivially, from Theorem 3,

$$Z((2m + \tfrac{1}{2})\,\pi) \geqslant N_m.$$

Hence

$$\Omega((2m + \tfrac{1}{2})\,\pi) \geqslant \pi N_m - \pi/2 \qquad \text{for} \quad m = 1, 2, 3, \ldots .$$

Now a celebrated result related to the Central Limit Theorem asserts [2]:

Let $\hat{N}_m$ be the number of sign changes in the sequence $\hat{S}_1 = Y_1$, $\hat{S}_2 = Y_1 + Y_2, \ldots, \hat{S}_m = Y_1 + \cdots + Y_m$. Then

$$\liminf_{m \to \infty} \frac{\hat{N}_m}{\log m} \geqslant \frac{1}{2} \qquad \text{(a.s.)}$$

(where $\log m$ is the usual natural logarithm). But

$$\hat{N}_m \leqslant N_m \leqslant \hat{N}_m + 1 \qquad \text{and so} \quad \liminf_{m \to \infty} \frac{N_m}{\log m} \geqslant \frac{1}{2} \qquad \text{(a.s.)}$$

From these inequalities it follows easily that

$$\liminf_{t \to \infty} \frac{\Omega((2m + \frac{1}{2})\,\pi)}{\log m} \geqslant \frac{\pi}{2} \qquad \text{(a.s.)}.$$

It is now not too difficult to replace the integral variable m by the real variable $t \geqslant 0$. From the geometry of the dynamics in the (x, y)-plane, we note that $\Omega(t)$ tends to increase with time t, but it might occasionally decrease by at most π; that is,

$$\Omega(t_2) \geqslant \Omega(t_1) - \pi \qquad \text{for} \quad t_2 \geqslant t_1 > 0.$$

For each given large $t > 0$ we can bracket t within an interval of duration 2π

$$t_1 < t \leqslant t_2,$$

where $t_1 = (2m + \frac{1}{2})\,\pi$ and $t_2 = [2(m+1) + \frac{1}{2}]\,\pi$ for a unique positive integer m. Then we compute the required limits,

$$\liminf_{t \to \infty} \frac{\Omega(t)}{\log t} \geqslant \liminf_{m \to \infty} \frac{\Omega((2m + \frac{1}{2})\,\pi) - \pi}{\log m} \cdot \frac{\log m}{\log(2m + \frac{5}{2})\,\pi}.$$

Hence we obtain the result

$$\liminf_{t \to \infty} \frac{\Omega(t)}{\log t} \geqslant \liminf_{m \to \infty} \frac{\Omega((2m + \frac{1}{2})\,\pi)}{\log m} \geqslant \frac{\pi}{2} \text{ (a.s.)},$$

as required. ∎

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