Non-local pseudo-differential operators

Ryuichi Ishimura

Department of Mathematics and Informatics, Faculty of Sciences, Chiba University, Yayoicho, Chiba, 263, Japan
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Abstract

The notion of non-local pseudo-differential operators, as well as their symbols and the operation on holomorphic functions, is established and the invertibility theorem for such operators is proved.

Résumé

La notion des opérateurs pseudo-différentiels non-locaux, leurs symboles et l’opération sur des fonctions holomorphes, est établie. Le théorème d’inversibilité pour tels opérateurs est démontré.

Keywords: Pseudo-differential operators; Symbolical calculus; Invertibility theorem

Mots-clés : Opérateurs pseudo-différentiels ; Calculs des symboles ; Théorème d’inversibilité

1. Introduction

After Malgrange’s thèse [13], many authors have studied the generalized differential operator, including partial differential operators of infinite order or convolution operators.

Among them, for the “local”-operator, the theory of the pseudo-differential operator, especially the study of the micro-differential operators, of Sato et al. [14] presented a quite general possibility for the study of the partial differential equations. In this direction, T. Aoki [1–3] developed the symbol calculus of the pseudo-differential operators and studied the invertibility of a micro-differential operator. Using this theory, Aoki obtained in [4], the existence and continuation theorem for a differential equation of infinite order.

* Corresponding author.
E-mail address: ishimura@math.s.chiba-u.ac.jp (R. Ishimura).
As for the non-local operator, for example, in [7,8], we established a general criterion for the analytic continuation of a convolution equation defined in the complex domain, using the method developed by Kiselman [11] (see also Sébbar [15]).

In this paper, we insist that the theory developed by Aoki is still powerful for the study of “non-local” equations. At first, we define the non-local pseudo-differential operator and then we will represent them by their symbols. This make us possible to consider the non-local pseudo-differential equations concretely. As an example, we will establish the integral representation formula for a non-local micro-differential operators by means of its symbol. Next, we also develope the symbol calculus as Aoki and finally, we will obtain the invertibility theorem for some of these operators.

2. Notations and recall

In this paper, we set $X := \mathbb{C}^n$. Let $O_X$ be the sheaf of holomorphic functions on $X$. We define the sphere at infinity $S^{2n-1}_\infty$ by $(\mathbb{C}^n \setminus \{0\})/\mathbb{R}^+$ and we consider the compactification by directions $D^{2n} := \mathbb{C}^n \cup S^{2n-1}_\infty$ of $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. For $\xi \in \mathbb{C}^n \setminus \{0\}$, we denote by $\xi \infty \in S^{2n-1}_\infty$ the class defined by $\xi$, that is,$$
ixi := (\mathbb{R}^+ \cdot \xi \in D^{2n}) \cap S^{2n-1}_\infty.
$$

For a subset $A \subset X$, we will also denote:

$$A\infty := \{\xi \infty \in S^{2n-1}_\infty \mid \mathbb{R}^+ \cdot \xi \subset A\}.$$  

By the same way, for a subset $\Omega \subset S^*X \simeq X \times S^{2n-1}_\infty$, we will also denote:

$$\Omega\infty := \{(z, \xi \infty) := (z, \xi \infty) \in S^*X \mid \{z\} \times (\mathbb{R}^+ \cdot \xi) \subset \Omega\}.$$  

In this paper, for any compact convex set $M \subset X$, we will make use of two types of supporting functions:

For any $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{C}^n$, 

$$H_M(\xi) := \sup_{z \in M} \text{Re}(z, \xi),  \quad (2.1)$$

$$I_M(\xi) := \inf_{z \in M} \text{Re}(z, \xi), \quad (2.2)$$

where $\langle z, \xi \rangle := \sum_{j=1}^n z_j \xi_j$ with $z = (z_1, z_2, \ldots, z_n)$, which will be also denoted by $z \cdot \xi$.

Also, for an open set $D \subset X$, we will often write, for the simplicity, $\tilde{D} := D + M$. For a set $C \subset \mathbb{C}^n$, let $C^0$ be the polar set $\{z \in X \mid \text{Re}(z, \xi) > 0 \text{ for any } \xi \in C\}$ and for a set $A \subset X$ we denote by $A^\circ = -A$ its anti-podal $\{-z \mid z \in A\}$. In this paper, we will denote $A + B := \{a + b \mid a \in A, b \in B\}$ and $A - B := \{a - b \mid a \in A, b \in B\}$ for any $A, B \subset X$. 
3. Definition of non-local pseudo-differential operators and some operations

Let \( M \subset X := \mathbb{C}^n \) be a compact convex set and \( p = (x_0, \xi_0) \in T^*X \). We set \( \Delta_M := \{(x, y) \in X \times X \mid z := y - x \in M \} \). \( \Delta_M \) will be also denoted by \( \Delta \). For any proper closed convex cone \( G \subset X \) with vertex at 0 such that \( G \subset \{\xi_0\}^{0u} \cup \{0\} \) and any \( \epsilon > 0 \), we set \( G_\epsilon = G_\epsilon(\xi_0) := (G + \epsilon\xi_0) \cap \{\xi_0\}^{0u} \), where, as defined in the preceding section, \( \{\xi_0\}^{0u} \) is the anti-podal of the polar set of the direction \( \xi_0 \), that is, \( \{\xi_0\}^{0u} := \{z \in X \mid \text{Re}(z, \xi_0) < 0\} \). For one of such \( G_\epsilon \), let \( S \subset X \times X \) be a closed set such that we have:

\[
C_\Delta(S) \subset X \times G_\epsilon, 
\]

where \( C_B(A) \) is the normal cone of \( A \subset X \) along the sub-manifold \( B \), which is a closed conic set in \( T_BX \) (see [10, Chapter IV]). For a typical example of \( S \), let \( \Gamma \subset X \) be a proper convex cone with vertex at 0 such that \( \Gamma \subset \{\xi_0\}^{0u} \cup \{0\} \) and set \( S := \{(x, y) \in X \times X \mid z := y - x \in \Gamma \} \). We denote by \( q_{i,j} \) the projection: \( X \times X \times X \to X \times X \) by the \( i \)th and \( j \)th components: \( q_{i,j}(x_1, x_2, x_3) = (x_i, x_j) \). We recall that a closed set \( S \subset X \times X \) is said to be a proper ordering if it satisfies:

(i) \( S \supset \Delta \),
(ii) \( q_{1,3}(q_{1,2}^{-1}S \cap q_{2,3}^{-1}S) \subset S \),
(iii) \( q_{1,3}^{-1}S \cap q_{2,3}^{-1}S \xrightarrow{q_{1,3}} S \) is a proper map (Definition 3.1.1 of [9]).

**Definition 3.1.** Let \( M \subset X \) be a compact convex set. A closed set \( Z \subset X \times X \) is said to be a proper semi-ordering with support \( M \) if it satisfies the following conditions:

there is a proper ordering \( S \) such that we have:

(0) \( S + \Delta \subset S \);
(i) \( S + \Delta_M \subset Z \);
(ii) \( q_{1,3}(q_{1,2}^{-1}Z \cap q_{2,3}^{-1}Z) \subset Z + \Delta_M \);
(iii) \( q_{1,3}^{-1}Z \cap q_{2,3}^{-1}Z \xrightarrow{q_{1,3}} Z + \Delta_M \) is a proper map.

A closed set \( Z \subset X \times X \) is said to be a proper ordering with support \( M \) if it is the intersection of finite number of proper semi-orderings with support \( M \), that is, there are finite number of proper semi-orderings \( Z_i \) (\( 1 \leq i \leq N \)) with support \( M \) such that \( Z = \bigcap_{i=1}^{N} Z_i \).

Any open set \( W \subset X \) is said to be \( Z \)-open if \( \{y \in X \mid (x, y) \in Z, x \in W\} \subset W \), that is, denoting \( q_{i} \) the \( i \)th projection from \( X \times X \) to \( X \), we have:

\[
q_2(Z \cap q_1^{-1}W) \subset W.
\]

**Definition 3.2.** Let \( Z, Z' \) be two proper orderings with compact convex support \( M, M' \), respectively. We will denote

\[
Z' \circ Z := q_{1,3}(q_{1,2}^{-1}Z \cap q_{2,3}^{-1}Z').
\]
The pair $(Z, Z')$ will be said to be composable with respect to $(M, M')$ if $Z' \circ Z$ is a proper ordering with support $M + M'$ and the map $q_{13}: q_{1,2}^{-1}Z \cap q_{2,3}^{-1}Z' \to Z' \circ Z$ is proper.

**Definition 3.3.** Let $Z, Z' \subset X \times X$ be two proper orderings and $M, M' \subset X$ compact convex sets. An open set $D \subset X$ is said to be a $(Z, Z')$-round open set with respect to $(M, M')$ if $\{ y \in X \mid (x, y) \in Z, (y, z) \in Z', x \in D, z \in \tilde{D} := D + M + M' \} \subset \tilde{D}$, that is, denoting $q_i$ the $i$th projection from $X \times X \times X$ to $X$,

$$q_2(q_{1,2}^{-1}Z \cap q_{2,3}^{-1}Z' \cap q_1^{-1}(D + M + M')) \subset D + M. \quad (3.3)$$

We simply say that $D$ is $Z$-round with respect to $M$ if $M = M'$, $Z = Z'$.

In the case that $M = \{0\}$, this is just to say that $D$ is $Z$-round (see [9]). Recall, in the case of $Z := \{(x, y) \in X \times X \mid z := y - x \in \Gamma\}$, $D$ is $\Gamma$-round iff $(D + \Gamma) \cap (D - \Gamma) = D$ (see [9]).

**Definition 3.4.** Let $Z$ be a proper ordering with support $M$. $Z$ is said to be at the direction $\xi_0$ if there is a corresponding proper ordering satisfying (3.1). For a proper ordering $Z$ with support $M$ at $\xi_0$ and a $Z$-round open set $D$ with $x_0 \in D$, now we define:

$$\mathcal{E}(Z; D) := H^n_Z(D \times \tilde{D}, \mathcal{E}_X^{0,n}) \quad (3.4)$$

here $\mathcal{E}_X^{0,n}$ is the sheaf of holomorphic $(0, n)$-forms. We call any $P \in \mathcal{E}(Z; D)$ a non-local pseudo-differential operator defined near the point $p$ (or rather defined near the direction $\xi_0$) and carried by the compact convex set $M$. Remarking that if $Z_1 \subset Z_2$, we have the natural morphism $\mathcal{E}(Z_1; D) \mathcal{E}(Z_2; D)$, we define:

$$\mathcal{E}_M^{\mathbb{R}}(D)_{\xi_0} := \lim_{\longrightarrow Z} \mathcal{E}(Z; D), \quad (3.5)$$

where $Z$ runs over the family of proper orderings with support $M$ at the direction $\xi_0$. We also define:

$$\mathcal{E}_M^{\mathbb{R}}(D)_{\xi_0} := \bigcup_M \mathcal{E}_M^{\mathbb{R}}(D)_{\xi_0}.$$  

In the case of $Z := \{(x, y) \in X \times X \mid z := y - x \in \Gamma + M\}$, we denote $\mathcal{E}_M(\Gamma; D)$ instead of $\mathcal{E}(Z; D)$.

**Proposition 3.5.** Let $Z, Z'$ be two proper orderings with compact convex supports, respectively, $M, M'$ at $\xi_0$ such that $(Z, Z')$ is composable with respect to $(M, M')$. Set $Z'' := Z' \circ Z$, then $Z''$ is a proper ordering with support $M + M'$. Let $D$ be a $(Z, Z')$-round open set with respect to $(M, M')$. Then we have the natural morphism, which we call the composition map:

$$\mathcal{E}(Z; D) \otimes \mathcal{E}(Z'; \tilde{D}) \to \mathcal{E}(Z''; D). \quad (3.6)$$
Proof. We set $D := D + M + M'$. Then, by the cup-product, we have the following map:

$$
\mathcal{E}(Z;D) \otimes \mathcal{E}(Z';\widetilde{D}) = H^n_Z(D \times \widetilde{D}, \mathcal{O}^{(0,n)}_{X \times X}) \otimes H^n_Z(\widetilde{D} \times \widetilde{D}, \mathcal{O}^{(0,n)}_{X \times X})
$$

$$
\rightarrow H^{2n}_{q_{1,2}\widetilde{Z} \cap q_{2,3}^{\ast}Z}(D \times \widetilde{D} \times \widetilde{D}, \mathcal{O}^{(0,n,n)}_{X \times X \times X}).
$$

Recalling Definition 3.2, we may use Proposition 3.1.4 of [9] and we have:

$$
H^2_{q_{1,2}\widetilde{Z} \cap q_{2,3}^{\ast}Z}(D \times \widetilde{D} \times \widetilde{D}, \mathcal{O}^{(0,n,n)}_{X \times X \times X}) \rightarrow H^{2n}(D \times \widetilde{D}, Rq^{(0,n,n)}_{1,3} \mathcal{O}^{(0,n,n)}_{X \times X \times X}).
$$

Finally, the integration along fibers $\int_{q_{1,3}} \mathrm{d}x_{2}$ gives:

$$
H^{2n}_{q_{1,2}\widetilde{Z} \cap q_{2,3}^{\ast}Z}(D \times \widetilde{D} \times \widetilde{D}, \mathcal{O}^{(0,n,n)}_{X \times X \times X}) \rightarrow H^{2n}(D \times \widetilde{D}, Rq^{(0,n,n)}_{1,3} \mathcal{O}^{(0,n,n)}_{X \times X \times X}) = \mathcal{E}(Z'\cup D).
$$

For any $P \in \mathcal{E}(Z;D)$ and $Q \in \mathcal{E}(Z';\widetilde{D})$, we denote by $Q \circ P \in \mathcal{E}(Z'';D)$ the image by the morphism of the proposition and call it the composition of $P$ and $Q$.

A non-local pseudo-differential operator may operate on holomorphic functions:

**Proposition 3.6.** For a proper ordering $Z$ with compact convex support $M$ at $\xi_0$, let $D$ be a $Z$-round open set in $X = \mathbb{C}^n$. Let $W, W_0 \subset X$ be two $Z$-open sets with $W_0 \subset W$ and $W \setminus W_0 \subset D$. Then, for any $k$, we have the natural morphism:

$$
\mathcal{E}(Z;D) \otimes H^k_{(W \setminus W_0) + M}((D + M) \cap (W + M), \mathcal{O}_X) \rightarrow H^k_{W \setminus W_0}(W, \mathcal{O}_X). \tag{3.7}
$$

**Proof.** We denote, for the simplicity, $\widetilde{D} := D + M$ and $T := W \setminus W_0$. Let $q_i$ be the $i$th projection from $X \times X$ to $X$. Set $W := q_2^{-1}(T + M) \cap Z$. The by the cup-product, we have:

$$
\mathcal{E}(Z;D) \otimes H^k_{T + M}(\widetilde{D} \cap (W + M), \mathcal{O}_X)
$$

$$
= H^n_Z(D \times \widetilde{D}, \mathcal{O}^{(0,n)}_{X \times X}) \otimes H^k_{T + M}(\widetilde{D} \cap (W + M), \mathcal{O}_X)
$$

$$
\rightarrow H^{n+k}_{W}(D \times (\widetilde{D} \cap (W + M)), \mathcal{O}^{(0,n)}_{X \times X}).
$$

As in the proof of the preceding proposition, we may easily see that $q_1^{-1}(T) \cap W$ is an open set in $W$ and the projection $q_1^{-1}(T) \cap W \rightarrow T$ is a proper map. Then we can apply Proposition 3.1.4 of [9] and we have:

$$
H^{n+k}_{W}(D \times (\widetilde{D} \cap (W + M)), \mathcal{O}^{(0,n)}_{X \times X}) \rightarrow H^{n+k}_{T}(D, Rq_1^{(0,n)} \mathcal{O}^{(0,n)}_{X \times X})
$$

$$
= H^{n+k}_{T}(D \cap W, Rq_1^{(0,n)} \mathcal{O}^{(0,n)}_{X \times X}).
$$

By the integration along the fibers of $q_1$, we have:

$$
H^{n+k}_{T}(D \cap W, Rq_1^{(0,n)} \mathcal{O}^{(0,n)}_{X \times X}) \rightarrow H^k_T(D \cap W, \mathcal{O}_X) = H^k_T(W, \mathcal{O}_X). \tag{
\diamondsuit}
$$
In particular, for the case $k = 1$, we have:

**Corollary 3.7.** In the situation of the proposition, let $P \in \mathcal{E}(Z; D)$. Then we have the operation

$$P : \mathcal{O}(D \cap W_0 + M) / \mathcal{O}(D \cap W + M) \to \mathcal{O}(D \cap W_0) / \mathcal{O}(D \cap W).$$

(3.8)

As a typical and the most important example of preceding discussion is the following case: let $\Gamma_1, \Gamma_2, \ldots, \Gamma_N$ be proper convex closed cones in $X$ with vertex at 0. Set

$$Z = Z \cap \bigcap_{l=1}^N (\Gamma_l + M) := \{(x, y) \in X \times X \mid y - x \in \bigcap_{l=1}^N (\Gamma_l + M)\}.$$

In fact, every $Z_l := \{(x, y) \in X \times X \mid y - x \in \Gamma_l + M\}$ (1 \leq l \leq N), is a proper semi-ordering with support $M$ and thus $Z = \bigcap_{l=1}^N Z_l$ is a proper ordering with support $M$, in the sense of Definition 3.1. (We remark, in general, $(\bigcap_{l=1}^N \Gamma_l) + M \subsetneq \bigcap_{l=1}^N (\Gamma_l + M)$.) In this case, the notion of $Z$-open or $Z$-round is simply renamed as $\bigcap_{l=1}^N (\Gamma_l + M)$-open or $\bigcap_{l=1}^N (\Gamma_l + M)$-round and for an open $\bigcap_{l=1}^N (\Gamma_l + M)$-round set $D$, we denote:

$$\mathcal{E}_M(\Gamma_1, \Gamma_2, \ldots, \Gamma_N; D) := \mathcal{E}(Z; D).$$

(3.9)

We may easily verify that this is a special case of non-local pseudo-differential operators defined in Definition 3.4. Also we verify easily that all the hypothesis of Propositions 3.5 and 3.6 are also valid in this case. Among these circumstances, we will concentrate ourselves mainly in some special case as follows. In the rest of this section, we suppose that $D$ is a convex set. Then $\tilde{D} = D + M$ is also a convex open set. Taking a coordinate on $X$, we can calculate (3.9) by means of Čech cohomology. By a rotation of coordinate, we may assume $\xi_0 = (1, 0, \ldots, 0)$. For any $\delta > 0$, set the cones

$$\Gamma_1, \delta \Gamma_1 := \{z \in X \mid \delta |\text{Im} z_1| \leq - \text{Re} z_1\},$$

$$\Gamma_j, \delta \Gamma_j := \{z \in X \mid \delta |z_j| \leq |z_1|\} \quad (2 \leq j \leq n),$$

and we calculate $\mathcal{E}_M(\Gamma_1, \Gamma_2, \ldots, \Gamma_n; D)$. Let take the following holomorphically convex domains:

$$V_1 := D \times \tilde{D} \setminus \{(x, y) \mid z = y - x \in (\Gamma_1 + M)\},$$

$$V_j := D \times \tilde{D} \setminus \{(x, y) \mid z = y - x \in (\Gamma_j + M)\} \quad (2 \leq j \leq n).$$

(3.10) (3.11) (3.12) (3.13)
is an open covering of \( D \times \tilde{D} \setminus Z \). Let denote \( V := \bigcap_{k=1}^{n} V_k \) and \( V_k := \bigcap_{l \neq k} V_l \) for \( k = 1, 2, \ldots, n \). \( V = D \times \tilde{D} \setminus \bigcup_{k=1}^{n} \{(x, y) \mid z = y - x \in (I_k + M) \} \) (2 \( \leq j \leq n \)). We easily see that

\[
H^2(Z; D \times \tilde{D}, \mathcal{O}^{(0,n)}_{X \times X}) \cong H^{n-1}(D \times \tilde{D} \setminus Z, \mathcal{O}^{(0,n)}_{X \times X}) \cong \frac{\mathcal{O}^{(0,n)}_{X \times X}(V)}{\sum_{k=1}^{n} \mathcal{O}^{(0,n)}_{X \times X}(V_k)}.
\] (3.14)

So, for any \( P \in \mathcal{E}_M(\Gamma_1, \Gamma_2, \ldots, \Gamma_n; D) \), it corresponds a representing \((0,n)\)-form \( K(x, z) dy = K(x, y) dy \in \mathcal{O}^{(0,n)}_{X \times X}(V) \) determined uniquely, modulo the denominator \( \sum_{k=1}^{n} \mathcal{O}^{(0,n)}_{X \times X}(V_k) \). We call \( K(x, y) dy \) the kernel of \( P \).

**Remark.** We remark that, in the situation of the Corollary 3.7, the operation of \( f(x) \in \mathcal{O}((D \cap W_0) + M) \), which is determined modulo \( \mathcal{O}(D \cap W) \), is given by:

\[
Pf(x) = \int_{\gamma + \{x\}} K(x, y) f(y) \, dy,
\] (3.15)

where the integral path \( \gamma \) will be defined explicitly in the next section (see (4.4)).

**Remark.** We remark in the above situation, \( \bigcap_{k=1}^{n} (I_k + M) \supset \bigcap_{k=1}^{n} I_k + M \) and therefore \( V \subset D \times \tilde{D} \setminus \{(x, y) \mid z = y - x \in (\bigcap_{k=1}^{n} I_k + M) \} \).

**Remark.** If \( \Gamma_1, \Gamma_2, \ldots, \Gamma_n \) and \( \Gamma'_1, \Gamma'_2, \ldots, \Gamma'_n \) be two classes of cones of the form (3.10)–(3.11) and \( M, M' \) two compact convex sets. Let consider \( Z := \bigcap_{k=1}^{n} (I_k + M) \) and \( Z' := \bigcap_{k=1}^{n} (I'_k + M') \). We set also \( Z'' := \bigcap_{k=1}^{n} (I_k + I'_k + M + M') \). Then we have:

\[
Z' \circ Z = Z_{\bigcap_{k=1}^{n} I_k + I'_k + M + M'} \subset Z'' = Z_{\bigcap_{k=1}^{n} I_k + I'_k + M + M'}.
\] (3.16)

Then we have the composition map

\[
\mathcal{E}_M(\Gamma_1, \Gamma_2, \ldots, \Gamma_n; D) \otimes \mathcal{E}_M(\Gamma'_1, \Gamma'_2, \ldots, \Gamma'_n; \tilde{D}) \rightarrow \mathcal{E}_{M+M'}(\Gamma_1 + \Gamma'_1, \Gamma_2 + \Gamma'_2, \ldots, \Gamma_n + \Gamma'_n, D)
\] (3.17)
as the composition of maps

\[
\mathcal{E}_M(\Gamma_1, \Gamma_2, \ldots, \Gamma_n; D) \otimes \mathcal{E}_M(\Gamma'_1, \Gamma'_2, \ldots, \Gamma'_n; \tilde{D}) \rightarrow \mathcal{E}(Z; D) \otimes \mathcal{E}(Z'; D) \rightarrow \mathcal{E}(Z''; D)
\]

\[
= \mathcal{E}_{M+M'}(\Gamma_1 + \Gamma'_1, \Gamma_2 + \Gamma'_2, \ldots, \Gamma_n + \Gamma'_n; D).
\]

here the first map is that of Proposition 3.5 and the second one is the natural map, counting account (3.4) and (3.16) in mind. In particular, we have:
\[ E_M(\Gamma_1, \Gamma_2, \ldots, \Gamma_n; D) \otimes E_M'(\Gamma_1, \Gamma_2, \ldots, \Gamma_n; \tilde{D}) \rightarrow E_{M+M'}(\Gamma_1, \Gamma_2, \ldots, \Gamma_n; D), \]

where of course \( \tilde{D} = D + M \).

4. Symbols of non-local pseudo-differential operators

Now we will make another interpretation of a non-local pseudo-differential operator by means of its symbol. Let \( \Omega \) be a conique neighbourhood of \( p = (x_0, \xi_0) \in T^*X \). For any \( r > 0 \), we will denote \( \Omega(r) = \{(x, \xi) \in \Omega \mid |\xi| > r \} \) and \( \Omega[r] = \{(x, \xi) \in \Omega \mid |\xi| \geq r \} \).

**Definition 4.1** (cf. [1]). For the compact convex set \( M \), set:

\[ S^M(\Omega) := \{ P(x, \xi) \in O(\Omega(r)) \mid \text{(with some } r > 0) \mid \text{for any conic set } \Omega' \subseteq \Omega, \text{ any } r' > r \text{ and any } \epsilon > 0, \text{ there exists } C_\epsilon > 0 \text{ such that we have} \]

\[ |P(x, \xi)| \leq C_\epsilon e^{H_M(\xi) + \epsilon r'|\xi|} \text{ (for every } (x, \xi) \in \Omega'[r']) \} \] (4.1)

and

\[ N^M(\Omega) := \{ P(x, \xi) \in S^M(\Omega) \mid P(x, \xi) \text{ being } \in O(\Omega(r)), \text{ for any conic set } \Omega' \subseteq \Omega \text{ and any } r' > r \text{, there exist } \epsilon_0 > 0 \text{ and } C > 0 \text{ such that we have} \]

\[ |P(x, \xi)| \leq C e^{H_M(\xi) - \epsilon_0 |\xi|} \text{ (for every } (x, \xi) \in \Omega'[r']) \} \] (4.2)

We remark that in the case of \( M = \{0\} \), we have \( S^{\{0\}}(\Omega) = S(\Omega) \) and \( N^{\{0\}}(\Omega) = N(\Omega) \) (see Aoki [1]). In this section, we use the situation of the last of the preceding section:

\[ \Gamma_{1,\delta} = \Gamma_1 := \{ z \in X \mid |\delta| |Im z_1| \leq -\text{Re} z_1 \}, \]

\[ \Gamma_{j,\delta} = \Gamma_j := \{ z \in X \mid |\delta| |z_j| \leq |z_1| \} \quad (2 \leq j \leq n), \]

and set for the simplicity \( G(M) := \bigcap_{k=1}^n (\Gamma_k + M) \). In the notations of the preceding section, take a non-local pseudo-differential operator \( P \in E_M(\Gamma_1, \Gamma_2, \ldots, \Gamma_n; D) \) with convex \( G(M) \)-round open set \( D \) and take one of its kernel \( K(x, y) \) dy defined by (3.14). Thanks of the \( G(M) \)-roundness, we see that \( D \) satisfies the following condition.

Let \( \pi_k \) be the \( k \)th projection on \( \mathbb{C}^n \) to \( \mathbb{C} \). Set \( A_k := \pi_k(\{ z \in \mathbb{C}^n \mid \text{Re}(z, \xi_0) \geq I_M(\xi_0) \} \cap G(M)), B_1 := \pi_1(G(M)) \) and \( C_j := \text{conv} A_j \) (\( 2 \leq j \leq n \)) where \( \text{conv} E \) is the convex-hull of a set \( E \subset \mathbb{C} \). We remark that

\[ A_1 = \{ z_1 = w_1 + x_1 \mid |\delta| |\text{Im} w_1| \leq -\text{Re} w_1, x \in M \} \cap \{ z \in \mathbb{C}^n \mid \text{Re}(z, \xi_0) \geq I_M(\xi_0) \}. \]
Now the assumption: there exists an open set $U_0 \subset D$, $U_0 \neq \emptyset$ such that we have:

$$\left[ \partial A_1 \cap \partial B_1 \right] \times \partial C_1 \times \cdots \times \partial C_n + U_0 \subset \tilde{D} = D + M,$$  \hspace{1cm} (4.3)

where $\partial C$ is the boundary set of a set $C \subset \mathbb{C}$. So the set $\{ \cdot \}$ in the left-hand side of (4.3) is a sub-set of the Shirov boundary of the product set $A_1 \times A_2 \times \cdots \times A_n$.

We shall correspond to the kernel $K(x, y) \, dy$, a symbol $P(x, \xi) \in S^M(\Omega)$ ($\Omega$ to be determined). To this purpose, let define an integral path $\gamma$ in $X$ such that for an open set $U \subseteq U_0$, $U \neq \emptyset$, the set $\{(x, x + z) \mid x \in U, \ z \in \gamma\}$ is included in the domain $V = \bigcap_{k=1}^n V_i$ where $K(x, y) \, dy$ is defined; there are two points $a, b \in \mathbb{C}$ near $\partial A_1$ such that $\text{Re} \, a, \text{Re} \, b < \inf_{\xi \in M} \text{Im} \, z_1, \text{Im} \, b > \sup_{\xi \in M} \text{Im} \, z_1$. Let take an oriented smooth Jordan path $\gamma_1 \subset \mathbb{C}$ having $a_1$ as the start point and $b_1$ as the terminal point so that $\tilde{\gamma}_1 := \{ (x, x + (z_1, z_2, \ldots, z_n)) \} \subset U \times \tilde{D} \mid z_1 \in \gamma_1, z_j \in \mathbb{C} \ (2 \leq j \leq n) \} \subset V_1$. For $z_1 \in \gamma_1$, let $\gamma_j = \gamma_j(z_1) \subset \mathbb{C}$ be an oriented smooth Jordan closed curve with positive orientation in $\mathbb{C}$ so that

$$\tilde{\gamma}_j := \{ (x, x + (z_1, z_2, \ldots, z_n)) \} \subset U \times \tilde{D} \mid z_j \in \gamma_j, z_i \in \mathbb{C} \ (i \neq j, j) \} \subset V_j.$$

(These are possible if we take $U$ small enough.) We remark that by the assumption (4.3), $\gamma_j$ may be taken independently of $z_1$. Setting $\gamma := \gamma_1 \times \gamma_2 \times \cdots \times \gamma_n$, now we define the symbol of $P$ as follows:

$$\sigma(P)(x, \xi) := \int\limits_{\gamma} e^{z \xi} K(x, x + z) \, dz.$$

\hspace{1cm} (4.4)

**Proposition 4.2.** In the preceding situation, we may suppose (4.3). We assume also $\xi_0 = (1, 0, \ldots, 0)$. Then there exists a conic neighbourhood $\Omega$ of $p := (x_0, \xi_0)$ such that $U \subset \pi(\Omega)$, where $\pi : T^*X \rightarrow X$ is the projection, and we have the following: for any $P \in \mathcal{E}_M(\Gamma_1, \Gamma_2, \ldots, \Gamma_n; D)$ and any kernel $K(x, y) \, dy \in \mathcal{O}_{X \times X}(V)$ of $P$, the symbol $\sigma(P)$ of $P$ defined by (4.4) is contained in $S^M(\Omega)$. Besides, the symbol $\sigma(P)$ is uniquely determined in $S^M(\Omega)$ modulo $N^M(\Omega)$.

**Proof.** The fact that $\sigma(P)$ satisfies (4.1) at $p = (x_0, \xi_0)$ is obvious and by a rotation of variables and by the fact that $\overline{\Omega} \cap S^{2n-1}_{\infty}$ is compact, we see $\sigma(P) \in S^M(\Omega)$. By the definition of $\gamma$, especially by the choice of $\gamma_1$, if $K(x, y) \, dy \in \mathcal{O}_{X \times X}(V_1)$, we have $\sigma(P) \in N^M(\Omega)$ and if $K(x, y) \, dy \in \mathcal{O}_{X \times X}(V_j)$ for some $j \ (2 \leq j \leq n)$, we have $\sigma(P) = 0$. \hfill $\Box$

Next we will prove that to any symbol $P(x, \xi) \in S^M(\Omega)$, we may associate a kernel $K(x, y) \, dy$ of a non-local pseudo-differential operator $P$ at the direction $\xi_0$ and this correspondence and that of the preceding proposition are inverse to each other.

Let $U \subset U_0$ be an open set and $\omega$ an open cone with vertex at 0 with $\omega \ni \xi_0$ and $U \times \omega \subset \Omega$. For any $\xi \in \omega$, set the open half-space $H_{\xi} := \{ z \in X \mid \text{Re} \{ z, \xi \} > H_M(\xi) \}$. 

For any \( z \in \mathcal{H}_\xi \), there is \( \delta > 0 \) so that \( \text{Re}(z, \xi) > H_M(\xi) + \delta|\xi| \). Then for any \( x \in U \) and \( \rho \in \mathbb{C} \) so that \( |\rho| > r \) (\( r \) being appeared in (4.1)) and \( \rho \xi \in \omega \), we set:

\[
L_\rho(x, z, \xi) := \int_\rho^\infty e^{-\tau z \cdot \xi} z^{n-1} P(x, \tau \xi) \, d\tau,
\]

where the path of integration is taken in the direction \( \rho \). By (4.1), if we take \( U \) small enough, this integral is absolutely and compactly convergent for \( (x, z, \xi) \in U \times \{(w, \zeta) \in X \times \omega \mid \text{Re}(w, \zeta) > H_M(\zeta)\} \). Now by a rotation of coordinate, we may assume \( \xi_0 = (1, 0, \ldots, 0) \). For \( \eta' = (\eta_1, \eta_2, \ldots, \eta_n) \) with \( |\eta'| < 1 \), set \( \eta := (1, \eta') := (1, \eta_1, \eta_2, \ldots, \eta_n) \).

For any small \( \varepsilon > 0 \) and \( z \) with \( z_j \neq 0 \) \((2 \leq j \leq n)\), we define the \((n - 1)\)-chain,

\[
\beta := \left[0, \frac{\varepsilon}{z_2}\right] \times \left[0, \frac{\varepsilon}{z_3}\right] \times \cdots \times \left[0, \frac{\varepsilon}{z_n}\right],
\]

where \( [0, w] := \{cw \mid 0 \leq c \leq 1\} \) for \( w \in \mathbb{C} \). Now we define the kernel of \( P(x, \xi) \) by the formula:

\[
K_\rho(x, y) := \frac{1}{(2\pi \sqrt{-1})^n} \left[ \int_\beta L_\rho(x, z, \eta) \, d\eta \right]_{z=x-y}
\]

with \( \eta = (1, \eta') \). By a translation, we may also assume \( 0 \in M \) and so for any \( z \notin \bigcup_{k=1}^n (\Gamma_k + M) \), we have \( z_j \neq 0 \) for \( 2 \leq j \leq n \). Remarking that a small oscillation of the integral direction \( [\rho, \infty[ \) in (4.5) gives an analytic continuation, we see easily that for \( (x, y) \in V \) for some \( \delta > 0 \) in the preceding notations, \( K_\rho(x, y) \), which will be now denoted by \( K(x, y) \), is holomorphic. We see easily that from \( P(x, \xi) \), the kernel \( K(x, y) \) is in fact defined uniquely modulo \( \sum_{k=1}^n \mathcal{O}_{X \times X}(V_k^*) \), because for \( \rho' \) another choice of \( \rho \) close to \( \rho \), we have:

\[
K_\rho(x, y) - K_{\rho'}(x, y) = \frac{1}{(2\pi \sqrt{-1})^n} \left[ \int_\beta \left( \int_{\rho'}^{\rho} e^{-\tau z \cdot \xi} z^{n-1} P(x, \tau \eta) \, d\tau \right) \, d\eta \right]_{z=x-y}
\]

\[
\in \sum_{k=1}^n \mathcal{O}_{X \times X}(V_k^*).
\]

If \( P(x, \xi) \in H^M(\Omega) \), taking (4.2) into account, we see also easily that there is \( \delta > 0 \) such that, taking \( D \) smaller so that \( D \) satisfying (4.3), we have \( K(x, y) \in \sum_{k=1}^n \mathcal{O}_{X \times X}(V_k^*) \). Thus we have proved:

**Proposition 4.3.** For any conic open neighbourhood \( \Omega \subseteq T^*X \) of \( p = (x_0, \xi_0) \), \( \xi_0 \) being \((1, 0, \ldots, 0)\), there are cones \( \Gamma_1, \Gamma_2, \ldots, \Gamma_n, \delta \) of the form defined in (3.10) and (3.11) with some \( \delta > 0 \) and a convex \( \bigcap_{k=1}^n (\Gamma_k + M) \)-round open set \( D \) satisfying (4.3) with a
non-empty open set \( U_0 \subset D \) such that to any symbol \( P(x, \xi) \in SM(\Omega) \), we can associate the kernel \( K(x, y) \) by the following formula: setting \( \xi := (1, \eta') \),

\[
K(x, y) := \frac{1}{(2\pi \sqrt{-1})^n} \int_\beta \int_\rho \int_\gamma \int_\rho e^{-\tau(y-x) \cdot \eta} \tau^{n-1} P(x, \tau \xi) \, d\tau \, d\eta' \, d\rho \, d\gamma
\]

(4.8)

Taking \( \rho \) and \( \bar{\beta} \) as in (4.5) and (4.6), and this \( K(x, y) \) is uniquely determined by \( P(x, \xi) \) in \( O_{X \times X}(V) / \sum_{k=1}^n O_{X \times X}(V_k) \). Thus any symbol \( P(x, \xi) \) defines a non-local pseudo-differential operator \( P \in \mathcal{E}_M(\Gamma_1, \Gamma_2, \ldots, \Gamma_n; D) \). Furthermore this \( P \) is equal to 0 if the symbol \( P(x, \xi) \) is contained in \( NM(\Omega) \).

Now we prove that the correspondences defined in Propositions 4.2 and 4.3 are inverse to each other: for \( K(x, y) \, dy \in O^{(0, n)}_{X \times X}(V) \) and \( P(x, \xi) \in SM(\Omega) \), we will prove:

(I) \[
\frac{1}{(2\pi \sqrt{-1})^n} \int_\beta \int_\rho \int_\gamma \int_\rho e^{-\tau(y-x) \cdot \eta} \tau^{n-1} K(x, x + w) \, d\tau \, d\eta' \, d\rho \, d\gamma \equiv K(x, y) \text{ modulo } \sum_{k=1}^n O_{X \times X}(V_k),
\]

(4.9)

(II) \[
\frac{1}{(2\pi \sqrt{-1})^n} \int_\gamma e^{z \cdot \xi} \, dz \int_\beta \int_\rho \int_\rho e^{-\tau z \cdot \eta} \tau^{n-1} P(x, \tau \eta) \, d\tau \, d\eta' \, d\rho \equiv P(x, \xi) \text{ modulo } NM(\Omega)
\]

(4.10)

with \( \eta := (1, \eta') \). Where \( D, \delta > 0 \) or \( \Omega \) are possibly taken smaller.

**Proof of (4.9).** In the sequel, for \( (x, y) \in V \), pose \( z := y - x \). Take \( \gamma_k \) near \( \partial \Gamma_k \). For \( z \) so that \( (z_1 - \gamma_1) \cap (\Gamma_1 + M) = \emptyset \) and \( z_j \) being in the exterior of the domain circled by \( \gamma_j \) (\( 2 \leq j \leq n \)), \( z \) being fixed, taking \( \rho \) in somewhat different direction from 1, we may assume that for any \( \tau \in [\rho, \infty] \) and any \( w_1 \in \gamma_1 \), we have \( \text{Re} \, \tau (z - w_1) > 0 \). So, in the definition of \( \bar{\beta} \), if we take \( \varepsilon > 0 \) small enough, then for any \( \eta' \in \bar{\beta} \) and \( \rho \in \gamma, \) we have \( \text{Re} \, \tau (z - \bar{w}) \cdot \eta = \text{Re} \, \tau (z_1 - w_1) + \text{Re} \, \tau (z' - w') \cdot \eta' > 0 \). Then in (4.9), we can change the integration order:

\[
\frac{1}{(2\pi \sqrt{-1})^n} \int_\beta \int_\rho \int_\gamma \int_\rho e^{-\tau(y-x) \cdot \eta} \tau^{n-1} K(x, x + w) \, d\tau \, d\eta' \, d\rho \, d\gamma
\]

\[
= \frac{1}{(2\pi \sqrt{-1})^n} \int_\beta \int_\rho \int_\gamma K(x, x + w) \, dw \int_\rho e^{-\tau(z-w) \cdot \eta} \tau^{n-1} \, d\tau.
\]

(4.11)
By a direct calculation, we have then
\[
\int_0^\infty \frac{e^{-\tau(z-w) \cdot \eta} \tau^{n-1} d\tau}{\rho} = \frac{(n-1)!}{((z-w) \cdot \eta)^n} + O((z-w) \cdot \eta)^0).
\]

Now for \(k = 1, 2, \ldots, n\), take small circles \(\gamma_k \subset \mathbb{C}\) around \(z_k\) with positive orientation and pose \(\gamma^0 := \gamma_1 \times \gamma_2 \times \cdots \times \gamma_n\). Setting
\[
\alpha := (\gamma_1 + \gamma_1^\circ) \times (\gamma_2 + \gamma_2^\circ) \times \cdots \times (\gamma_n + \gamma_n^\circ)
- \gamma_1^\circ \times (\gamma_2 + \gamma_2^\circ) \times \cdots \times (\gamma_n + \gamma_n^\circ) - \cdots - (\gamma_1 + \gamma_1^\circ) \times (\gamma_2 + \gamma_2^\circ) \times \cdots \times \gamma_n^\circ
+ \cdots + (-1)^{n-1}(\gamma_1 + \gamma_1^\circ) \times \gamma_2^\circ \times \cdots \times \gamma_n^\circ + \cdots
+ (-1)^n \gamma_1^\circ \times \cdots \times \gamma_{n-1}^\circ \times (\gamma_n + \gamma_n^\circ),
\]
we have:
\[
\gamma = \gamma_1 \times \gamma_2 \times \cdots \times \gamma_n
= ((\gamma_1 + \gamma_1^\circ) - \gamma_1^\circ) \times ((\gamma_2 + \gamma_2^\circ) - \gamma_2^\circ) \times \cdots \times ((\gamma_n + \gamma_n^\circ) - \gamma_n^\circ)
= \alpha + (-1)^n \gamma^0.
\]

On \(\alpha\), it is easy to verify that the integral
\[
\int_0^\infty \frac{e^{-\tau(z-w) \cdot \eta} \tau^{n-1} d\tau}{\rho} \equiv 0 \text{ modulo } \sum_{k=1}^n O_X \times (V_k).
\]

And because the integral \(\int_{\hat{\beta}} d\eta' \int_{\gamma^0} d\gamma \int_{\gamma^0} d\gamma' \int_{\gamma^0} d\gamma\) corresponding to the holomorphic part \(O((z-w) \cdot \eta)^0\) is \(= 0\), we have that (4.11) is equal to:
\[
\frac{(n-1)!}{(2\pi \sqrt{-1})^n} \int_{\hat{\beta}} d\eta' \int_{\gamma^0} \frac{K(x, x + w)}{((w-z) \cdot \eta)^n} dw \text{ modulo } \sum_{k=1}^n O_X \times (V_k).
\]

Then we have:
\[
\frac{(n-1)!}{(2\pi \sqrt{-1})^n} \int_{\hat{\beta}} d\eta' \int_{\gamma^0} \frac{K(x, x + w)}{((w-z) \cdot \eta)^n} dw
= \frac{(n-1)!}{(2\pi \sqrt{-1})^n} \int_{\hat{\beta}} K(x, x + w) dw \int_{\gamma^0} d\eta' \int_{\gamma^0} \frac{d\gamma'}{((w-z) \cdot \eta)^n}
\]
\[ = \frac{(n-1)!}{(2\pi \sqrt{-1})^n} \int_{\gamma'} K(x, x+w) \, dw \int_0^{z_{a1}^{-1}x} \frac{dz_{21}^{-1}x}{z_{21}^{-1}x} \int_0^{z_{21}^{-1}x} \frac{d\eta_2}{((w_1-z_1)+(w'-z') \cdot \eta)^n}. \]  

(4.12)

Because, setting \( u' := (u_3, \ldots, u_n) \) etc.,

\[ = \frac{1}{n-1} \frac{1}{w_2-z_2} \left[ \frac{1}{((w_1-z_1)+(w''-z'') \cdot \eta'')^{n-1}} \right] \]

and the first term of right-hand side is holomorphic at \( z_1 = 0 \), we have that (4.12) is equal to, by modulo \( \sum_{k=1}^n O_X \times X(V_k) \).

\[ \frac{(n-2)!}{(2\pi \sqrt{-1})^n} \int_{\gamma'} \frac{K(x, x+w)}{w_2-z_2} \, dw \int_0^{z_{a2}^{-1}x} \frac{dz_{22}^{-1}x}{z_{22}^{-1}x} \int_0^{z_{22}^{-1}x} \frac{d\eta_3}{((w_1-z_1)+(w'-z') \cdot \eta)^{n-1}}. \]

Continuing this procedure, (4.12) is equal to, by modulo \( \sum_{k=1}^n O_X \times X(V_k) \).

\[ \frac{1}{(2\pi \sqrt{-1})^n} \int_{\gamma'} \frac{K(x, x+w)}{(w_2-z_2) \cdots (w_n-z_n)} \frac{1}{w_1-z_1} \, dw \equiv K(x, x+z) \equiv K(x, y). \]

\[ \Box \]

**Proof of (4.10).** Let a point \( c \in M \) such that \( H_M(\xi_0) = \text{Re}(c, \xi_0) \). Then on \( \gamma_1 \), there is a point \( d \in \gamma_1 \) so that \( d - c_1 \in \mathbb{R}_+ \). We divide \( \gamma_1 \) into two parts:

\[ \gamma_1^+ := \{ z_1 \in \gamma_1 | \text{Im}(z_1 - c_1) \geq 0 \}, \]  

(4.13)

\[ \gamma_1^- := \{ z_1 \in \gamma_1 | \text{Im}(z_1 - c_1) < 0 \}, \]  

(4.14)

\[ \gamma_1 = \gamma_1^+ \cup \gamma_1^- . \]

If we take \( \delta > 0 \) smaller, we may take two directions \( \rho_+ \) and \( \rho_- \) near the direction 1 so that setting:
we have \( \Re z_1 \cdot \tau > \Re c_1 \cdot \tau = H_M(\tau \xi_0) \) for any \( z_1 \in \gamma_1^\pm, \tau \in \Sigma \pm \). We may assume that \( |\rho_+| = |\rho_-| \) and so with \( r_1 > r, -1 \ll \theta_- < 0 < \theta_+ \ll 1, \rho_+ = r_1 e^{\sqrt{-1} \theta_+}, \rho_- = r_1 e^{\sqrt{-1} \theta_-} \).

Set \( \Sigma_0 := \{ r_1 e^{i \theta} | \theta_- < \theta < \theta_+ \} \)

Putting \( \gamma' := \gamma_2 \times \cdots \times \gamma_n \), now (4.10) is equal to:

\[
\sum_{i=\pm} \frac{1}{(2\pi \sqrt{-1})^n} \int_{\gamma_1^\pm} e^{-z \cdot \xi} d\zeta \int_{\Sigma_i} e^{-\tau \cdot \eta} e^{n-1} P(x, \tau \eta) d\tau
\]

\[
= \frac{1}{(2\pi \sqrt{-1})^n} \sum_{i=\pm} \int_{\gamma_1^\pm} e^{-z \cdot \xi} d\zeta \int_{\Sigma_i} e^{-\tau \cdot \eta} e^{n-1} P(x, \tau \eta) d\tau
\]

If we consider by modulo \( N_M(\Omega) \), this is also equal to

\[
\equiv \frac{1}{(2\pi \sqrt{-1})^n} \sum_{i=0, \pm \Sigma_i} \int_{\gamma_1^\pm} e^{-z \cdot \xi} d\zeta \int_{\Sigma_i} e^{-\tau \cdot \eta} e^{n-1} P(x, \tau \eta) d\tau
\]

\[
= \frac{1}{(2\pi \sqrt{-1})^n} \sum_{i=0, \pm \Sigma_i} \int_{\gamma_1^\pm} e^{-z \cdot \xi} d\zeta \int_{\Sigma_i} e^{-\tau \cdot \eta} e^{n-1} P(x, \tau \eta) d\tau
\]

\[
(4.18)
\]

Here we have:

\[
\int_{\gamma_1^\pm} e^{-z_1 \cdot (\tau - \xi_1)} d\zeta_1 = \left[ \frac{1}{\tau - \xi_1} e^{-z_1 \cdot (\tau - \xi_1)} \right]_{z_1=a}^{z_1=d} = \frac{1}{\tau - \xi_1} e^{-a(\tau - \xi_1)} - \frac{1}{\tau - \xi_1} e^{-d(\tau - \xi_1)}
\]

and

\[
\int_{\Sigma_i} d\tau \int_{\gamma_1^\pm} e^{-\alpha(\tau - \xi_1) - z' \cdot (\tau \eta - \xi')} d\tau ' e^{n-1} e^{-a(\tau - \xi_1) - z' \cdot (\tau \eta - \xi')} d\eta ' \in N_M(\Omega).
\]

This is same for \( i = + \), so by the Cauchy integral formula with weight, we have that modulo \( N_M(\Omega) \), (4.18) is equal to:
Because \( \lambda(P(x,\xi)) \) is holomorphic on the conic neighbourhood \( \Omega(r) \) of \( \rho = (x_0, \xi_0) \) and \( \eta := (1,0,\ldots,0) \), taking \( \varepsilon > 0 \) small enough so that \( 0 < |z_j \eta_j| \ll 1 \) (for \( 2 \leq j \leq n \)), the function \( P(x, \xi \eta) = P(x; \xi_1, \xi_1 \eta_2, \ldots, \xi_1 \eta_n) \) is holomorphic at \( \eta' = 0 \). Then setting \( \eta'' := (\eta_3, \ldots, \eta_n) \), (4.19) is equal to:

\[
\frac{1}{(2\pi \sqrt{-1})^{n-1}} \int_{\gamma'_{n}} e^{\varepsilon z' \xi_{n}} \frac{1}{\tau - \xi_{1}} \int_{\gamma_{n}} \int_{\Sigma_{+} - \Sigma_{0} - \Sigma_{-}} \frac{P(x, \tau \eta)\tau^{n-1}}{\tau - \xi_{1}} e^{-\varepsilon z' \eta'} d\tau d\eta'\]

This last is equal to:

\[
\frac{\xi_{1}^{n-1}}{(2\pi \sqrt{-1})^{n-1}} \int_{\gamma'_{n}} d\xi_{n} \int_{\gamma_{n}} e^{-\varepsilon z_{2} \xi_{1} \eta_{2}} d\eta_{2}, (4.19)
\]

Because \( P(x, \xi) \) is holomorphic on the conic neighbourhood \( \Omega(r) \) of \( \rho = (x_0, \xi_0) \) and \( \eta := (1,0,\ldots,0) \), taking \( \varepsilon > 0 \) small enough so that \( 0 < |z_j \eta_j| \ll 1 \) (for \( 2 \leq j \leq n \)), the function \( P(x, \xi \eta) = P(x; \xi_1, \xi_1 \eta_2, \ldots, \xi_1 \eta_n) \) is holomorphic at \( \eta' = 0 \). Then setting \( \eta'' := (\eta_3, \ldots, \eta_n) \), (4.19) is equal to:

\[
\int_{\gamma'_{n}} e^{\varepsilon z_{2} \xi_{1} \eta_{2}} d\xi_{2} \int_{\gamma_{n}} e^{-\varepsilon z_{2} \xi_{1} \eta_{2}} P(x, \xi_{1} \eta_{2}, \xi_{1} \eta'') d\eta_{2}. (4.20)
\]

The change of variable \( \lambda := z_{2} \xi_{1} \eta_{2} \) gives:

\[
\frac{\xi_{1}}{2\pi \sqrt{-1}} \int_{\gamma_{2}} e^{\varepsilon z_{2} \xi_{2}} d\xi_{2} \int_{\gamma_{2}} e^{-\varepsilon z_{2} \xi_{1} \eta_{2}} P(x, \xi_{1}, \xi_{1} \eta_{2}, \xi_{1} \eta'') d\eta_{2}
\]

\[
= \frac{1}{2\pi \sqrt{-1}} \int_{\gamma_{2}} e^{\varepsilon z_{2} \xi_{2}} d\xi_{2} \int_{\gamma_{2}} e^{-\lambda} \frac{\xi_{1} e^{-\lambda}}{z_{2}} P(x, \xi_{1}, \frac{\lambda}{z_{2}}, \xi_{1} \eta'') d\lambda
\]

\[
= \frac{1}{2\pi \sqrt{-1}} \int_{\gamma_{2}} e^{-\lambda} d\lambda \int_{\gamma_{2}} e^{\varepsilon z_{2} \xi_{2}} \frac{P(x, \xi_{1}, \frac{\lambda}{z_{2}}, \xi_{1} \eta'')}{z_{2}} d\xi_{2}.
\]
If we make the change of variable \( w_2 := 1/z_2 \) and set a circle \( \gamma_2^{-1} := \{ w_2 \mid 1/w_2 \in \gamma_2 \} \) with the opposite direction to \( \gamma_2 \) and \( \tilde{\gamma}_2 := -\gamma_2^{-1} \), this is equal to:

\[
= \frac{1}{2\pi \sqrt{-1}} \int_{0}^{\xi_1 e} e^{-\lambda} \, d\lambda \oint_{\gamma_2^{-1}} w_2 P(x, \xi_1, \lambda w_2, \xi_1 \eta'') e^{\xi_2/w_2} \left( -\frac{dw_2}{w_2^2} \right)
\]

\[
= \frac{1}{2\pi \sqrt{-1}} \int_{0}^{\xi_1 e} e^{-\lambda} \, d\lambda \oint_{\tilde{\gamma}_2} P(x, \xi_1, \lambda w_2, \xi_1 \eta'') \sum_{i=0}^{\infty} \frac{\xi_i}{i!} w_2^{-i} \, dw_2
\]

\[
= \frac{1}{2\pi \sqrt{-1}} \sum_{i=0}^{\infty} \xi_i \frac{\xi_i}{(i!)^2} \int_{0}^{\xi_1 e} e^{-\lambda} \, d\lambda \oint_{\tilde{\gamma}_2} P(x, \xi_1, 0, \xi_1 \eta'') \, dw_2
\]

\[
= \sum_{i=0}^{\infty} \frac{\partial_i P(x, \xi_1, 0, \xi_1 \eta'')}{i!} \xi_i \frac{\xi_i}{(i!)^2} \int_{0}^{\xi_1 e} \lambda^i e^{-\lambda} \, d\lambda.
\]

Because, by the integrations by parts, we have \( \int_{0}^{\xi_1 e} \lambda^i e^{-\lambda} \, d\lambda \equiv i! \) modulo \( N^M(\Omega) \), this is equal to:

\[
= \sum_{i=0}^{\infty} \frac{\partial_i P(x, \xi_1, 0, \xi_1 \eta'')}{i!} \xi_i = P(x, \xi_1, \xi_2, \xi_1 \eta'').
\]

Repeating this procedure, finally we have (4.20) is equal to \( P(x, \xi) \) modulo \( N^M(\Omega) \), this proves the (4.10). \( \square \)

**Remark 4.4.** In the case of local pseudo-differential operators, the above correspondences were first defined by K. Kataoka and he proved that they have been inverse to each other (see [5]).

At the end of this section, we see the composition of two non-local pseudo-differential operators in terms of their symbols, that is:

**Proposition 4.5** (The Hörmander–Leibniz formula). Let \( M_1, M_2 \subset X \) be two compact convex sets, \( \Gamma_1, \ldots, \Gamma_n \) closed conic sets defined in (3.10)–(3.11) for some \( \delta > 0 \) and \( D \) a \( (Z^n \setminus (\Gamma_1 + M_1), Z^n \setminus (\Gamma_1 + M_2)) \)-round open set with respect to \( (M_1, M_2) \). Then for any \( P \in \mathcal{E}_{M_1}(\Gamma_1, \ldots, \Gamma_n; D + M_2) \) and any \( Q \in \mathcal{E}_{M_2}(\Gamma_1, \ldots, \Gamma_n; D) \), there exist respective symbols \( \sigma(P)(x, \xi) = P(x, \xi) \in S^{M_1}(\Omega_1 + (M_2 \times \{0\})) \) and \( \sigma(Q)(x, \xi) = Q(x, \xi) \in S^{M_2}(\Omega_2) \),
such that we have the following formula which is in $S^{M_1+M_2}(\Omega_1 \cap \Omega_2)$ and converging compactly in $\Omega_1 \cap \Omega_2$:

$$\sigma(Q \circ P) = \sum_{\alpha \in \mathbb{Z}_n^+} \frac{1}{\alpha!} \partial_\xi^\alpha Q(x, \xi) \cdot \partial_x^\alpha P(x, \xi). \quad (4.21)$$

**Proof.** We will work in the preceding notations. We may assume $\Omega_1 = \Omega_2 =: \Omega$. Let $L(x, y) dy \in O^{0, \rho_2(V^2)}(X)$ and $K(x, y) dy \in O^{0, \rho_1(V^1)}(X)$ be, respectively, a kernel of $P$ and $Q$ ($V^i$ being as before $\bigcap_{i=1}^{n} V_k^i$ where $V_k^i$ is defined by $\Gamma_k$, $M_1$ ($i = 1, 2$) and $D$ or $D + M_2$ in (3.12)–(3.13)). By Definition 3.3, we may take an integral path $\lambda$ as same type as $\gamma$ but for $\Gamma_k + M_2$ where $\gamma$ corresponds to $\Gamma_k + M_1 + M_2$. Then we have, modulo $NM_1 + M_2(\Omega)$:

$$\sigma(Q \circ P)(x, \xi) = \int \gamma e^{z \cdot \xi} d\gamma \int_L(x, w) K(w, x + z) dw$$

$$= \int \lambda L(x, w) dw \int \gamma \frac{e^{(u+w-x) \cdot \xi}}{(x-u)} K(w, w + u) du$$

$$\equiv \int \gamma L(x, w) dw \int \frac{e^{u \cdot \xi}}{\gamma} K(w, w + u) du$$

$$= \int \lambda L(x, x + v) P(x + v, \xi) dv$$

$$= \sum_{\alpha \geq 0} \frac{1}{\alpha!} \partial_\xi^\alpha P(x, \xi) \int \lambda \frac{e^{v \cdot \xi}}{\alpha!} \partial_x^\alpha L(x, x + v) dv$$

$$\equiv \sum_{\alpha \geq 0} \frac{1}{\alpha!} \partial_\xi^\alpha P(x, \xi) \int \lambda \frac{e^{v \cdot \xi} \partial_x^\alpha L(x, x + v)}{\partial_\xi^\alpha} dv$$

$$= \sum_{\alpha \geq 0} \frac{1}{\alpha!} \partial_\xi^\alpha P(x, \xi) \int \lambda e^{v \cdot \xi} L(x, x + v) dv$$

$$= \sum_{\alpha \geq 0} \frac{1}{\alpha!} \partial_\xi^\alpha Q(x, \xi) \cdot \partial_x^\alpha P(x, \xi). \quad \square$$

**Remark.** For any $P, Q$, the Leibniz–Hörmander formula is not true, in general, for any their symbols, but is only true for some special symbols, corresponding to $P, Q$. In the framework of the formal symbols, we can establish the Leibniz–Hörmander formula for any symbols (see Proposition 6.11).
5. Non-local micro-differential operators and non-local differential operators

In the circumstance of Definition 3.4, we impose further assumption on \( Z \). Let \( Z \subset X \times X \) be a proper ordering with compact convex support \( M \). As well as \( X \), the space \( X \times X \) has the action of \( \mathbb{C} \) defined by \( c \cdot (x, y) := (c \cdot x, c \cdot y) \) for \( c \in \mathbb{C} \) and \((x, y) \in X \times X \). An action by \( c \) is said to be a rotation if \(|c| = 1\). A set in \( X \) or \( X \times X \) is said to be of Reinhardt if it is invariant by any rotation.

**Definition 5.1.** Let \( Z \) be invariant by rotations, i.e., of Reinhardt. For any of such \( Z \) and \( Z \)-round open set \( D \), we call any \( P \in \mathcal{E}(Z; D) \) a non-local micro-differential operator defined near the direction \( \xi_0 \) and carried by the compact convex set \( M \). By the same way, we define the \( \mathbb{C} \)-action on \( X \times X \). The theory of non-local operator will be well-developed only globally in some sense and so we will treat mainly the case of \( D = \mathbb{C}^n \). In this case:

**Definition 5.2.** For proper orderings \( Z_0, Z_1, Z_2 \) with support in \( M \) such that \( Z_1 \) is of Reinhardt and \( Z_2 \) is invariant by any \( c \in \mathbb{C} \) with \(|c| \leq 1\). We will denote:

\[
\mathcal{E}^R(Z_0) := \mathcal{E}(Z_0; \mathbb{C}^n),
\]

\[
\mathcal{E}^\infty(Z_1) := \mathcal{E}(Z_1; \mathbb{C}^n).
\]

In the situation of (3.9), denoting \( G := \{\Gamma_1, \Gamma_2, \ldots, \Gamma_N\} \) for the simplicity, we will denote:

\[
\mathcal{E}^R_M(G) = \mathcal{E}^R_M(\Gamma_1, \Gamma_2, \ldots, \Gamma_N) := \mathcal{E}_M(\Gamma_1, \Gamma_2, \ldots, \Gamma_N; \mathbb{C}^n)
\]

(cf. (3.5)) and

\[
\mathcal{E}^\infty_M(G) = \mathcal{E}^\infty_M(\Gamma_1, \Gamma_2, \ldots, \Gamma_N) := \mathcal{E}_M(\Gamma_1, \Gamma_2, \ldots, \Gamma_N; \mathbb{C}^n)
\]

(5.3)

if any \( \Gamma_j \) is of Reinhardt.

We remark that if \( M_1 \subset M_2 \), evidently \( \mathcal{E}^R_M(G) \subset \mathcal{E}^R_{M_2}(G) \) and \( \mathcal{E}^\infty_M(G) \subset \mathcal{E}^\infty_{M_2}(G) \).

In this section, for any non-local micro differential operator \( P \in \mathcal{E}_M^\infty(\Gamma_1, \Gamma_2, \ldots, \Gamma_n) \), where each \( \Gamma_k \) are, as in (3.10)–(3.11), of the form:

\[
\Gamma_1 := \{z \in X \mid z_1 \neq 0\},
\]

\[
\Gamma_j = \Gamma_j,\delta := \{z \in X \mid \delta|z_j| \leq |z_1|\} \quad (2 \leq j \leq n),
\]

with \( \delta > 0 \), we will calculate explicitly (4.8). Let \( K(x, y) \) dy be a kernel of \( P \). By the same reasoning as [14], remark that in this case, \( K \) can be expanded into the series:

\[
K(x, y) = \sum_{\alpha=(a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n, a_2, \ldots, a_n \geq 0} a_{\alpha}(x)\Phi_{\alpha}(x - y),
\]

(5.7)
where \( \Phi_\nu(z) \) is defined in [14, p. 337] as follows: for \( \nu \in \mathbb{Z} \),

\[
\Phi_\nu(\tau) := \begin{cases} 
\frac{1}{2\pi \sqrt{-1}} \frac{v!}{(-v-1)^{\nu+1}} \left( -\tau \cdot \left( \sum_{q=1}^{v-1} \frac{1}{q} \right) \right) & \nu \geq 0, \\
\frac{1}{2\pi \sqrt{-1}} \frac{\tau^{-v-1}}{(-v-1)!} \left( \ln \tau - \left( \sum_{q=1}^{v-1} \frac{1}{q} \gamma \right) \right) & \nu < 0
\end{cases}
\] (5.8)

and \( \Phi_\alpha(z) := \Phi_{\alpha_1}(z) \cdots \Phi_{\alpha_n}(z) \) (where \( \gamma \) is the Euler number). In this case, set the formal power series:

\[
\hat{P}(x, \xi) = \sum_{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}^n, \alpha_2, \ldots, \alpha_n \geq 0} a_{\alpha}(x) \xi^\alpha = \sum_{i=-\infty}^{\infty} \hat{P}_i(x, \xi)
\] (5.9)

and call the formal series (5.9) the formal symbol of a non-local micro-differential operator \( P \in \mathcal{E}_M^\infty(\Gamma_1, \Gamma_2, \ldots, \Gamma_n) \), where we posed \( \hat{P}_i(x, \xi) := \sum_{|\alpha| = i} a_{\alpha}(x) \xi^\alpha \), the “homogeneous part” of degree \( i \) in the variable \( \xi \). We emphasize that the negative part (i.e., \( i < 0 \)) of formal symbol is not a true symbol defined in Proposition 4.2. In fact, the formal symbol does not converge for \( i < 0 \), in general.

**Proposition 5.3.** Let

\[
\omega(\xi) = \sum_{k=1}^{n} (-1)^{n-k} \frac{1}{\xi_k} d\xi_1 \wedge \cdots \wedge d\xi_{k-1} \wedge d\xi_{k+1} \wedge \cdots \wedge d\xi_n
\]

be the Leray form (see [12]). Then for a non-local micro-differential operator \( P \in \mathcal{E}_M^\infty(\Gamma_1, \Gamma_2, \ldots, \Gamma_n) \), (denoting for the simplicity \( z := y - x \)), a kernel \( K(x, y) \) of \( P \) is given by:

\[
\sum_{i=-\infty}^{\infty} \frac{1}{(2\pi \sqrt{-1})^n \beta} \int_{\beta} \hat{P}_i(x, \xi) \frac{(n+i-1)!}{(\zeta \cdot \xi)^{n+i}} \omega(\xi)|_{z=y-x} \\
+ \sum_{i=-\infty}^{\infty} \frac{1}{(2\pi \sqrt{-1})^n \beta} \int_{\beta} \hat{P}_i(x, \xi) \frac{(-1)^{-(n-i)}}{(-n+i)!} (\zeta \cdot \xi)^{-(n-i)} \\
\times (-\ln(z \cdot \xi)) \omega(\xi)|_{z=y-x},
\] (5.10)

where the \((n-1)\)-chain \( \beta \) is just the \( \tilde{\beta} \) in the inhomogeneous coordinate \( \eta' := \xi'/\xi_1 \); \( \beta := \{ \xi = (\xi_1, \eta') | \xi_1 \neq 0, \eta' := \xi'/\xi_1 \in \tilde{\beta} \} \).

**Proof.** By (5.7) and (5.9), it suffices to prove Proposition in the case where \( K(x, y) = \Phi_\alpha(x - y) \) and so \( \hat{P}(x, \xi) = \xi^\alpha \). Then direct calculations shows (5.10). \( \square \)
The operation of $P$ on a holomorphic function, (3.15), is given by the following formula:

**Corollary 5.4** (cf. Sternin and Shatalov [16, p. 282]). Set $\Gamma := \bigcap_{1 \leq k \leq n} \Gamma_k$. Let $\omega \subset X$ be a $\Gamma$-open set. Then for any $f(x) \in \mathcal{O}_X(\omega + M)$, the action of the micro-differential operator $P$ on $f$ is given by:

$$
P f(x) = \frac{1}{(2\pi \sqrt{-1})^n} \sum_{i=-n+1}^{\infty} \int_{H_x} \hat{P}_i(x, \xi) \frac{(n+i-1)!}{((y-x) \cdot \xi)^{n+i}} f(y) \omega(\xi) \wedge dy$$

$$+ \frac{1}{(2\pi \sqrt{-1})^n} \sum_{i=-\infty}^{-n+1} \int_{H_x} \hat{P}_i(x, \xi) \frac{(-1)^{-(n-i)}}{(-n-i)!} ((y-x) \cdot \xi)^{-(n-i)}$$

$$\times \ln \left( \frac{1}{(y-x) \cdot \xi} \right) f(y) \omega(\xi) \wedge dy$$

$$= \frac{1}{(2\pi \sqrt{-1})^n} \sum_{i=-n+1}^{\infty} \int_{H_x} (n+i-1)! \text{Res} \left\{ \hat{P}_i(x, \xi) \frac{f(y)}{((y-x) \cdot \xi)^{n+i}} \omega(\xi) \wedge dy \right\}$$

$$+ \frac{1}{(2\pi \sqrt{-1})^n} \sum_{i=-\infty}^{-n+1} \int_{H_x} \hat{P}_i(x, \xi) \frac{(-1)^{-(n-i)}}{(-(n-i))!} ((y-x) \cdot \xi)^{-(n-i)}$$

$$\times \ln \left( \frac{1}{(y-x) \cdot \xi} \right) f(y) \omega(\xi) \wedge dy, \quad (5.11)$$

where $H_x$ is the Leray’s cycle defined in [12, p. 150, n° 54], notation and $h_x$ is the homology class so that we have $\delta h_x = H_x$ with $\delta$ the Leray coboundary operator. And here Res means the residue.

As a special type of non-local micro-differential operator, we define the non-local differential operator:

**Definition 5.5.** We define the space of non-local differential operators carried by $M$:

$$\mathcal{D}_M^{\infty}(\mathbb{C}^n) := H^n_{\Delta M}(X \times X, \mathcal{O}_{\mathbb{C}^n}^{(0,n)}(X \times X)); \quad (5.12)$$

$\mathcal{D}_M^{\infty}(\mathbb{C}^n)$ is just the non-local micro-differential operators having only the positive index parts in (5.7):

**Proposition 5.6.** We have the following canonical identification:

$$\mathcal{D}_M^{\infty}(\mathbb{C}^n) \simeq \left\{ P \in \mathcal{D}_M^{\infty}(\Gamma_1, \Gamma_2, \ldots, \Gamma_n) \mid P \text{ has the kernel of the form} \right\}$$

$$K(x, y) dy = \sum_{\alpha \geq 0} a_{\alpha}(x) \Phi_{\alpha}(x - y) dy \right\}. \quad (5.13)$$
6. Invertibility of a non-local pseudo-differential operator

Example. Let \( n = 1 \) and \( D := \frac{d}{dx} \). For a difference operator \( P = e^D \in D^{-\infty}_{\infty}(\mathbb{C}) \) with shift 1, “the inverse shift” \( P^{-1} := e^{-D} \) is evidently contained in \( D^{-\infty}_{-1}(\mathbb{C}) \) and we have \( e^D \circ e^{-D} = e^{-D} \circ e^D = \text{id}_{D^{-\infty}_{1}(\mathbb{C})} = \text{id}_{D^{-\infty}_{\infty}(\mathbb{C})} \). We remark here \( \{1\} + \{-1\} = \{0\} \). We note also \( \sigma(P) = e^\xi \in S^{[1]} \), \( \sigma(P^{-1}) = e^{-\xi} \in S^{[-1]} \) and \( \sigma(e^D \circ e^{-D}) = e^\xi e^{-\xi} = 1 \in S^{[0]} \).

Definition 6.1. Let \( \Omega \subset T^* X \) be an open conic set, \( M \subset X \) a compact convex set and \( P(x, \xi) \in S^M(\Omega) \) a symbol on \( \Omega \). Suppose as in Definition 4.1, \( P(x, \xi) \in \mathcal{O}_X(\Omega(r)) \) with \( r > 0 \). Any point \( p \in \Omega \) is said to be non-\( M \)-characteristic with respect to the corresponding non-local pseudo-differential operator \( P \) carried by \( M \), or \( P \) is non-\( M \)-characteristic at \( p \), if there exists, an open conic neighbourhood \( \Omega' \subset \Omega \) of \( p \) and \( r' > r \) so that we have:

\[
\text{for any } \varepsilon > 0, \text{ there exists } C_\varepsilon > 0 \text{ such that } |P(x, \xi)| \geq C_\varepsilon e^{H_M(\xi) - \varepsilon|\xi|} \quad (\text{for every } (x, \xi) \in \Omega'(r')). \tag{6.1}
\]

We set:

\[
\text{Car}^M(P) := \Omega(r) \setminus \{q \in \Omega(r) \mid P \text{ is non-}M\text{-characteristic at } q\},
\]

\[
\text{Car}^M_\infty(P) := \{(x, \xi \infty) \in X \times S_{\infty}^{2n-1} \mid (x, \xi) \in \text{Car}^M(P)\}. \tag{6.2}
\]

A point \( q \in \Omega(r) \) is also said to be \( M \)-characteristic with respect to \( P \) if \( q \in \text{Car}^M(P) \).

In this section, we will study the invertibility of a non-local pseudo-differential operator.

Remark. We remark that if \( p = (x, \xi) \) is non-\( M \)-characteristic with respect to \( P \), then there exists an open conic neighbourhood \( \Omega' \subset \Omega \) of \( p \) such that \( 1/P(x, \xi) \in S^{-M}(\Omega') \).

More strongly, there is \( r' > r \) so that \( 1/P(x, \xi) \in \mathcal{O}_X(\Omega'(r')) \) and we have:

\[
\begin{cases}
\text{for any conic set } \Omega'' \subset \Omega', \text{ any } r'' > r' \text{ and any } \varepsilon > 0, \\
\text{there exists } C_\varepsilon > 0 \text{ such that } \left|\frac{1}{P(x, \xi)}\right| \leq C_\varepsilon e^{-H_M(\xi) + \varepsilon|\xi|} \quad (\text{for every } (x, \xi) \in \Omega''[r'']). \tag{6.3}
\end{cases}
\]

In fact, we have \( -H_M(\xi) = I_{-M}(\xi) \leq H_{-M}(\xi) \).

The purpose of this section is the following theorem:

Theorem 6.2. Let \( M \subset X \) be a compact convex set, \( \Omega \subset T^* X \) a conic open set of the form \( \mathbb{C}^n \times \omega \) with an open cone \( \omega \) with vertex at 0 and \( P(x, \xi) \in S^M(\Omega) \) a symbol satisfying (4.1) for any \( \Omega' \subset \Omega \) of the form \( \mathbb{C}^n \times \omega' \) with \( \omega' \subset \omega \). Suppose that the corresponding non-local pseudo-differential operator \( P \) carried by \( M \) is non-\( M \)-characteristic on \( \Omega \).
Then there is a non-local pseudo-differential operator, denoted by $P^{-1}$ defined on $\mathbb{C}^n \times \omega_1$ with an open cone $\omega_1 \subset \omega$ and carried by $(-M)$ such that we have $P \circ P^{-1} = P^{-1} \circ P = \text{id}$ with the identity operator $\text{id}$ of the space of non-local pseudo-differential operators carried by the compact convex set $M - M$.

We will prove the theorem using the method of the symbol calculus developed by T. Aoki in [1–3]. For this purpose, we first introduce the formal symbols. (Recall the notations in Definition 4.1.)

**Definition 6.3** (cf. Aoki [2]). Let $M \subset X$ be a compact convex set and $\Omega \subset T^*X$ a conic open set.

\[
\hat{S}^M(\Omega) := \left\{ P(t; x, \xi) = \sum_{\nu=0}^{\infty} t^\nu P_{\nu}(x, \xi) \mid \text{there exists } r > 0 \text{ such that } P_{\nu}(x, \xi) \in \mathcal{O}(\Omega ((\nu + 1)r)) \ (\forall \nu \in \mathbb{Z}_+) \text{ and we have the following:} \right. \\
\left. \quad \text{for any conic set } \Omega^\prime \Subset \Omega, \text{ there exist } d > r \text{ and } A > 0 \text{ with } 0 < A < 1 \text{ such that for any } \varepsilon > 0, \text{ there exists } C_\varepsilon > 0 \text{ so that we have} \right. \\
\left. \quad \left| P_{\nu}(x, \xi) \right| \leq C_\varepsilon A^\nu e^{H_M(\xi) + \varepsilon |\xi|} \right. \\
\left(\text{for every } \nu \geq 0 \text{ and any } (x, \xi) \in \Omega^\prime [(\nu + 1)d] \right) \right\}
\]  

(6.4)

and

\[
\hat{N}^M(\Omega) := \left\{ P(t; x, \xi) = \sum_{\nu=0}^{\infty} t^\nu P_{\nu}(x, \xi) \in \hat{S}^M(\Omega) \mid P_{\nu}(x, \xi) \text{ being } \in \mathcal{O}(\Omega ((\nu + 1)r)), \right. \\
\left. \quad \text{for any } q = (y, \eta) \in \Omega(r), \text{ there exist a conic neighbourhood } \Omega^\prime \Subset \Omega \right. \\
\left. \quad \text{of } q, \ d > r \text{ and } A > 0 \text{ with } 0 < A < 1 \text{ such that we have the condition,} \right. \\
\left. \quad 0 \leq \mu := H_M(|\eta|) - I_M(|\eta|) < -\frac{\ln A}{d} \right. \\
\left. \quad \text{and} \right. \\
\left. \quad \text{for any } \varepsilon > 0, \text{ there exists } C_\varepsilon > 0 \text{ so that we have,} \right. \\
\left. \quad \left| \sum_{\nu=0}^{m-1} P_{\nu}(x, \xi) \right| \leq C_\varepsilon A^m e^{H_M(\xi) + \varepsilon |\xi|} \right. \\
\left(\text{for every } m \geq 1 \text{ and any } (x, \xi) \in \Omega^\prime [md] \right) \right\}
\]  

(6.5)

and

(6.6)

We call any $P(t; x, \xi) \in \hat{S}^M(\Omega)$ a formal symbol and $P(t; x, \xi) \in \hat{N}^M(\Omega)$ a formal null symbol.
We remark that in the case of $M = \{0\}$, we have $\hat{S}^{0}(\Omega) = \hat{S}(\Omega)$ and $\hat{N}^{0}(\Omega) = \hat{N}(\Omega)$ (see Aoki [2]). We may canonically embed $S^{M}(\Omega)$ into $\hat{S}^{M}(\Omega)$:

**Proposition 6.4.** We have the natural embedding:

$$ P(x, \xi) \mapsto \left[ P(x, \xi) + 0t + 0t^{2} + \cdots \right]; \quad S^{M}(\Omega) \hookrightarrow \hat{S}^{M}(\Omega). $$

**Lemma 6.5.** Let $P_{\nu}(x, \xi) \in C(\Omega(r))$ be a sequence satisfying the following condition:

For any conic set $\Omega' \subset \Omega$, $r' > r$, there exists $B > 0$ such that for any $\varepsilon > 0$, there exists $C_{\varepsilon} > 0$ so that we have

$$ |P_{\nu}(x, \xi)| \leq C_{\varepsilon} \frac{B^{\nu+1}}{|\xi|^{\nu}} e^{H_{M}(\xi)+\varepsilon|\xi|} $$

(for every $\nu \geq 0$ and any $(x, \xi) \in \Omega'[r']$);

then we have:

$$ P(t; x, \xi) := \sum_{\nu=0}^{\infty} t^{\nu} P_{\nu}(x, \xi) \in \hat{S}^{M}(\Omega). $$

**Proof.** Set $d := \max(r, B)$ and $A := B/d$. Because on $\Omega'[\nu+1d]$, we have $|\xi| \geq (\nu+1)d$, there is $C' > 0$ so that we have,

$$ |P_{\nu}(x, \xi)| \leq C_{\varepsilon} \left( \frac{B}{d} \right)^{\nu} e^{H_{M}(\xi)+\varepsilon|\xi|} \leq C' A^{\nu} e^{H_{M}(\xi)+\varepsilon|\xi|}. \quad \square $$

**Proposition 6.6.** We have:

$$ S^{M}(\Omega) \cap \hat{N}^{M}(\Omega) = N^{M}(\Omega). $$

**Proof.** Let $q = (y, \eta) \in \Omega$ be fixed and take a small conic neighbourhood $\Omega' \subset \Omega$ of $q$, which will be chosen later.

(i) Let take any $P(x, \xi) \in S^{M}(\Omega) \cap \hat{N}^{M}(\Omega)$. Taking a small conic neighbourhood $\Omega' \subset \Omega$ of $q$, then there exist $d > r$ and $A > 0$ with $0 < A < 1$ and $\mu := H_{M}(\eta/|\eta|) - I_{M}(\eta/|\eta|) < - \ln A/d$ such that for any $\varepsilon > 0$, we can take $C_{\varepsilon} > 0$ so that

$$ |P(x, \xi)| \leq C_{\varepsilon} A^{\nu} e^{H_{M}(\xi)+\varepsilon|\xi|} \quad (for \ every \ \nu \geq 1 \ and \ (x, \xi) \in \Omega'[md]). $$

For any $\varepsilon_{1} > 0$, if we take $\Omega'$ small enough, we may assume that we have $H_{M}(\xi) \leq I_{M}(\xi) + \mu|\xi| + \varepsilon_{1}|\xi|$. Let take any $r' > d$ and any $(x, \xi) \in \Omega'[r']$. Take also $\delta > 0$ with $0 < \delta < - \ln A/d + \mu$. We set $\varepsilon_{0} := - (\delta + (\mu/d) \ln A) - \varepsilon_{1} > 0$. Take $m := [\nu/d]$ for $\xi$ with $[\cdot]$ the Gauss’ notation. Then we have $(d(m+1) \geq |\xi| \geq md$ and so $(x, \xi) \in \Omega'[md].$ We have:
\[ |P(x, \xi)| \leq C_\delta A^m \exp(H_M(\xi) + \delta |\xi|) \]
\[ = C_\delta \exp(m \ln A + H_M(\xi) + \delta |\xi|) \]
\[ \leq C_\delta \exp \left( \frac{|\xi|}{d} - 1 \right) \ln A + H_M(\xi) + \delta |\xi| \]
\[ \leq C_\delta e^{-\ln A} \exp((\delta + \ln A/d + \mu \varepsilon_1)|\xi| + I_M(\xi)) \]
\[ = C_\delta A^{-1} \exp(-\varepsilon_0 |\xi| + I_M(\xi)). \]

This means \( P(x, \xi) \in N^M(\Omega) \).

(ii) Conversely, let \( P(x, \xi) \in N^M(\Omega) \) and any \( m \geq 1 \). Then for any \( d > r \) with \( d \geq 1 \), there exist \( \varepsilon_0 > 0 \), \( C > 0 \) such that \( 0 < \varepsilon_0 < 1 \) and
\[ |P(x, \xi)| \leq Ce^{H_M(\xi)}(A')^{|\xi|} \leq Ce^{H_M(\xi)}(A')^md = CA^m e^{H_M(\xi)}. \]

and thus we have \( P(x, \xi) \in \hat{N}_M(\Omega). \)

By Propositions 6.4 and 6.6, we have the natural injection:
\[ \iota: S^M(\Omega)/N^M(\Omega) \rightarrow \hat{S}^M(\Omega)/\hat{N}^M(\Omega). \] (6.8)

In fact, \( \iota \) is “bijective” in the following sense:

**Proposition 6.7.** For any formal symbol \( P(t; x, \xi) = \sum_{\nu=0}^{\infty} t^\nu P_\nu(x, \xi) \in \hat{S}^M(\Omega) \), there exist a conic open neighbourhood \( \Omega' \Subset \Omega \) of \( p \) and a symbol \( P(x, \xi) \in S^M(\Omega') \) such that we have:
\[ P(t; x, \xi) - P(x, \xi) \in \hat{N}^M(\Omega'). \]

**Proof.** We work on the situations of Proposition 4.3 and Definition 6.3: Set for any \( \nu \),
\[ K_\nu(x, y) := \frac{1}{(2\pi \sqrt{-1})^n} \int_{\tilde{\beta}} d\eta' \int_0^\infty e^{-t(y-x) \cdot \xi} \tau^{n-1} P_\nu(x, \tau \xi) d\tau, \]
where \( \tilde{\beta} \) being defined in Proposition 4.3 by (4.8) and set
\[ \tilde{P}_\nu(x, \xi) := \int_y e^{i \xi \cdot z} K_\nu(x, x + z) dz, \]
where \( \gamma \) being as in (4.4). By Propositions 4.3 and 4.2, there is a conic open neighbourhood \( \Omega' \Subset \Omega \) of \( p \), independent of \( \nu \), such that we have \( \tilde{P}_\nu(x, \xi) \in S^M(\Omega') \) and for any \( r > 0 \), we have \( \tilde{P}_\nu(x, \xi) \in \mathcal{O}(\Omega'[r]) \). We see easily that (we may assume) for any \( \varepsilon > 0 \), there exists \( C > 0 \) such that for any \( \nu \), we have:
\[
|\tilde{P}_\nu(x, \xi)| \leq CA^\nu e^{H_U(\xi) + \varepsilon |\xi|} \quad (\forall (x, \xi) \in \Omega'[r])
\]
with \( A \) being same for \( P_\nu(x, \xi) \) appeared in (6.4). Then the series \( \sum_{\nu=0}^\infty \tilde{P}_\nu(x, \xi) \) is absolutely and compactly convergent on \( \Omega'[r] \) and thus defines a symbol \( P(x, \xi) \in S^M(\Omega') \); in fact, we have:
\[
\sum_{\nu=0}^\infty |\tilde{P}_\nu(x, \xi)| \leq \frac{C}{1 - A} e^{H_M(\xi) + \varepsilon |\xi|} \quad (\forall (x, \xi) \in \Omega'[r]).
\]
And, by the same way, we have:
\[
|P(x, \xi) - \sum_{\nu=0}^{m-1} \tilde{P}_\nu(x, \xi)| \leq \frac{C}{1 - A} A^m e^{H_M(\xi) + \varepsilon |\xi|} \quad (\forall (x, \xi) \in \Omega'[r]),
\]
and this means
\[
P(x, \xi) = \sum_{\nu=0}^\infty t^\nu \tilde{P}_\nu(x, \xi) \in \tilde{N}^M(\Omega').
\]
We note also, by (4.10), for any \( \nu \),
\[
P_\nu(x, \xi) \equiv \tilde{P}_\nu(x, \xi) \mod N^M(\Omega').
\]
More precisely, by the proof of (4.10), we can prove easily
\[
|P_\nu(x, \xi) - \tilde{P}_\nu(x, \xi)| \leq CA^\nu e^{U(\xi) - \varepsilon_0 |\xi|}
\]
with some \( \varepsilon_0 > 0 \) independent of \( \nu \). Thus we conclude:
\[
\sum_{\nu=0}^\infty t^\nu P_\nu(x, \xi) = \sum_{\nu=0}^\infty t^\nu \tilde{P}_\nu(x, \xi) \in \tilde{N}^M(\Omega')
\]
and we have finally
\[
P(x, \xi) = \sum_{\nu=0}^\infty t^\nu P_\nu(x, \xi) \in \tilde{N}^M(\Omega'). \quad \square
\]
The following lemma is easy to prove:
Lemma 6.8. (1) Let $P = P(x, \xi) \in S^M(\Omega)$ be satisfied $P(x, \xi) \in \mathcal{O}(\Omega(r))$.

(a) For any $\Omega' \subset \Omega$, $r' > r$ and $\varepsilon > 0$, there exists $C > 0$ such that we have the following: for any $\Omega'' \subset \Omega'$ and $r'' > r'$, there is $\delta > 0$ independent of $P$ so that for every $\alpha, \beta \in \mathbb{Z}^n_+$, we have:

$$|\partial_\xi^\alpha \partial_x^\beta P(x, \xi)| \leq C \frac{\alpha! \beta!}{|\xi|^{\alpha + \beta} e^{HM(\xi) + \varepsilon|\xi|}} \quad (\forall (x, \xi) \in \Omega''[r'']).$$

(b) If $P \in N^M(\Omega)$, then for any $\Omega' \subset \Omega$ and $r' > r$, there exists $\varepsilon_0 > 0$ and $C > 0$ such that we have the following: for any $\Omega'' \subset \Omega'$ and $r'' > r'$, there is $\delta > 0$ independent of $P$ so that for every $\alpha, \beta \in \mathbb{Z}^n_+$, we have:

$$|\partial_\xi^\alpha \partial_x^\beta P(x, \xi)| \leq C \frac{\alpha! \beta!}{|\xi|^{\alpha + \beta} e^{IM(\xi)}} \quad (\forall (x, \xi) \in \Omega''[r'']).$$

(2) Let $P = P(t; x, \xi) = \sum_{\nu=0}^{\infty} t^n P_\nu(x, \xi) \in \widehat{S}^M(\Omega)$ be as in Definition 6.4 with the same $\Omega'$, $r > 0$, $d > 0$, $0 < A < 1$.

(a) For any $\varepsilon > 0$, there exists $C > 0$ such that we have the following: for any $\Omega'' \subset \Omega'$ and $d'' > d$, there is $\delta > 0$ independent of $P$ so that for every $\alpha, \beta \in \mathbb{Z}^n_+$, we have for any $\nu$,

$$|\partial_\xi^\alpha \partial_\nu^\beta P_\nu(x, \xi)| \leq C \frac{\alpha! \beta! A^n}{|\xi|^{\alpha + \beta} e^{\hat{M}(\xi) + \varepsilon|\xi|}} \quad (\forall (x, \xi) \in \Omega''[(\nu + 1)d']).$$

(b) If $P \in \widehat{N}^M(\Omega)$ satisfies (6.5) and (6.6) with same $q$, $\Omega'$, $d > r$, $A$ and $\mu$, then for any $\varepsilon > 0$, there exists $C > 0$ such that we have the following: for any $\Omega'' \subset \Omega'$ and $d'' > d$, there is $\delta > 0$ independent of $P$ so that for every $\alpha, \beta \in \mathbb{Z}^n_+$, we have for any $m \geq 1$:

$$\sum_{\nu=0}^{m} \partial_\xi^\alpha \partial_\nu^\beta P_\nu(x, \xi) \leq C \frac{\alpha! \beta! A^m}{|\xi|^{\alpha + \beta} e^{\hat{M}(\xi) + \varepsilon|\xi|}} \quad (\forall (x, \xi) \in \Omega''[md']).$$

We need the following lemma:

Lemma 6.9. Let $P = P(t; x, \xi) = \sum_{\nu=0}^{\infty} t^n P_\nu(x, \xi) \in \widehat{S}^M(\Omega)$. For an $(n \times n)$-matrix with complex components $L = (a_{i,j})_{1 \leq i,j \leq n} \in \mathbf{Mat}_{n \times n}(\mathbb{C})$, we consider the following constant coefficients "formal differential operator",

$$E := e^{t \partial_t} \mathcal{L} \partial_x := \sum_{B=(\beta_i)} \frac{t^{|\beta|}}{B!} \prod_{i,j=1}^{n} (a_{i,j} \partial_{x_i} \partial_{x_j})^{\beta_i,j}.$$

Then $E$ operates on $P$ by
\[
\sigma(E \circ P)(t; x, \xi) := \sum_{l=0}^{\infty} \sum_{|B|=l} \frac{1}{B!} \prod_{i,j=1}^{n} (a_{i,j} \partial_{\xi_i} \partial_{x_j})^{\delta_{i,j}} P_{\nu}(x, \xi)
\]

which is well-defined in \(\hat{S}^M(\Omega)\). Furthermore, if \(P(t; x, \xi) \in \hat{N}^M(\Omega)\), then we have \(\sigma(E \circ P) \in \hat{N}^M(\Omega)\).

**Proof.** We suppose that \(P\) satisfies the conditions of (6.4) in Definition 6.3 with the same constants or domains appeared there. Let take any \(\Omega' \supseteq \Omega\) and set \(b := \max |a_{i,j}|\). Then, by the preceding lemma, (using the constants of the lemma), we have, for any \(\Omega'' \supseteq \Omega'\) and \(d' > d\) there exists \(\delta > 0\) such that for any \(m \geq 0\),

\[
\left| (a_{i,j} \partial_{\xi_i} \partial_{x_j})^m P_{\nu}(x, \xi) \right| \leq b^m |\partial^m \xi \partial^m x P_{\nu}(x, \xi)| \leq b^m C \left( \frac{(m!)^2 A^v}{|\xi|^m \delta^m} e^{H_M(\xi) + \epsilon |\xi|} \right)
\]

for any \((x, \xi) \in \Omega''[(v + 1)d']\). Thus we have on \(\Omega[(l + 1)d']\),

\[
\left| \sum_{v+|B|=l} \frac{1}{B!} \prod_{i,j=1}^{n} (a_{i,j} \partial_{\xi_i} \partial_{x_j})^{\delta_{i,j}} P_{\nu}(x, \xi) \right| \leq \sum_{v+|B|=l} b^{|B|} |A^v| \frac{|B!| \delta^{|B|}}{|\xi|^{|B|} |\delta^{|B|} e^{H_M(\xi) + \epsilon |\xi|} |}
\]

\[
\leq C e^{H_M(\xi) + \epsilon |\xi|} \sum_{v=0}^{l} 2^{l-v+2n+1} b^{l-v}(l-v)! A^v |\xi|^{-v} \delta^{2(l-v)}
\]

\[
\leq 2^{2n+1} C e^{H_M(\xi) + \epsilon |\xi|} A^l \sum_{v=0}^{l} \left( \frac{2b(l-v)}{|\xi| \delta^2 A} \right)^{l-v}
\]

\[
\leq 2^{2n+1} C e^{H_M(\xi) + \epsilon |\xi|} A^l \sum_{v=0}^{l} \left( \frac{2b}{(l+1)d \delta^2 A} \right)^{v}
\]

Taking \(d'\) bigger, we conclude \(\sigma(E \circ P) \in \hat{S}^M(\Omega)\). By the same way, if \(P(t; x, \xi) \in \hat{N}^M(\Omega)\), we see \(\sigma(E \circ P) \in \hat{N}^M(\Omega)\).

As announced before (Remark of the end of Section 4), now we state the formula for the composition of two non-local pseudo-differential operators in terms of their formal symbols, that is:

**Proposition 6.10** (Leibniz–Hörmander). Let \(M_1, M_2 \subset X\) be two compact convex sets and \(\Gamma_1, \ldots, \Gamma_n\) closed conic sets defined in (3.10)–(3.11) for some number \(\delta > 0\). Let
$D$ be a $(Z_{\sum_{\alpha=1}^n (\Gamma_1 + M_1)}: Z_{\sum_{\alpha=1}^n (\Gamma_1 + M_2)})$-round open set with respect to $(M_1, M_2)$. Then for any $P \in \mathcal{E}_{M_1}(\Gamma_1, \ldots, \Gamma_n; D + M_2)$ and $Q \in \mathcal{E}_{M_2}(\Gamma_1, \ldots, \Gamma_n; D)$ with formal symbols $P(t; x, \xi) \in \hat{S}^{M_1}(\Omega_1 + (M_2 \times \{0\}))$ and $Q(t; x, \xi) \in \hat{S}^{M_2}(\Omega_2)$, respectively. We consider the "formal differential operator":

$$e^{t\partial_x \cdot \partial_y} := \sum_{\alpha \in \mathbb{Z}^n} \frac{t^{|\alpha|}}{\alpha!} \partial_x^\alpha \partial_y^\alpha.$$  

Then the symbol of the composition operator $Q \circ P$ is given by:

$$e^{t\partial_x \cdot \partial_y} Q(t; x, \xi)P(t; y, \eta)|_{y=x, \eta=\xi} := \sum_{\alpha \in \mathbb{Z}^n} \frac{t^{|\alpha|}}{\alpha!} \partial_x^\alpha Q(t; x, \xi) \partial_y^\alpha P(t; x, \xi). \quad (6.9)$$

**Proof.** Set $M := M_1 + M_2$. We may assume $\Omega_1 = \Omega_2 =: \Omega$. By the preceding lemma, $e^{t\partial_x \cdot \partial_y} Q(t; x, \xi)P(t; y, \eta)|_{y=x, \eta=\xi} := \sum_{\alpha \in \mathbb{Z}^n} \frac{t^{|\alpha|}}{\alpha!} \partial_x^\alpha Q(t; x, \xi) \partial_y^\alpha P(t; x, \xi)$ is well-defined in $\hat{S}^M(\Omega)$. By Propositions 4.4 and 6.7, we may assume that there are symbols $P(x, \xi) \in S^M_1(\Omega + (M_2 \times \{0\}))$ and $Q(x, \xi) \in S^M_2(\Omega)$ defining same operators as $P(t; x, \xi)$ and $Q(t; x, \xi)$, i.e., operators $P$ and $Q$, respectively, such that we have:

$$\sigma(Q \circ P)(x, \xi) = \sum_{\alpha \in \mathbb{Z}^n} \frac{t^{|\alpha|}}{\alpha!} \partial_x^\alpha Q(x, \xi) \partial_y^\alpha P(x, \xi)$$

$$= e^{t\partial_x \cdot \partial_y} Q(x, \xi)P(y, \eta)|_{y=x, \eta=\xi}.$$  

Because we have easily,

$$Q(t; x, \xi)P(t; y, \eta) - Q(x, \xi)P(y, \eta) \in \hat{N}^{M_2 \times M_1}(\Omega \times (\Omega + (M_2 \times \{0\}))),$$

by Lemma 6.10, we obtain the desired formula (6.9).

We recall a theorem due to Aoki [1, Theorem 3.3.1]:

**Theorem 6.11** [1]. Suppose that a symbol $P(x, \xi) \in S^{[0]}(\Omega)(= S(\Omega))$, being a symbol of $P \in \mathcal{E}_{R}(\Omega)$, satisfies the following: for every $\Omega' \Subset \Omega$, there exist $C_1, C_2 > 0$ and $r' > 0$ such that we have:

$$C_1 \leq |P(x, \xi)| \leq C_2 \quad \text{on } \Omega'[r'].$$

Then there exist a formal symbol in $\hat{S}^{[0]}(\Omega)$ corresponding to the pseudo-differential operator $Q$ which is the inverse of the operator $P$ in $\mathcal{E}_{R}(\Omega)$: i.e., we have $Q \circ P = P \circ Q = 1$ in $\mathcal{E}_{R}(\Omega)$.
Recall also a lemma of [1, p. 24, sublemma]:

**Lemma 6.12.** Let \( f(w) \) be a holomorphic function on \( \{ w \in \mathbb{C} \mid |w - w_0| < r \} \) which is continuous and never vanishes on \( \{ w \in \mathbb{C} \mid |w - w_0| \leq r \} \). Then we have:

\[
\frac{f'(w)}{f(w)} = \frac{1}{\pi} \int_0^{2\pi} \ln|f(w_0 + r\exp(-i\theta) + w| \frac{w_0 + r\exp(-i\theta) + w}{(w_0 + r\exp(-i\theta) - w)^2} \, d\theta
\]
on \( \{ w \in \mathbb{C} \mid |w - w_0| \leq r \} \).

We need also the following proposition due to Aoki:

**Proposition 6.13.** Let \( S(t; x, \xi) = 1 + \sum_{\nu=1}^{\infty} t^\nu S_\nu(x, \xi) \in \hat{S}(\Omega) \) satisfies the following condition: for any \( \Omega' \subset \Omega \), there are \( d > 0, A > 0, C > 0, a > 0 \) and a function \( A(s) : \mathbb{R}^+ \to \mathbb{R}^+ \) such that \( 0 < A < 1 \) and \( 0 < a < (1 - A)/(2CA) \) and we have:

\[
\lim_{s \to \infty} \frac{A(s)}{s} = 0
\]

and

\[
|S_\nu(x, \xi)| \leq CA^\nu\left( \frac{A(|\xi|)}{|\xi|} + a \right) \text{ on } \Omega'[(v + 1)d].
\]

Then for any symbol \( S(x, \xi) \in S(\Omega) \) with \( S(x, \xi) - S(t; x, \xi) \in \hat{N}(\Omega) \) and \( \Omega' \subset \Omega \), there exist \( C_1, C_2 > 0 \) and \( r' > 0 \) such that \( 0 < C_1 < C_2 \) and \( 0 < a < (1 - A)/(2CA) \) and we have:

\[
C_1 \leq |S(x, \xi)| \leq C_2 \text{ on } \Omega'[r'].
\]

**Proof.** For any \( \Omega' \subset \Omega \), there exist \( d' > d \) and \( B > 0 \) with \( 0 < B < 1 \) satisfying the following: for any \( \varepsilon > 0 \), there are \( C' > 0 \) such that we have for any \( m \),

\[
|S(x, \xi) - 1 - \sum_{\nu=1}^{m-1} S_\nu(x, \xi)| \leq C'B^m e^{\varepsilon|\xi|}
\]
on \( \Omega'[md'] \). Taking \( m := [|\xi|/d + 1] \), there exist \( \varepsilon_0, C'' > 0 \) such that we have on \( \Omega'[md'] \),

\[
|S(x, \xi) - 1| \leq C'' e^{-\varepsilon_0|\xi|} + \sum_{\nu=1}^{m-1} CA^\nu\left( \frac{A(|\xi|)}{|\xi|} + a \right)
\]

\[
\leq C'' e^{-\varepsilon_0|\xi|} + C \frac{A}{1 - A} \left( \frac{A(|\xi|)}{|\xi|} + a \right)
\]

\[
\leq C'' e^{-\varepsilon_0|\xi|} + C \frac{A}{1 - A} \frac{A(|\xi|)}{|\xi|} + \frac{1}{2}.
\]
Because $\Lambda(\xi)/|\xi|$ become small arbitrary, we have the proposition.

Now we are ready to prove Theorem 6.2:

**Proof of Theorem 6.2.** We set $Q(x, \xi) := 1/P(x, \xi)$. By the remark of Definition 6.1, we may assume that $Q(x, \xi) \in S^{-M}(\Omega)$ and $P(x, \xi), Q(x, \xi) \in O(\Omega(r))$ with $r \gg 1$. Then $p(x, \xi) := \ln P(x, \xi) \in O(\Omega(r))$ is well-defined. In the sequel, for any symbol $S(x, \xi)$, let denote by $S$ the corresponding non-local pseudo-differential operator and same for a formal symbol. For example, by $Q$ we denote the operator corresponding to $Q(x, \xi)$.

Let $R(t; x, \xi) := e^{t\partial_t \partial_y \partial_x} Q(x, \xi)$ and let $T(t; x, \xi)$ be the formal symbol of $R \circ P$: i.e., by Proposition 6.10,

$$T(t; x, \xi) = e^{t\partial_t \partial_y \partial_x} e^{p(x, \xi) - p(x, \eta)} \bigg|_{y = x, \eta = \xi} = e^{t\partial_t \partial_y \partial_x} e^{p(x, \eta) - p(x, \xi)} \bigg|_{y = x, \eta = \xi}.$$

There exist holomorphic mappings $\psi(y, x, \eta, \xi) := (\psi_1(y, x, \eta), \psi_2(y, x, \eta), \ldots, \psi_n(y, x, \eta))$ and $\phi(x, \eta, \xi) := (\phi_1(x, \eta, \xi), \phi_2(x, \eta, \xi), \ldots, \phi_n(x, \eta, \xi))$ such that we have:

$$p(y, \eta) - p(x, \eta) = \langle y - x, \psi(y, x, \eta) \rangle; \quad \text{(6.10)}$$

$$p(x, \eta) - p(x, \xi) = \langle \eta - \xi, \phi(x, \eta, \xi) \rangle. \quad \text{(6.11)}$$

Then we have:

$$T(t; x, \xi) = e^{t\partial_t \partial_y \partial_x} e^{(y - x, \psi(y, x, \eta) + (\eta - \xi, \phi(x, \eta, \xi)))} \bigg|_{y = x, \eta = \xi} = e^{t\partial_t \partial_y \partial_x} e^{\langle \eta - \xi, \phi(x, \eta, \xi) \rangle} + e^{\langle \eta - \xi, \phi(x, \eta, \xi) \rangle} \bigg|_{y = x, \eta = \xi}.$$

There exist a holomorphic mapping,

$$\Psi(y, x, \eta, \xi) := (\Psi_1(y, x, \eta, \xi), \Psi_2(y, x, \eta, \xi), \ldots, \Psi_n(y, x, \eta, \xi))$$

such that

$$e^{\langle y - x, \psi(y, x, \eta) \rangle} - 1 = e^{\langle \eta - \xi, \phi(x, \eta, \xi) \rangle} \Psi(y, x, \eta, \xi).$$

Then remarking $e^{i(\theta_k \partial_t + \partial_\eta)}(y_k - x_k) = 0$ for any $k$, we have,
\[ T(t; x, \xi) = e^{(\partial_t; \partial_x + \partial_{\xi})} [y - x, \Psi(y, x, \eta, \xi)] e^{(\eta - \xi, \phi(x, \eta, \xi))] |_{y=x, \eta=\xi} \]

\[ = \left[ (y - x, e^{(\partial_t; \partial_x + \partial_{\xi})} \Psi(y, x, \eta, \xi)) + e^{(\eta - \xi, \phi(x, \eta, \xi))] |_{y=x, \eta=\xi} \right] \]

\[ = e^{(\partial_t; \partial_x + \partial_{\xi})} \left[ e^{(\eta - \xi, \phi(x, \eta, \xi))] |_{y=x, \eta=\xi} \right] \]

\[ = \sum_{\alpha, \beta \geq 0} \frac{t^{|\beta|}}{\alpha! \beta!} \partial_{\xi}^{\beta} [e^{(\eta - \xi, \phi(x, \eta, \xi))] |_{\eta=\xi} \]

In the case where we do not have \( \beta \geq \alpha \), we have:

\[ \partial_{\xi}^{\beta} [e^{(\eta - \xi, \phi(x, \eta, \xi))] |_{\eta=\xi} = 0. \]

So we may assume \( \beta \geq \alpha \).

(i) The case \( \beta > \alpha = 0 \). Then we have \( \partial_{\xi}^{\beta} [1] = 0 \).

(ii) The case \( \beta \geq \alpha > 0 \). Then we have:

\[ \partial_{\xi}^{\beta} [e^{(\eta - \xi, \phi(x, \eta, \xi))] |_{\eta=\xi} = \sum_{0 \leq |\gamma| \leq \beta} \frac{(-1)^{|\gamma|}}{\alpha! \beta!} \partial_{\xi}^{\beta - \alpha} [\phi(x, \eta, \xi)^{\alpha}] |_{\eta=\xi}(\gamma \neq \alpha), \]

\[ = \left\{ \begin{array}{ll}
0 & (\gamma \neq \alpha), \\
\frac{(-1)^{|\gamma|}}{\alpha! \beta!} \partial_{\xi}^{\beta - \alpha} [\phi(x, \eta, \xi)^{\alpha}] |_{\eta=\xi} & (\gamma = \alpha).
\end{array} \right. \]

(iii) The case \( \alpha = \beta = 0 \). Then \([1] |_{\eta=\xi} = 1 \).

Thus we have:

\[ T(t; x, \xi) = 1 + \sum_{0 < |\alpha| \leq \beta} \frac{(-1)^{|\alpha|}}{\alpha! (\beta - \alpha)!} \partial_{\xi}^{\beta - \alpha} [\phi(x, \eta, \xi)^{\alpha}] |_{\eta=\xi} \]

or equivalently, setting:

\[ T_v(t; x, \xi) = \left\{ \begin{array}{ll}
1 & (v = 0), \\
\sum_{|\beta| = 0 < |\alpha| \leq \beta} \frac{(-1)^{|\alpha|}}{\alpha! (\beta - \alpha)!} \partial_{\xi}^{\beta - \alpha} [\phi(x, \eta, \xi)^{\alpha}] |_{\eta=\xi} & (v > 0),
\end{array} \right. \]

we have:

\[ T(t; x, \xi) = \sum_{v=0}^{\infty} t^v T_v(t; x, \xi). \]
Let take any $\Omega_1 \Subset \Omega$ of the form $\mathbb{C}^n \times \omega_1$ with $\omega_1 \Subset \omega$ (recall $\Omega = \mathbb{C}^n \times \omega$) and set the distance $\delta_0 := \text{dist}(\partial \omega \cap \{ |\xi| = 1 \}, \partial \omega_1 \cap \{ |\xi| = 1 \})$ and $\delta := \min(\delta_0, 1/2)$. In the sequel, we set:

$$H := \max_{|\xi| = 1} |H_M(\xi)|.$$  

Because for any $k$, we have:

$$\partial_\xi p(x, \xi) = \frac{\partial_\xi P(x, \xi)}{P(x, \xi)},$$

by using Lemma 6.12, for any $r' > r$, we have the following: there is a continuous function $a(x) \geq 0$ and for any $\varepsilon > 0$, there is $K > 0$ so that

$$|\partial_\xi p(x, \xi)| \leq \frac{\varepsilon |\xi| + |H_M(\xi)| + \delta H |\xi| + K + a(x)}{|\xi| \delta^3}$$

$$\leq \frac{\varepsilon |\xi| + H |\xi| + a(x)}{|\xi| \delta^4}$$
on $\Omega_1[r']. \quad (6.14)$

We recall a lemma of Aoki (see [2]):

**Lemma.** There is a function $A(s) : \mathbb{R}_+ \to \mathbb{R}_+$ (where $\mathbb{R}_+ := ]0, \infty[ \subset \mathbb{R}$), such that we have:

$$\begin{cases}
\text{for any } c > 0, \text{ we have } A(cs) \leq cA(s), \\
\lim_{s \to \infty} \frac{A(s)}{s} = 0, \\
|\partial_\xi p(x, \xi)| \leq \frac{A(|\xi|) + H |\xi| + a(x)}{\delta^4 |\xi|} \quad \text{on } \Omega_1[r'].
\end{cases}$$

By (6.11), we have for any $k$,

$$\phi_k(x, \eta, \xi) = \int_0^1 \partial_\xi p(x, s\eta + (1-s)\xi) \, ds,$$

so we have:

$$|\phi(x, \eta, \xi)| \leq \frac{2^A(|\xi| + |\eta|) + H (|\xi| + |\eta|) + a(x)}{\delta^4 |\xi|}.$$  

Then by the inequality of Cauchy and (6.12), for any $x \in \mathbb{C}^n$ and any $R > 1$, setting $a_R = a_R(x) := \max_{|y| \leq R} a(x + y)$, we have:
Taking (6.15), we have:

\[
|T_\nu(x, \xi)| \leq \sum_{|\beta| = \nu} \sum_{\alpha \leq \beta} \frac{\beta!}{\alpha! R^{\beta/(\delta|\xi|)}|\beta-\alpha|} \times \left[ 4 \frac{A((1+\delta)|\xi|)}{\delta^4|\xi|} + (1+\delta)H|\xi| + aR^2 \right]^{\alpha|} \\
\leq \sum_{l=1}^{\nu} \frac{2^l n! v!}{l! R^l (\delta|\xi|)^{v-l}} \left[ 6 \frac{A(|\xi|) + H|\xi| + aR}{\delta^4|\xi|} \right]^{l} \\
= \sum_{l=1}^{\nu} \frac{v!}{l! (\delta|\xi|)^{v-l}} \left[ 12n \frac{A(|\xi|) + H|\xi| + aR}{\delta^4|\xi|} \right]^{l} \\
\leq R^{-v} \left[ 12n \frac{A(|\xi|) + H|\xi| + aR}{\delta^4|\xi|} \right]^{m} \times \sum_{l=1}^{\nu} \frac{v!}{l! (\delta|\xi|)^{v-l}} \left[ 12n \frac{A(|\xi|) + H|\xi| + aR}{\delta^4|\xi|} \right]^{l-1}.
\]

Taking \(d \geq 1\) large enough so that \(d \geq \max(3, 2r)\) and we have:

\[
12n \frac{A(|\xi|)}{\delta^4|\xi|} < 1 \quad \text{for any } \xi \in \omega_1 \text{ with } |\xi| \geq d.
\] (6.15)

Then setting \(m = l - 1\), we have, for any \(\xi \in \omega_1\) with \(|\xi| \geq (\nu + 1)d\),

\[
|T_\nu(x, \xi)| \leq R^{-v} \left[ 12n \frac{A(|\xi|) + H|\xi| + aR}{\delta^4|\xi|} \right]^{m} \sum_{l=1}^{\nu} \left( \frac{v-1}{m} \right) \frac{v}{m+1} \frac{(v-1-m)!}{((\nu+1)d)^{v-m}} \\
\times \left[ 12n \frac{A(|\xi|) + H|\xi| + aR}{\delta^4|\xi|} \right]^{l} \\
\leq v R^{-v} \left[ 12n \frac{A(|\xi|) + H|\xi| + aR}{\delta^4|\xi|} \right]^{m} \sum_{l=1}^{\nu} \left( \frac{v-1}{m} \right) \frac{1}{d^{v-1-m}} \\
\times \left[ 12n \frac{A(|\xi|) + H|\xi| + aR}{\delta^4|\xi|} \right]^{l} \\
= v R^{-v} \left( \frac{1}{d} + \frac{12n A(|\xi|) + H|\xi| + aR}{\delta^4|\xi|} \right)^{v-1} \left[ 12n \frac{A(|\xi|) + H|\xi| + aR}{\delta^4|\xi|} \right]^{m} \\
\leq v R^{-v} \left( \frac{1}{d} + \frac{12n A(|\xi|) + H|\xi| + aR}{\delta^4|\xi|} \right)^{v-1} \left[ 12n \frac{A(|\xi|) + H|\xi| + aR}{\delta^4|\xi|} \right]^{m} \\
= v R^{-v} \left( \frac{1}{d} + \frac{12n A(|\xi|) + H|\xi| + aR}{\delta^4|\xi|} \right)^{v-1} \left[ 12n \frac{A(|\xi|) + H|\xi| + aR}{\delta^4|\xi|} \right].
\]

By (6.15), we have:
\[
\frac{1}{d} + \left[ 12n \frac{A(|\xi|) + H|\xi| + aR}{\delta^4|\xi|} \right] < 1 + \frac{1}{d} + \frac{12n}{\delta^4} \left( H + \frac{aR}{d} \right) = 1 + \frac{12n}{\delta^4} H + \frac{1 + 12naR/\delta^4}{d}.
\]

Therefore setting,
\[
A := \frac{2 + 12nH/\delta^4 + (1 + 12naR/\delta^4)/d}{R},
\]
taking \( R \gg 1 \), we have \( 0 < A < 1/2 \) for \( d \gg 1 \) and
\[
|T_v(x, \xi)| \leq \frac{A^v}{R} \left[ 12n \frac{A(|\xi|) + H|\xi| + aR}{\delta^4|\xi|} \right] \leq \frac{12n}{\delta^4} R A^v \left[ \frac{A(|\xi|)}{|\xi|} + H + \frac{aR}{d} \right].
\]

Then by Proposition 6.14, taking \( \Omega \) a little bit smaller and taking \( R > 1 \) large enough, there are \( T(x, \xi) \in S(\Omega) \) and \( C', C'' > 0 \) such that we have \( T(x, \xi) - T(t; x, \xi) \in \tilde{N}(\Omega) \) and
\[
C' \leq |T(x, \xi)| \leq C''
\]
on \( \Omega'[r'] \). In fact, if we set \( C := \frac{12nd}{\delta^4R} \) and \( a := H + aR/d \) in Proposition 6.13, taking \( R > 1, d > 1 \) large enough, we have:
\[
A(1 + 2Ca) = \frac{2 + 12nH/\delta^4 + (1 + 12naR/\delta^4)/d}{R} \left( 1 + \frac{24nd}{\delta^4R} \left( H + \frac{aR}{d} \right) \right) < 1
\]
that is \( 0 < a < (1 - A)/(2CA) \). Thus by Theorem 6.11, \( T \) is invertible in \( E^R(\Omega) \). Then we have:
\[
T^{-1} \circ \left( e^{\partial t \cdot \partial_x Q} \right) \circ P = 1.
\]
And by the same way, we have also the left inverse of \( P \) and thus the theorem was proved. \( \square \)

**Remark.** By the above proof, we can prove that thus constructed \( P^{-1}(t; x, \xi) \) satisfies the following estimate which is stronger than (6.4),
\[
\sum_{v=0}^{\infty} r^v R_v(x, \xi) = \mathcal{O}(\Omega(\Omega'(v + 1)r)) \quad (\forall v \in \mathbb{Z}_+)
\]
and \( r > 0 \), we have the following: for any conic set \( \Omega' \subset \Omega \), there exist \( d > r \) and \( A > 0 \) with \( 0 < A < 1 \) such that for any \( \varepsilon > 0 \), there exists \( C_\varepsilon > 0 \) so that we have:
\[
|R_v(x, \xi)| \leq C_\varepsilon A^v e^{-M(\xi)r}\varepsilon \quad \text{for every } v \geq 0 \text{ and any } (x, \xi) \in \Omega'(v + 1)d].
\]
In fact, we can easily prove Lemmas 6.8, (1)(a), (2)(a), 6.9 and Proposition 6.11 with this estimate, instead of the estimate of type (6.4).

Example. For example, let consider the case where the operator $P$ is a non-local pseudodifferential operator having a constant coefficient symbol $P(\xi) \in S^M(\Omega)$, $\Omega = \mathbb{C}^n \times \omega$, non-characteristic at $p = (x_0, \xi_0) \in \Omega$, so $P$ is non-characteristic at any $(x, \xi_0) \in X \times \{\xi_0\}$. In this case, by the proof of Theorem 6.2, we find that $P$ has the inverse operator $P^{-1}$ with symbol $1/P(\xi)$. We remark also that to say the symbol $P(\xi)$ is with constant coefficients is equivalent to say $\hat{P}(\xi)$ is with constant coefficients. Then in the situation of Proposition 4.3, for any $\bigcap_{k=1}^n (\Gamma_k, \delta + M - M)$-round open set $D$, $\bigcap_{k=1}^n (\Gamma_k, \delta + M - M)$-open sets $U, U_0$ with $U_0 \subset U$, $U \setminus U_0 \subset D$ and any $f(x) \in O((U_0 \cap D) + M - M)$, setting $\eta := (1, \eta')$ for the simplicity, we have:

$$P^{-1} f(x) = \frac{1}{(2\pi \sqrt{-1})^n} \int_{\gamma} \int_{\beta} \int_\rho \int_\tau e^{-\tau \cdot \eta} \tau^{n-1} f(x+z) \frac{1}{P(\tau \eta)} \, d\tau$$

$$\in O((U_0 \cap D) + M)$$

and thus this gives a solution of the equation,

$$Pu(x) \equiv f(x) \text{ on } U_0 \cap D \text{ mod } O((U \cap D) + M).$$

References