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Anti-periodic solutions for evolution equations associated with maximal monotone mappings

Yuqing Chen [a], Juan J. Nieto [b,*], Donal O'Regan [c]

[a] Faculty of Applied Mathematics, Guangdong University of Technology, Guangzhou, Guangdong 510006, PR China
[b] Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Santiago de Compostela, Santiago de Compostela 15782, Spain
[c] Department of Mathematics, National University of Ireland, Galway, Ireland

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ABSTRACT

In this work, we study the anti-periodic problem for a nonlinear evolution inclusion where the nonlinear part is an odd maximal monotone mapping and the forcing term is an anti-periodic mapping. Several existence results are obtained under suitable conditions. An example is presented to illustrate the results.

1. Introduction

The study of anti-periodic solutions for nonlinear evolution equations is closely related to the study of periodic solutions, and it was initiated by [1]. Anti-periodic problems have been studied by many authors; see [2–23] and references therein. In [24] Okochi showed that

$$\begin{cases} x'(t) \in -\partial\phi(x(t)) + f(t), & \text{a.e. } t \in R, \\ x(t) = -x(t+T), & t \in R \end{cases} \tag{E1.1}$$

has a solution, where $\phi : D(\phi) \subseteq H \to H$ is an even lower semi-continuous convex function, and $f(t) : R \to H$ satisfies $f(t+T) = -f(t)$ and $f(\cdot) \in L^2(0, T)$. It is of interest to ask whether

$$\begin{cases} x'(t) \in -Ax(t) + f(t), & \text{a.e. } t \in R, \\ x(t) = -x(t+T), & t \in R \end{cases} \tag{E1.2}$$

has a solution, when $A : D(A) \subseteq H \to 2^H$ is an odd maximal monotone mapping. The purpose of this work is to study this problem and we show that this equation has a solution under a linear growth condition on A. Also we consider the anti-periodic problem

$$\begin{cases} x'(t) \in -Ax(t) + \partial G(x(t)) + f(t), & \text{a.e. } t \in R, \\ x(t) = -x(t+T), & t \in R, \end{cases} \tag{E1.3}$$

where $G : H \to H$ is a continuously differentiable mapping such that ∂G is a bounded mapping, i.e. ∂G maps bounded subsets to bounded subsets and $f(\cdot) \in L^2([0, T]; H)$. Under a linear growth condition on A and the condition that $D(A)$ is compactly embedded into H, we prove an existence result for (E1.3).

* Corresponding author. Tel.: +34 881 813 177; fax: +34 981 597 054.
E-mail addresses: ychen64@163.com (Y. Chen), juanjose.nieto.roig@usc.es (J.J. Nieto), donal.oregan@nuigalway.ie (D. O'Regan).

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2. Anti-periodic solutions for nonlinear equations associated with maximal monotone mappings

Lemma 2.1. *Let H be a real Hilbert space, and let $A : D(A) \subseteq H \to 2^H$ be an odd maximal monotone mapping, where $D(A)$ is symmetric and $0 \in D(A)$, and $f(\cdot) : R \to H$ is a function satisfying $\int_0^T \|f(t)\|^2 dt < +\infty$. In addition suppose $\alpha D(A) \subseteq D(A)$ for some $\alpha \in (0, 1)$. Then there exists a unique $y_\alpha \in \overline{D(A)}$ such that*

$$\begin{cases} x'(t) \in -Ax(t) + f(t), & \text{a.e. } t \in (0, +\infty), \\ x(0) = y_\alpha, & -\alpha x(T) = y_\alpha \end{cases} \tag{E2.1}$$

has a unique solution.

Proof. It is well known that

$$\begin{cases} x'(t) \in -Ax(t) + f(t), & \text{a.e. } t \in (0, +\infty), \\ x(0) = y \in \overline{D(A)} \end{cases} \tag{E2.2}$$

has a unique solution $x(t, y)$ for each $y \in \overline{D(A)}$ and it depends continuously on the initial value y; see for example [25].

We define a mapping $K_\alpha : \overline{D(A)} \to \overline{D(A)}$ by

$$K_\alpha y = -\alpha x(T, y), \quad y \in \overline{D(A)}.$$

For $y_1, y_2 \in \overline{D(A)}$, we have

$$\frac{d\|x(t, y_1) - x(t, y_2)\|^2}{dt} = 2(x(t, y_1) - x(t, y_2), x'(t, y_1) - x'(t, y_2)) \le 0.$$

Thus $\|x(T, y_1) - x(T, y_2)\| \le \|y_1 - y_2\|$, so we have

$$\|K_\alpha y_1 - K_\alpha y_2\| \le \alpha \|y_1 - y_2\|, \quad \text{for all } y_1, y_2 \in \overline{D(A)}.$$

Banach's contraction principle guarantees that there exists a unique $y_\alpha \in \overline{D(A)}$ such that

$$K_\alpha y_\alpha = y_\alpha.$$

That is, $x(t, y_\alpha)$ is a solution of (E2.1). The uniqueness is obvious. □

Theorem 2.2. *Let H be a real Hilbert space, and let $A : D(A) \subseteq H \to 2^H$ be an odd maximal monotone mapping, where $D(A)$ is symmetric and convex, and $f(\cdot) : R \to H$ is a function satisfying $f(t + T) = -f(t)$ for $t \in R$ and $\int_0^T \|f(t)\|^2 dt < +\infty$. In addition suppose $\|g\| \le M\|x\|$ for all $x \in D(A)$, $g \in Ax$, where $M > 0$ is a constant such that $MT < 2$. Then*

$$\begin{cases} x'(t) \in -Ax(t) + f(t), & \text{a.e. } t \in R, \\ x(t) = -x(t + T), & t \in R \end{cases} \tag{E2.3}$$

has a solution.

Proof. Since $D(A)$ is symmetric and convex, $0 \in D(A)$. Take a sequence $\alpha_n \in (0, 1)$, $n = 1, 2, \ldots$, such that $\alpha_n \to 1$. By Lemma 2.1, there exist $y_n \in \overline{D(A)}$ such that

$$\begin{cases} x'(t) \in -Ax(t) + f(t), & \text{a.e. } t \in (0, +\infty), \\ x(0) = y_n, & -\alpha_n x(T) = y_n \end{cases}$$

has a unique solution $x(t, y_n)$.

We claim that $\{y_n\}_{n=1}^\infty$ is bounded in H. Indeed, there exist $f_n(t) \in Ax(t, y_n)$ for a.e. $t \in (0, +\infty)$, $n = 1, 2, \ldots$, such that

$$x'(t, y_n) = -f_n(t) + f(t), \quad \text{a.e. } t \in (0, T).$$

Take the inner product with $x'(t, y_n)$ and integrate over $[0, T]$ and we get

$$\int_0^T \|x'(t, y_n)\|^2 dt = -\int_0^T (f_n(t), x'(t, y_n)) dt + \int_0^T (f(t), x'(t, y_n)) dt.$$

From this and the assumption on A, it immediately follows that

$$\int_0^T \|x'(t, y_n)\|^2 dt \le M \left(\int_0^T \|x(t, y_n)\|^2 dt\right)^{\frac{1}{2}} \left(\int_0^T \|x'(t, y_n)\|^2 dt\right)^{\frac{1}{2}} + \left(\int_0^T \|f(t)\|^2 dt\right)^{\frac{1}{2}} \left(\int_0^T \|x'(t, y_n)\|^2 dt\right)^{\frac{1}{2}}. \tag{2.1}$$

Since $-\alpha_n x(T, y_n) = y_n$, we have

$$x(t, y_n) = -\frac{\alpha_n}{1+\alpha_n}\int_0^T x'(s, y_n)\mathrm{d}s + \int_0^t x'(s, y_n)\mathrm{d}s = \frac{1}{1+\alpha_n}\int_0^t x'(s, y_n)\mathrm{d}s - \frac{\alpha_n}{1+\alpha_n}\int_t^T x'(s, y_n)\mathrm{d}s.$$

As a result

$$\max_{t\in[0,T]} \|x(t, y_n)\| \le \frac{1}{1+\alpha_n}\sqrt{T}\left(\int_0^T \|x'(t, y_n)\|^2\mathrm{d}t\right)^{\frac{1}{2}}. \tag{2.2}$$

From (2.1) and (2.2), we obtain

$$\left(1 - \frac{MT}{1+\alpha_n}\right)\int_0^T \|x'(t, y_n)\|^2\mathrm{d}t \le \left(\int_0^T \|f(t)\|^2\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^T \|x'(t, y_n)\|^2\mathrm{d}t\right)^{\frac{1}{2}}. \tag{2.3}$$

Notice that $MT < 2$, and $\alpha_n \to 1$. For n sufficiently large, there exists $\beta_0 > 0$ such that $1 - \frac{MT}{1+\alpha_n} \ge \beta_0$, and from this and (2.3), we infer that $\{\int_0^T \|x'(t, y_n)\|^2\mathrm{d}t\}_{n=1}^{\infty}$ is bounded, and so by (2.2), $\{y_n\}$ is bounded. Thus the claim is true.

For simplicity we may assume that $y_n \rightharpoonup y_0 \in H$. (Otherwise, we may take a subsequence.) Notice that the convexity of $D(A)$ implies that $y_0 \in \overline{D(A)}$. Next, we prove that $x(t, y_0)$ is a solution of (E2.3). To achieve the goal we note that

$$(y_n + x(T, y_n) - y - x(T, y), y_n - y) \ge 0, \quad \text{for all } y \in \overline{D(A)},$$

since $\|x(T, y_n) - x(T, y)\| \le \|y_n - y\|$.

Letting $n \to \infty$, and noting that $y_n + x(T, y_n) \to 0$, we get

$$(-y - x(T, y), y_0 - y) \ge 0, \quad \text{for all } y \in \overline{D(A)}. \tag{2.4}$$

For $t \in (0, 1)$, set $y_t = ty_0 - (1-t)x(T, y_0)$, and note that $y_t \in \overline{D(A)}$ since $D(A)$ is convex. Now from (2.4) we get

$$(-ty_0 + (1-t)x(T, y_0) - x(T, ty_0 - (1-t)x(T, y_0)), y_0 + x(T, y_0)) \ge 0.$$

Again, notice that $x(t, y)$ depends continuously on the initial value $y \in \overline{D(A)}$, so by letting $t \to 1^-$, we get

$$-\|y_0 + x(T, y_0)\|^2 \ge 0.$$

Thus $y_0 = -x(T, y_0)$, so $x(t, y_0)$ is a solution of (E2.3). □

In the following, let $L^2([0, T]; H) = \{f(t) : [0, T] \to H; \int_0^T \|f(s)\|^2\mathrm{d}s < +\infty\}$, and the norm in $L^2([0, T]; H)$ is denoted by $\|f(\cdot)\|_{L^2} = (\int_0^T \|f(s)\|^2\mathrm{d}s)^{\frac{1}{2}}$. We let $C_a = \{v(t) : R \to H \text{ is continuous and } v(t) = -v(t+T), t \in R\}$, and $W_a = \{u(\cdot) \in C_a : u'(\cdot) \in L^2([0, T]; H)\}$. Now C_a is a Banach space under the norm $|v(\cdot)|_\infty = \max_{t\in[0,T]} \|u(t)\|$, and by Lemma 2.1 in [12] (see also [9]), W_a is a Banach space under the norm $\|u(\cdot)\|_a = \|u'(\cdot)\|_{L^2}$.

Theorem 2.3. *Let $A : D(A) \subseteq H \to 2^H$ be an odd maximal monotone mapping, where $D(A)$ is symmetric and convex, $G : H \to R$ is a continuously differentiable even function such that ∂G is a bounded mapping, i.e. ∂G maps bounded subsets to bounded subsets in H, and $f(t) : R \to H$ satisfies $f(t+T) = -f(t)$, for a.e. $t \in R$ and $\int_0^T \|f(t)\|^2\mathrm{d}t < +\infty$. Also suppose $D(A)$ is compactly embedded into H, and $\|g\| \le M\|x\|$ for all $x \in D(A)$, $g \in Ax$, where $M > 0$ is a constant such that $MT < 2$. Then the anti-periodic problem*

$$\begin{cases} u'(t) \in -Au(t) + \partial G(u(t)) + f(t), & \text{a.e. } t \in R, \\ u(t) = -u(t+T), & t \in R \end{cases} \tag{E2.4}$$

has a solution $u(\cdot) \in W_a$.

Proof. For each $v(\cdot) \in C_a$, we consider the anti-periodic problem

$$\begin{cases} u'(t) + u(t) \in -Au(t) + \partial Gv(t) + v(t) + f(t), & \text{a.e. } t \in R, \\ u(t) = -u(t+T). \end{cases} \tag{E2.5}$$

To prove that (E2.5) has a unique solution, we consider the initial value problem

$$\begin{cases} u'(t) + u(t) \in -Au(t) + \partial Gv(t) + v(t) + f(t), & \text{a.e. } t \in R, \\ u(0) = y \in \overline{D(A)}. \end{cases} \tag{E2.6}$$

Note that (E2.6) has a unique solution $x(t, y)$ for each $y \in \overline{D(A)}$. We define $K : \overline{D(A)} \to \overline{D(A)}$ by $Ky = -x(T, y)$. It is easy to see that K is a contraction since $I + A$ is strongly monotone. Therefore there exists $y_0 \in \overline{D(A)}$ such that $Ky_0 = -x(T, y_0)$.

Thus $x(t, y_0)$ is a solution to (E2.5). The uniqueness follows since $I + A$ is strongly monotone. We denote the unique solution of (E2.5) by $Kv(\cdot)$ for each $v(\cdot) \in C_a$.

From (E2.5), taking the inner products with $(Kv)'(t)$ and integrate over $[0, T]$, we get

$$\|(Kv)'(\cdot)\|_{L^2} \le M\|Kv(\cdot)\|_{L^2} + \|\partial Gv(\cdot)\|_{L^2} + \|v(\cdot)\|_{L^2} + \|f(\cdot)\|_{L^2}. \tag{2.5}$$

From Lemma 2.1 in [12] we have

$$|Kv(\cdot)|_\infty \le \frac{\sqrt{T}}{2}\left(\int_0^T |(Kv)'(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}}.$$

Thus

$$|(Kv)'(\cdot)|_{L^2} \le \left(1 - \frac{MT}{2}\right)^{-1}[\|\partial Gv(\cdot)\|_{L^2} + \|v(\cdot)\|_{L^2} + \|f(\cdot)\|_{L^2}]. \tag{2.6}$$

From (2.6) and the bounded assumption of ∂G we infer that K maps bounded subsets of C_a to bounded subsets of W_a. Since $D(A)$ is compactly embedded into H, K is a compact mapping.

Next, we prove that $K : C_a \to C_a$ is continuous. Let $v_n(\cdot) \to v(\cdot) \in C_a$ as $n \to \infty$. (Consequently, $v_n(\cdot) \to v(\cdot) \in L^2([0, T]; H)$.) The same reasoning as in (2.6) yields

$$|(Kv_n)'(\cdot) - (Kv)'(\cdot)|_{L^2} \le \left(1 - \frac{MT}{2}\right)^{-1}[\|\partial Gv_n(\cdot) - \partial Gv(\cdot)\|_{L^2} + \|v_n(\cdot) - v(\cdot)\|_{L^2}]. \tag{2.7}$$

Thus $\|(Kv_n)'(\cdot) - (Kv)'(\cdot)\|_{L^2} \to 0$. Therefore $\|Kv_n(\cdot) - Kv(\cdot)\|_\infty \to 0$, and K is continuous.

Finally, we prove that $Kv(\cdot) \ne \|\lambda v(\cdot)\|$ for all $\lambda \ge 1$, and $v(\cdot) \in C_a$ with $|v(\cdot)|_\infty = r_0$, where $r_0 > (1 - \frac{MT}{2})^{-1}\frac{\sqrt{T}}{2}(\int_0^T |f(t)|^2 \mathrm{d}t)^{\frac{1}{2}}$ is a constant.

If this is not true, there exist $\lambda_0 \ge 1$, $v_0(\cdot) \in C_a$ with $|v_0(\cdot)|_\infty = r_0$ such that $Kv_0(\cdot) = \lambda_0 v_0(\cdot)$, i.e. $v_0(t) = -v_0(t + T)$ and

$$\lambda_0(v_0'(t) + v_0(t)) \in -A\lambda_0 v_0(t) + \partial Gv_0(t) + v_0(t) + f(t), \quad \text{a.e. } t \in R,$$

i.e. there exists $g(t) \in A\lambda v_0(t)$, for a.e. $t \in R$ such that

$$\lambda_0(v_0'(t) + v_0(t)) = -g(t) + \partial Gv_0(t) + v_0(t) + f(t), \quad \text{a.e. } t \in R. \tag{2.8}$$

From (2.7), take the inner product with $v_0'(t)$, integrate over $[0, T]$ and note that $\int_0^T (\partial Gv_0(t), v_0'(t))\mathrm{d}t = 0$; we get

$$\lambda_0\left(\int_0^T |v_0'(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}} \le \lambda_0 M\sqrt{T}|v_0(\cdot)|_\infty + \left(\int_0^T |f(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}}. \tag{2.9}$$

Recall that

$$|v_0(\cdot)|_\infty \le \frac{\sqrt{T}}{2}\left(\int_0^T |v_0'(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}},$$

so we conclude from (2.8) that

$$\lambda_0|v_0(\cdot)|_\infty \le \lambda_0\frac{MT}{2}|v_0(\cdot)|_\infty + \frac{\sqrt{T}}{2}\left(\int_0^T |f(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}}.$$

Therefore it follows that

$$|v_0(\cdot)|_\infty \le \left(1 - \frac{MT}{2}\right)^{-1}\frac{\sqrt{T}}{2}\left(\int_0^T |g(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}},$$

which contradicts $|v_0(\cdot)|_\infty = r_0 > (1 - \frac{MT}{2})^{-1}\frac{\sqrt{T}}{2}(\int_0^T |f(t)|^2 \mathrm{d}t)^{\frac{1}{2}}$.

The homotopy property of the Leray–Schauder degree implies that $deg(I - K, B(0, r_0), 0) = 1$. Thus $Kv(\cdot) = v(\cdot)$ has a solution $v(\cdot)$ in $B(0, r_0)$, which is easily seen to be a solution of (E2.4). □

Corollary 2.4. *Let $\beta : R \to 2^R$ be an odd maximal monotone mapping, and $|g| \le M|x|$ for all $x \in R$ and $g \in \beta(x)$, where $MT < 2$, and $f(\cdot) : R \to R$ satisfy $f(t + T) = -f(t)$ for $t \in R$ and $\int_0^T f(t)^2 \mathrm{d}t < +\infty$. Then*

$$\begin{cases} u'(t) \in -\beta(u(t)) + 2u(t)e^{u^2(t)} + f(t), & \text{a.e. } t \in R, \\ u(t) = -u(t + T), & t \in R \end{cases} \tag{E2.7}$$

has a solution.

Remark. For A the sub-differential of a lower semi-continuous convex function, results similar to Theorem 2.3 were obtained in [5,13].

3. An example

Example 3.1. Let $\Omega \subset R^n$ be an open bounded subset with smooth boundary. Assume that $a_i, b_i : R \to [0, +\infty)$ are continuous functions for $i = 1, 2, \ldots, n$. Suppose the following conditions are satisfied:

(1) $c_1 \le a_i(x) \le c_2$ for all $(t, x) \in R^2$, where $c_1, c_2 > 0$ are constants;
(2) $\sum_i [b_i(x_i) - b_i(y_i)](x_i - y_i) \ge 0$, where $x = (x_i), y = (y_i) \in R^N$;
(3) $|b_i(t)| \le \beta|t|$ for all $t \in R$ and $i = 1, 2, \ldots, n$;
(4) $b_i(-t) = -b_i(t)$ for $t \in R$, and $i = 1, 2, \ldots, n$.

Let $A : H_0^1(\Omega) \to H^*$ be defined as

$$(Au, v) = \int_\Omega \left[\sum_{i=1}^n a_i(x) b_i(D_i u) D_i v\right] \mathrm{d}x$$

for all $u, v \in H_0^1(\Omega)$. Then we have the following:

(a) $\|Au\| \le c_2 \beta \sqrt{\int_\Omega (\sum_{i=1}^n |D_i u|^2) \mathrm{d}x}$ for all $u \in H_0^1(\Omega)$,
(b) $(Au - Av, u - v) \ge 0$ for $t \in R, u, v, \in H_0^1(\Omega)$, and A is continuous and monotone and so it is maximal monotone,
(c) $A(-u) = -Au$, for $u \in H_0^1(\Omega)$.

Consider

$$\begin{cases} u'(t, x) = -\sum_{i=1}^n D_i[a_i(x) b_i(D_i u)] + f(t, x), & \text{a. e. } t \in R, x \in \Omega, \\ u(t, x) = 0, \quad x \in \partial\Omega, \text{ a.e. } t \in R, \\ u(t, x) = -u(t + T, x), \quad t \in R, x \in \Omega, \end{cases} \tag{E3.1}$$

where $f(t, x) : R \times R^n \to R$ is continuous and $f(t + T, x) = -f(t, x)$.

Suppose that $c_2 \beta T < 2$. Then by Theorem 2.2, (E3.1) has a generalized solution $u(t, x)$, i.e. $u(t, x) = -u(t + T, x)$ for a.e. $t \in R, x \in \Omega$, and

$$\int_\Omega u'(t, x) v(x) \mathrm{d}x = \int_\Omega [a_i(x) D_i(u(t, x)) D_i v(x) + f(t, x) v(x)] \mathrm{d}x,$$

for a.e. $t \in R$, and $v(\cdot) \in H_0^1(\Omega)$.

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