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Generalized Jordan algebras

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Abstract

We study commutative algebras which are generalizations of Jordan algebras. The associator is defined as usual by $(x, y, z) = (xy)z - x(yz)$. The Jordan identity is $(x^2, y, x) = 0$. In the three generalizations given below, t, β, and γ are scalars. $((xx)y)x + t((xx)x)y = 0$, $((xx)x)(yx) - (((xx)x)y)x = 0$, $\beta((xx)y)x + \gamma((xx)x)y - (\beta + \gamma)((yx)x)x = 0$. We show that with the exception of a few values of the parameters, the first implies both the second and the third. The first is equivalent to the combination of $((xx)x)x = 0$ and the third. We give examples to show that our results are in some reasonable sense, the best possible.

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1. Introduction

In this paper we study commutative algebras over a field F of characteristic $\neq 2, 3$. The algebras are neither necessarily associative nor finite dimensional. The associator (x, y, z) is defined by $(x, y, z) = (xy)z - x(yz)$. We shall study algebras satisfying the following three identities where t, β, γ are scalars in F:

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$$((xx)y)x + t((xx)x)y = 0, \tag{1}$$

$$((xx)x)(yx) - (((xx)x)y)x = 0, \tag{2}$$

$$\beta((xx)y)x + \gamma((xx)x)y - (\beta + \gamma)((yx)x)x = 0. \tag{3}$$

When $t \neq -1$, (1) implies

$$((xx)x)x = 0. \tag{4}$$

The identity (2) is called 3-Jordan. In [5] Hentzel and Peresi proved that a simple commutative algebra with an idempotent element that satisfies (2) is either a Jordan or a pseudo-composition algebra. The special case of (3) when $\beta = 3$ and $\gamma = -1$ is called the Almost Jordan identity (see (5) below)

$$3((xx)y)x = 2((yx)x)x + ((xx)x)y. \tag{5}$$

This identity has attracted a considerable attention (see [8,9,4,10]). It is known that the Jordan identity consists of two separate pieces. One of them is (5) and the other is fourth power-associativity. Therefore, any commutative algebra satisfying (5) is Jordan if and only if it is fourth power associative.

In a commutative algebra satisfying (5) the span of the elements of the form (xx, x, x) is a trivial ideal. Consequently, any semi-prime algebra satisfying (5) is a Jordan algebra.

In the study of degree four identities not implied by commutativity, Osborn [7] classified those that were compatible with possessing a unit element. Carini et al. [1] extended this work by dropping the restriction on the existence of the unit element. The identity (3) appeared as one of the additional degree four identities. In an algebra with the unit element, (3) implies $(\beta + 3\gamma)(y, x, x) = 0$. This degree three identity, along with commutativity and characteristic $\neq 3$ implies associativity when $\beta + 3\gamma$ is not zero.

In [3] Correa et al. showed that both identities (3) and (4) imply that the multiplication operator is nilpotent.

2. Main section

We shall first prove that (1) implies (2). We let $f(a, b, c, d)$ be the linearized form of (1). That is

$$f(a, b, c, d) = ((bc)a)d + ((cd)a)b + ((db)a)c + t((bc)d)a + t((cd)b)a + t((db)c)a.$$

In the function $f(a, b, c, d)$, the first argument "a" represents the element y in (1) and the second, third, and fourth arguments, "b", "c", and "d" represent the three elements "x" of (1).

Consequently $f(a, b, c, d)$ is symmetric on the second, third and fourth arguments.

Theorem 1. *Let A be a commutative algebra over a field F of characteristic $\neq 2, 3$. If A satisfies*

$$((xx)y)x + t((xx)x)y = 0$$

with $t \neq 1$ or -1, then A is a 3-Jordan algebra.

Proof. We want to show that (1) implies (2). We express the dependence relations generated by the function $f(a, b, c, d)$ in matrix form relative to the terms: $((xx)x)(yx)$, $x((xx)(xy))$, $x(x((xx)y))$, $x(x(x(xy)))$, $(((xx)x)y)x$.

	$((xx)x)(yx)$	$x((xx)(xy))$	$x(x((xx)y))$	$x(x(x(xy)))$	$(((xx)x)y)x$
$f(x,x,x,xy)=$	1	t	0	$2+2t$	0
$f(yx,x,x,x)=$	$3t$	3	0	0	0
$xf(x,x,x,y)=$	0	0	t	$2+2t$	1
$xf(y,x,x,x)=$	0	0	3	0	$3t$
$J=$	$3(1-t^2)$	0	0	0	$-3(1-t^2)$

where $J = 3\text{Row}(1) - t\text{Row}(2) - 3\text{Row}(3) + t\text{Row}(4)$.

So when $t \neq 1$ or -1, $((xx)x)(yx) = (((xx)x)y)x$ and the algebra is a 3-Jordan algebra, i.e. (2) holds. □

We now show that (1) is equivalent to the combination of (3) and (4).

Theorem 2. *Let A be a commutative algebra satisfying*

$$((xx)y)x + t((xx)x)y = 0.$$

Then A satisfies

$$\beta((xx)y)x + \gamma((xx)x)y - (\beta+\gamma)((yx)x)x = 0$$

with $\beta = 3 - t$ *and* $\gamma = 3t - 1$.

Proof. We write the relations implied by the identities on the terms $((xx)y)x$, $((yx)x)x$, $((xx)x)y$. Recall that the function $f(a,b,c,d)$ is the linearized form of (1).

	$((xx)y)x$	$((yx)x)x$	$((xx)x)y$
$f(y,x,x,x)$	3	0	$3t$
$f(x,x,x,y)$	t	$2+2t$	1
Identity(3)	β	$-\beta-\gamma$	γ
Row(1) − Row(2)	$3-t$	$-2-2t$	$3t-1$

From Row(1) − Row(2) we see that A satisfies a (β,γ)-type identity for $\beta = 3-t$ and $\gamma = 3t-1$. □

Theorem 3. *Let A be a commutative algebra over a field of characteristic* $\neq 2, 3$ *which satisfies identities* (3) *and* (4). *If* $3\beta + \gamma \neq 0$, *then A satisfies* (1) *for* $t = \frac{\beta+3\gamma}{3\beta+\gamma}$.

Proof. The linearized form of (4) is $((xx)y)x + 2((yx)x)x + ((xx)x)y = 0$.

We display this linearized form (3) and also (1) in the following matrix to show the dependence relations between the monomials $((xx)y)x$, $((yx)x)x$, $((xx)x)y$.

	$((xx)y)x$	$((yx)x)x$	$((xx)x)y$
Identity(4)	1	2	1
Identity(3)	β	$-\beta-\gamma$	γ
Identity(1)	1	0	t
K	$3\beta+\gamma$	0	$\beta+3\gamma$

where $K = (\beta+\gamma)\text{Row}(1)+2\text{Row}(2)$.

Dividing Row(4) by $3\beta+\gamma$ produces an identity of the form (1) with the value of t being $\frac{\beta+3\gamma}{3\beta+\gamma}$. □

3. Examples

We now consider the exceptions to Theorem 1. In Theorem 1 neither of the cases $t = 1$ and $t = -1$ are 3-Jordan. We compare the cases $t = 1$ and $t = -1$ of identity (1), with the cases $\beta + \gamma = 0, \beta + 3\gamma = 0$ of identity (3). Finally, we show that identity (5) does not imply 3-Jordan.

We first look at the exceptional cases to Theorem 1. Using (1) with $t = 1$ gives an algebra satisfying (4) and also $((yx)x)x = 0$. This last identity was studied by Correa and Hentzel in [2] where it was shown that commutative, finitely generated algebras satisfying $((yx)x)x = 0$ were solvable. We used the computer program ALBERT [6] to construct a 13 dimensional algebra satisfying (1) with $t = 1$ which was not 3-Jordan.

We used the computer program ALBERT [6] to construct a 17 dimensional algebra which satisfies (1) with $t = -1$ which was not 3-Jordan.

We now compare identity (1) with identities (3) and (4). We see that (1) with $t = -1$ is identical to (3) with $\beta + \gamma = 0$. None of these equations imply (4).

When $t \neq -1$, then (1) implies (4). There is an exact correspondence between (1) with $t \neq -1$ and the combination of (4) and (3) with $\beta + \gamma \neq 0$ and $3\beta + \gamma \neq 0$. The correspondence is $t = \frac{\beta+3\gamma}{3\beta+\gamma}$ or inversely, $\beta = \frac{3-t}{4}, \gamma = \frac{3t-1}{4}$.

The case $3\beta + \gamma = 0$ corresponds to the identity $((xx)x)y = 0$ which corresponds to no value of t. This is due to the choice of term of (1) to which we assigned the coefficient t. If we had chosen (1) to be $t'((xx)y)x + ((xx)x)y = 0$, then $((xx)x)y = 0$ would correspond to $t' = 0$.

The following example was given by Osborn [8]. It is an algebra satisfying identity (5) which was not fourth power-associative and hence not Jordan. This example is not a 3-Jordan algebra either. Identity (5) corresponds to $\beta + 3\gamma = 0$.

Example 1. Let A be a commutative real algebra with basis s, t and multiplication table $s^2 = s + t$, $st = ts = \frac{1}{2}t$ and all other products being zero. A is not a Jordan algebra and not fourth power-associative because $(s^2, s, s) = \frac{-1}{4}t$. We now show that A is not a 3-Jordan algebra. Taking $x = s + t$ and $y = 2s$ makes $((xx)x, y, x) = \frac{-3}{4}t$.

The following example shows that the identities (3) and (4) do not imply the Jordan identity.

Example 2. Let $A = \langle x, x^2, x^3, y\rangle$ be a commutative four dimensional algebra with the following multiplication table: $xx = x^2$; $xx^2 = x^2x = x^3$; $x^2x^2 = x^3$; $x^2y = yx^2 = x^3$; $yy = x^2 + x^3$; all other products being zero. Then A is an algebra satisfying (3) and (4) for every β and γ. A is not a Jordan algebra because $(x^2, x, x) = -x^3$. A is also not four power-associative for the same reason.

References

[1] L. Carini, I.R. Hentzel, G.M. Piacentini-Cattaneo, Degree four identities not implied by commutativity, Comm. Algebra, 16 (2) (1988) 339–356.

[2] I.C. Correa, I.R. Hentzel, Commutative finitely generate algebras satisfying $((yx)x)x = 0$ are solvable, Rocky Muntains J. Math, accepted for publication.

[3] I. Correa, I.R. Hentzel, A. Labra, On the nilpotence of the multiplication operator in commutative right nil algebras, Comm. Algebra 30 (7) (2002) 3473–3488.

[4] I.R. Hentzel, L.A. Peresi, Almost Jordan rings, Proc. Amer. Math. Soc. 104 (2) (1988) 343–348.

[5] I.R. Hentzel, L.A. Peresi, A variety containing Jordan and pseudo-composition algebras, East-West J. Math. 6 (2–3) (2004) 141–151.

[6] D.P. Jacobs, D. Lee, S.V. Muddana, A.J. Offutt, K. Prabhu, T. Whiteley, Albert's user guide, Department of Computer Science, Clemson University, 1993.
[7] J.M. Osborn, Identities of non-associative algebras, Canad. J. Math. 17 (1965) 78–92.
[8] J.M. Osborn, Commutative algebras satisfying an identity of degree four, Proc. Amer. Math. Soc. 16 (1965) 1114–1120.
[9] H. Petersson, Zur Theorie der Lie-Tripel-Algebren, Math. Z. 97 (1967) 1–15.
[10] A.V. Sidorov, On Lie triple algebras, Algebra Logika (Trans.) 20 (1) (1981) 101–108.