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Comparison of Numerical Schemes for Solving the Advection Equation

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Abstract—We report on the dispersion and dissipation properties of numerical schemes aimed at solving the one-dimensional advection equation. The study is based on the consistency error, which is explicitly calculated for various standard finite-difference schemes. The oscillation and damping features of the numerical solutions are shown to be explained via a generalized Airy-like function. In the specific case of the advection of a step function, the solutions of the equivalent equations are systematically calculated and shown to recover the numerical solutions. A particular emphasis is put on one third-order accurate scheme, which involves a weak smearing of the step.

Keywords—Advection equation, Numerical schemes, Spurious oscillations.

1. INTRODUCTION

In this paper, we study the 1D advection equation $\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial x} = 0$ as a prototypical hyperbolic equation, where u denotes the advected quantity, t the time, x the space, and V a constant velocity. Choosing $u_0(x)$ as the initial condition leads to the exact solution $u(x,t) = u_0(x - Vt)$. Discretising this equation for numerical solving induces an inaccuracy that we have examined, with particular emphasis on the properties of dispersion and dissipation of the numerical solution.

2. METHOD OF ANALYSIS

We use the method of differential approximation [1]. For practical purpose, we keep only one or two terms for the differential approximation (we refer to [2,3] for mathematical justifications).

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Thus, for a scheme of order $n-1$, the difference equation is replaced by the following differential approximation:

$$\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial x} = \frac{c}{n}\frac{\partial^n u}{\partial x^n} + \frac{d}{n+1}\frac{\partial^{n+1} u}{\partial x^{n+1}}. \tag{1}$$

For an even-order scheme, the first term induces dispersion and the second one induces dissipation. For an odd-order scheme, the first term is dissipative and the second one is dispersive.

By using Fourier transform, we get an integral representation of the solution, which reads for a scheme of even order $n-1$ (mainly dispersive)

$$u(x,t) = u(x,0) * \int_{-\infty}^{\infty} dk e^{ik(x-Vt)} e^{\pm ictk^n/n} e^{-dtk^{n+1}/(n+1)}, \tag{2}$$

and for a scheme of odd-order $n-1$ (mainly dissipative)

$$u(x,t) = u(x,0) * \int_{-\infty}^{\infty} dk e^{ik(x-Vt)} e^{-ctk^n/n} e^{\pm idtk^{n+1}/(n+1)}, \tag{3}$$

the * designates a convolution product

The integral in the previous expressions are generalized Airy functions. For the dispersive case (n odd, i.e., even-order scheme), they are defined as

$$\mathrm{Ai}^{n,b}(y) = \int_{-\infty}^{\infty} dk e^{iky+ik^n/n-bk^{n+1}/(n+1)} \tag{4}$$

and, for the dissipative case (n even, i.e., odd-order scheme), as

$$\mathrm{Ai}^{n,b'}(y) = \int_{-\infty}^{\infty} dk e^{iky-k^n/n+ib'k^{n+1}/(n+1)}. \tag{5}$$

The number b measures the ratio dissipation-to-dispersion of a dispersive scheme and b' the dispersion-to-dissipation ratio for a dissipative scheme.

For large values of their arguments, these functions have accurate asymptotic expansions in terms of exponential and trigonometric functions, which rule both the wavelength and the damping of the numerically generated oscillations, as exposed in the following section.

3. PREDICTION OF THE ADVECTION OF A STEP FUNCTION BY THE NUMERICAL SCHEMES

We study the behaviour of generic numerical schemes advecting a step. We compute an asymptotic expansion of the generalized Airy functions describing the behaviour of these schemes with large arguments, in order to predict the amplitude and damping of the spurious numerical oscillations far from the step.

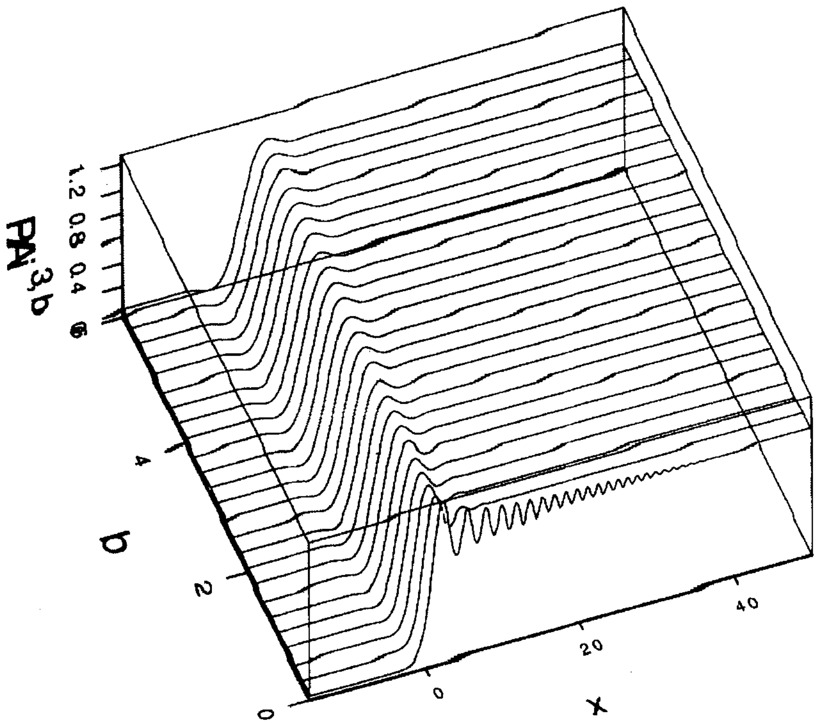
3.1. Second-Order Accurate Schemes

The main extra term introduced by the scheme is now dispersive. Advecting a step $H(x)$, we get

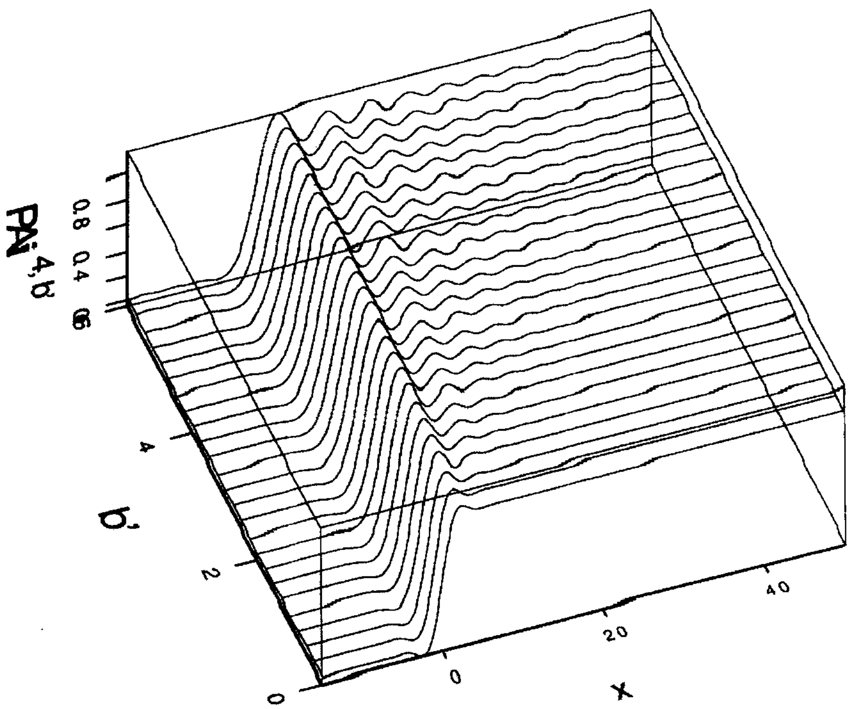
$$u(x,t) = \frac{1}{(ct)^{1/3}} P\mathrm{Ai}^{3,0}\left[\frac{(x-Vt)}{(ct)^{1/3}}\right], \tag{6}$$

$P\mathrm{Ai}$ is a primitive of the Airy function. The behaviour of $P\mathrm{Ai}$ for large x, and more specifically the oscillations far from the step, are predicted by using the steepest descent method. The integral can be written as the sum of a pole contribution, and the dicretisation induced critical point contribution

$$I_C \approx -\frac{1}{2\sqrt{\pi}} x^{-3/4} e^{-2x^{3/2}/3}. \tag{7}$$



(a) Influence of dissipation in the function $PAi^{3,b}(x)$. The extremes are -0.038 and 1.275.



(b) Influence of dispersion in the function $PAi^{4,b'}(x)$. The extremes are -0.052 and 1.175.

Figure 1.

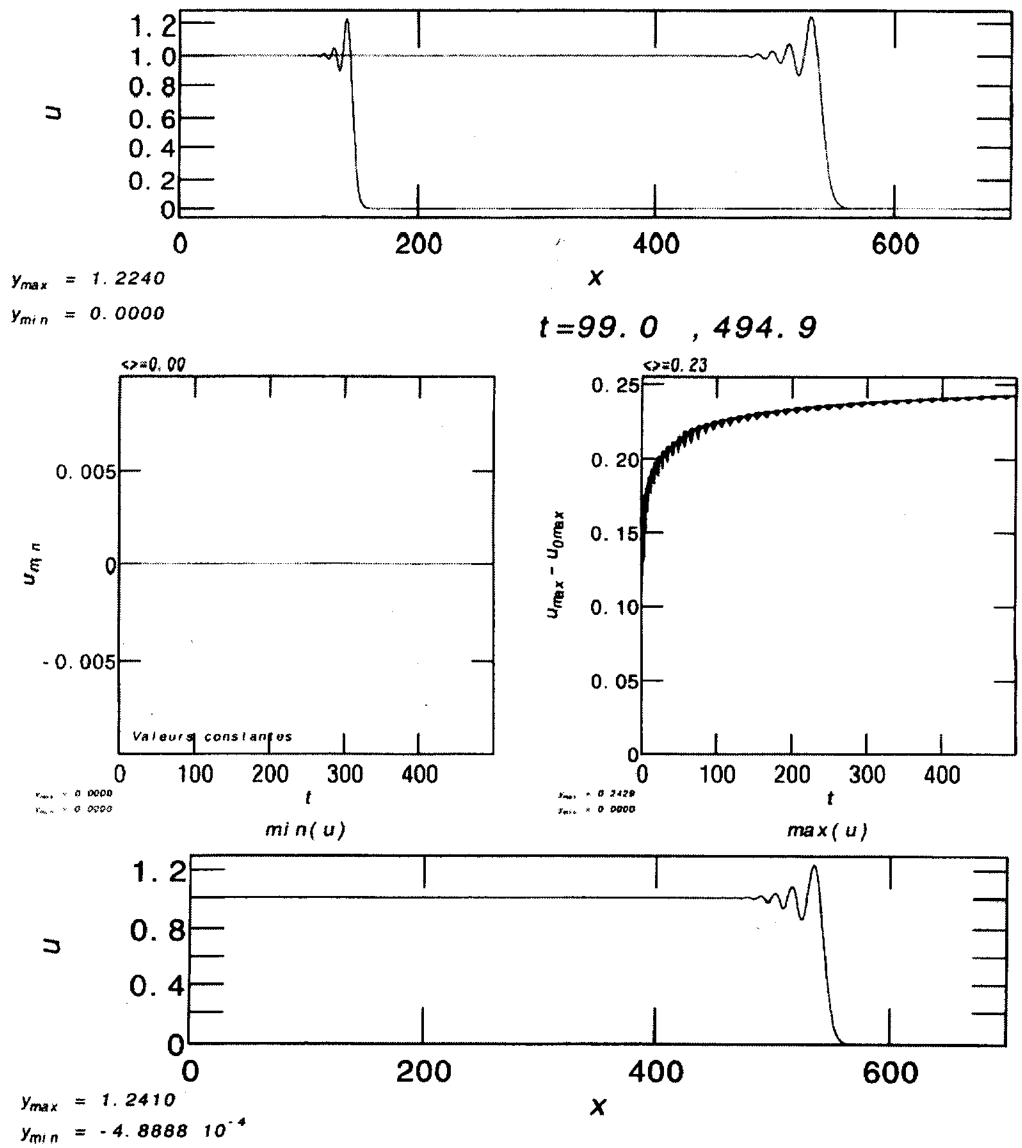


Figure 2. Lax-Wendroff scheme. (a) Numerical results. (b) Response calculated with the function $PAi^{3,b}(x)$.

The solution decreases exponentially, without oscillations, for $x > 0$. The numerical solution converges without oscillating towards the true solution.

For $x < 0$, the sum of the contributions of the two real critical points turns out to be

$$IC \approx -\frac{1}{\sqrt{\pi}}\,|x|^{-3/4}\sin\left(\frac{2}{3}\,|x|^{3/2} - \frac{\pi}{4}\right). \tag{8}$$

The decrease is rather slow ($|x|^{-3/4}$) and in addition involves spurious oscillations. The second-order accurate scheme with positive c is monotonous for $x > 0$ but generates oscillations for $x < 0$. For negative values of c, the results are similar, but with oscillations for $x > 0$ and exponential decrease for $x < 0$.

The long range oscillatory behaviour is a specific feature of second-order accurate schemes without dissipation, such as the leapfrog scheme, as we will see in Part 5. However, for schemes such as Lax-Wendroff ($c > 0$) or Beam-Warming ($c < 0$), we have to deal with the fourth-order

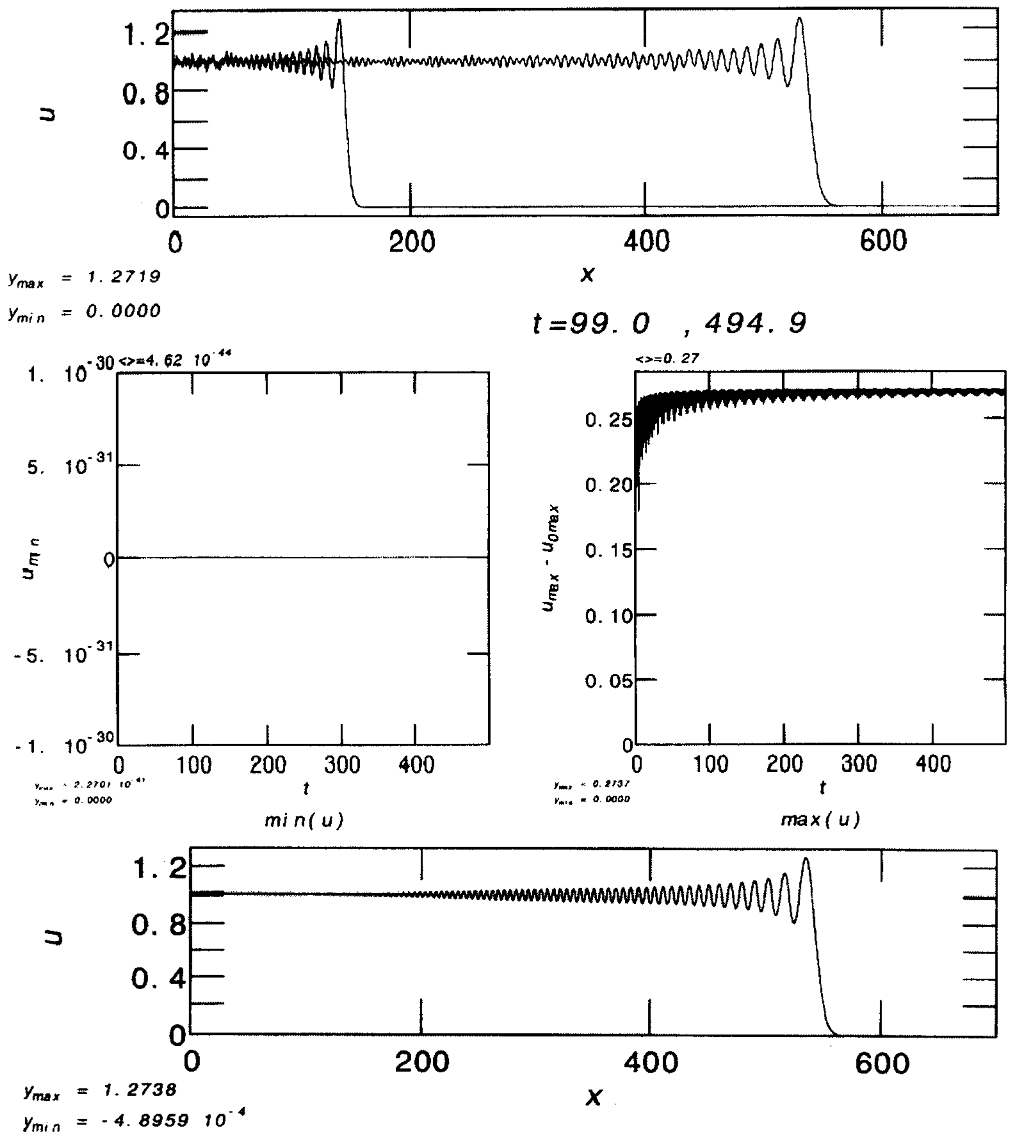


Figure 3. Leapfrog scheme. (a) Numerical results. (b) Response calculated with the function $P\mathrm{Ai}^{3,b}(x)$.

dissipation term and to consider functions $\mathrm{Ai}^{3,b\neq 0}$. The dissipation-to-dispersion ratio b is equal to $d/4c^{4/3}t^{1/3}$. The stationary points, which now satisfy $x + k^2 + ibk^3 = 0$, have a nonvanishing imaginary part. Because of the nonvanishing imaginary part $ibx/2$ of these points, the oscillations are now damped. The damping increases with b, as shown in Figure 1.

We should notice that for these second-order accurate schemes, b decreases with time. Therefore, the spurious oscillations become less damped with increasing time.

3.2. Third-Order Schemes

We now get for the step advection

$$\int_{-\infty}^{\infty} dk \, \frac{1}{2i\pi k} \, e^{ixk-(k^4/4)} = P\mathrm{Ai}^{4,0}(x). \tag{9}$$

The asymptotic expansion of the function $P\mathrm{Ai}^{4,0}(x)$ for large x appears as the sum of two critical points contributions for $x < 0$. For $x > 0$, as before, we have to include the pole contribution.

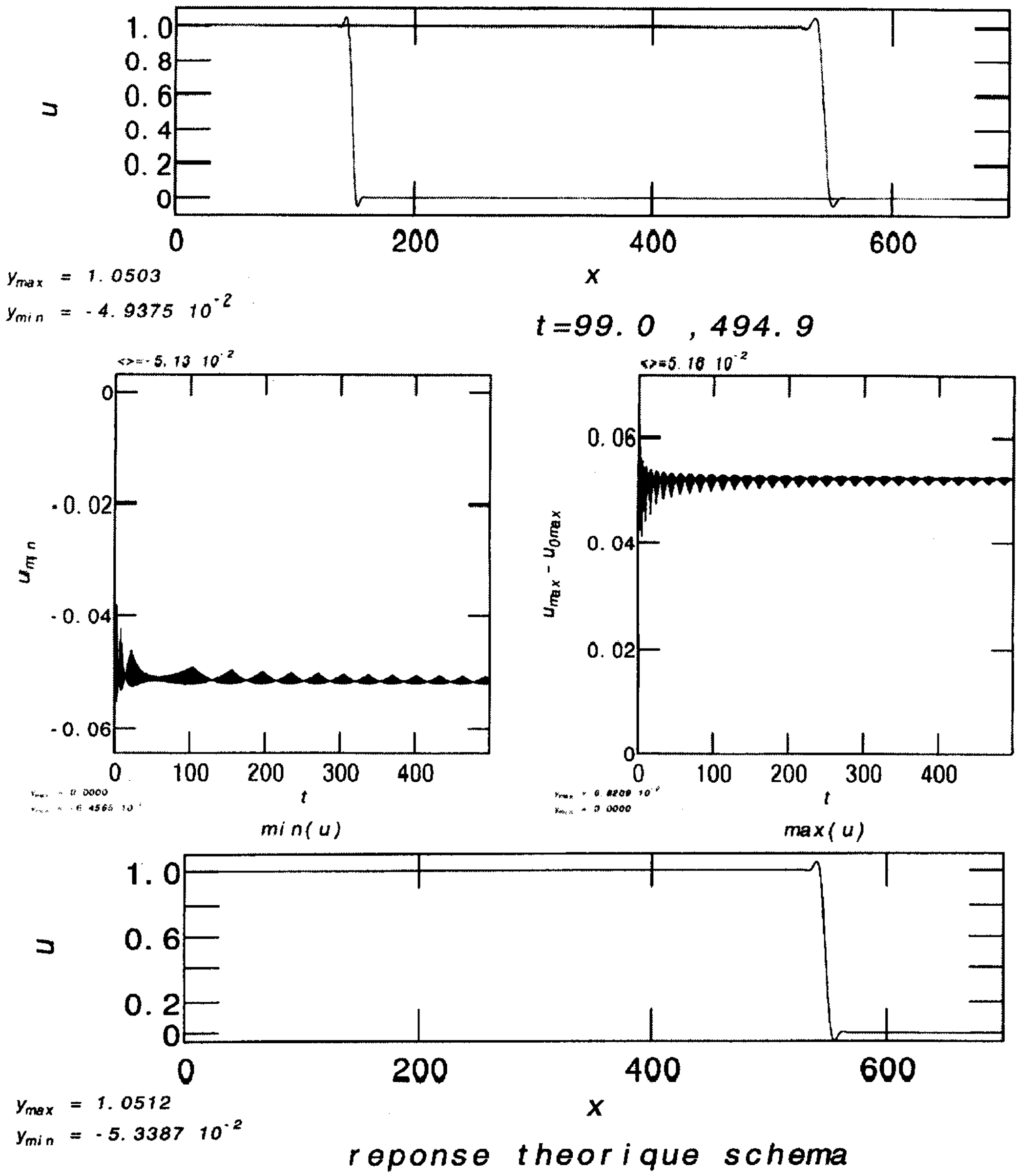


Figure 4. Corrected Lax-Wendroff scheme. (a) Numerical results. (b) Response calculated with the function $P\mathrm{Ai}^{4,b'}(x)$.

For $x > 0$, we get

$$P\mathrm{Ai}^{4,0}(x) \approx 1 + \sqrt{\frac{2}{3\pi}}\, x^{-2/3} e^{-3x^{4/3}/8} \sin\left(\frac{3\sqrt{3}}{8}\, x^{4/3} - \frac{\pi}{3}\right) \tag{10}$$

and for $x < 0$

$$P\mathrm{Ai}^{4,0}(x) \approx -\sqrt{\frac{2}{3\pi}}\, |x|^{-2/3} e^{-3|x|^{4/3}/8} \sin\left(\frac{3\sqrt{3}}{8}\, |x|^{4/3} - \frac{\pi}{3}\right). \tag{11}$$

Oscillations occur for third-order schemes. However, they are exponentially damped, and therefore, less severe than for the second-order schemes (see (8)).

Table 1.

	C_2	C_3	C_4	C_5	C_6
UP	$\eta - 1$	$\frac{(2\eta^2 - 3\eta + 1)}{2}$			
LW		$\frac{(-\eta^2 + 1)}{2}$	$\frac{\eta(-\eta^2 + 1)}{2}$		
WB		$\frac{(-\eta^2 + 3\eta - 2)}{2}$	$\frac{(-\eta^3 + 4\eta^2 - 5\eta + 2)}{2}$		
Fromm		$\frac{(-2\eta^2 + 3\eta - 1)}{4}$	$\frac{(-\eta^3 + 2\eta^2 - 2\eta + 1)}{2}$		
LF		$\frac{(-\eta^2 + 1)}{2}$		$\frac{(9\eta^4 - 10\eta^2 + 1)}{24}$	
Balakin				$\frac{(-\eta^4 + 5\eta^2 - 4)}{24}$	$\frac{\eta(-\eta^4 + 5\eta^2 - 4)}{24}$
L2			$\frac{(\eta^3 - 2\eta^2 - \eta + 2)}{6}$	$\frac{(2\eta^4 - 5\eta^3 + 5\eta - 2)}{12}$	

The effect of the dispersion terms is governed by the behaviour of the function $\mathrm{Ai}^{4,b'}$. For small enough dispersion-to-dissipation ratio b', the critical points move in the complex plane, but remain far enough from the real axis, so that the behaviour remains the same, with exponentially damped oscillations. However, for b' large enough, some critical points approach the real axis, generating weakly damped oscillations (Figure 1).

3.3. Higher-Order Schemes

We obtain the same type of behaviour for higher-order schemes: exponential damping of oscillations for odd-order schemes on both sides of the step, weakly damped oscillations on one side of the step for even-order schemes.

4. EQUIVALENT PDE FOR SOME TYPICAL SCHEMES

We have chosen some representative schemes: upwind (first-order), Lax-Wendroff, Beam-Warming, and Fromm, representative of second-order schemes with dissipation, leapfrog for nondissipative second-order schemes, Balakin (order 4), and a third-order scheme derived from Lax-Wendroff.

We have gathered in Table 1 the coefficients C_{n+1} which multiply the operators $\Delta x^n/(n+1)$ $\partial^{n+1}/\partial x^{n+1}$ in the consistency error ε_g. The following contractions have been used: UP (for upwind scheme), LW (Lax-Wendroff), WB (Warming-Beam), Fr (Fromm), LF (leap-frog), Ba (Balakin), L2 (Lax-Wendroff corrected).

The smaller the absolute value of the coefficients, the better the scheme. The dissipative feature is provided by the terms proportional to Δx, Δx^3, and Δx^5 which have to be, negative, positive, and negative, respectively. For $\eta = 1$, all the terms nullify, except for the IC scheme: the schemes become simple translation and are exact solvers of the advection equation.

5. NUMERICAL RESULTS

We have examined the various schemes studied in Section 4, and obtain excellent agreement. We present a limited number of results to save space.

5.1. Second-Order Schemes

LAX-WENDROFF SCHEME. The solution is a primitive of the Airy function $P\mathrm{Ai}^{3,b}[(x - Vt)/(ct)^{1/3}]$, with $c > 0$. Oscillations can, therefore, be seen in the wake of the front; their wavelength increases with time according to $t^{1/3}$, and the front width as well. The ratio b of dissipation to dispersion decreases according to $t^{-1/3}$, so that the oscillation amplitude increases. We have: $|b|(t = 500) = 8.5\,10^{-2}$. At time $t = 500$, the amplitude of the oscillations maximises at 24% of the plateau, close to the 27.7% value got without dissipation. The Airy function $P\mathrm{Ai}^{3,b}$ reproduces very well the pattern obtained numerically (Figure 2).

LEAP-FROG SCHEME. The solution is given by the function $P\mathrm{Ai}^{3,0}[(x - Vt)/(ct)^{1/3}]$, the space scale varies along time. The nondissipative feature of this scheme induces a large spreading of the oscillations (Figure 3). The latter increase rapidly to saturate at time $t = 20$ with 27% amplitude, which corresponds to the maximum value (1.275) of the function $P\mathrm{Ai}^{3,0}$. This function reproduces correctly the oscillations close to the front. At early times, the oscillations are damped by the mesh. Nevertheless, a high frequency signal can be diagnosed, which is badly reproduced by the differential approximation.

5.2. Third-Order Scheme

LAX-WENDROFF CORRECTED SCHEME. With correction of type with noncentered operator, a hollow and a hump can be seen on both the front and the back of the step, respectively, with very weak amplitude: 0.068. In this case, we have $|b'| = 3.2\,10^{-3}$, which indicates a weakly dispersive scheme. The solution is well described by the Airy function $P\mathrm{Ai}^{4,b'}$ (Figure 4).

6. CONCLUSION

In this paper, we have investigated the behaviour of the numerical solution, provided by various methods, in the specific case of the advection of a sharp step. This behaviour is shown and verified, via some examples, to be fully reproduced by primitives of functions which generalize the standard Airy function.

For even-order schemes, these functions, called dispersive Airy functions, present weakly damped spurious oscillations. With time, the dissipation-to-dispersion ratio decreases and induces a slower decrease of the oscillations when parting from the step front. For large time, the solution converges towards a purely dispersive Airy function.

Odd-order accurate schemes are governed by dissipative Airy functions,with strongly damped oscillations. Weak overshoots (5% for zero dispersion for the third-order scheme), appear on each side of the step. Moreover, the oscillations damp with time.

These various comments put emphasis on odd-order accurate schemes. Although they are not monotonic (except for first order), their dissipative feature induce less oscillations than even-order schemes and these oscillations damp with time, instead of amplifying as for odd-order schemes.

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