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Dentability and Extreme Points in Banach Spaces*

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It is shown that a Banach space E has the Radon–Nikodym property (equivalently, every bounded subset of E is dentable) if and only if every bounded closed convex subset of E is the closed convex hull of its strongly exposed points. Using recent work of Namioka, some analogous results are obtained concerning weak* strongly exposed points of weak* compact convex subsets of certain dual Banach spaces.

The notion of a dentable subset of a Banach space was introduced by Rieffel [16] in conjunction with a Radon–Nikodym theorem for Banach space-valued measures. Subsequent work by Maynard [12] and by Davis and Phelps [6] (also by Huff [8]) has shown that those Banach spaces in which Rieffel's Radon–Nikodym theorem is valid are precisely the ones in which every bounded closed convex set is dentable (definition below). Diestel [7] observed that the classes of spaces (e.g., reflexive spaces, separable conjugate spaces) which are known to have this property (the "Radon–Nikodym property") appear to coincide with those which are known to have the property that every bounded closed convex subset is the closed convex hull of its extreme points (the "Krein–Milman property") and he raised the question as to whether the two properties are equivalent. Quite recently, Lindenstrauss showed (Theorem 2) that the Radon–Nikodym property does indeed imply the Krein–Milman property. (The converse remains an open question.) In the present paper we prove a stronger result, in which "extreme point" is replaced by "strongly exposed point." Our methods also permit us to extend a recent theorem by Namioka [13], give an alternative proof of a result of Collier and Edelstein [5] and generalize some results of John and Zizler [9].

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We want to thank Professors Richard Bourgin, William J. Davis, Joseph Diestel and Ka-Sing Lau for helpful and stimulating conversations related to the subject matter of this paper. An important step in the argument (Lemma 7) is due to Errett Bishop, who communicated it to us in a letter a number of years ago. Above all, we are deeply indebted to Isaac Namioka. In addition to suggesting numerous simplifications in the proofs and better formulations of the theorems themselves, he pointed out that the technique of Lemma 6 (which we had used earlier in an unnecessary detour in the proof of Theorem 5) could be combined with Bishop's lemma to prove Theorem 9, which is the main result of this paper. Moreover, it was his enthusiasm and suggestions which led us to formulate our results so as to make them applicable to dual spaces, thus leading to an extension of part of his own recent work [13].

We first establish some notation. Throughout this paper, the letter C will denote a nonempty, bounded closed and convex subset of the Banach space E. If $f \in E^*$ we let

$$M(f, C) = \sup\{f(x) : x \in C\}$$

and

$$M(C) = \sup\{\|x\| : x \in C\}.$$

If $\|f\| = 1$ and $\alpha > 0$, then we define

$$S(f, \alpha, C) = \{x : x \in C \quad \text{and} \quad f(x) \geqslant M(f, C) - \alpha\}.$$

Such a set is called a *slice* of C and the fact that $\|f\| = 1$ and $\alpha > 0$ will always be understood when we refer to a "slice of C." We let U denote the unit ball of E, i.e.,

$$U = \{x \in E : \|x\| \leqslant 1\}.$$

DEFINITION. Let E be a Banach space and let F be a linear subspace of E^* which separates points of E. A subset $A \subset E$ is said to be *F-dentable* if for each $\epsilon > 0$ there is a point $x \in A$ such that x is not in the $\sigma(E, F)$-closed convex hull of $A \backslash (x + \epsilon U)$. In the case when $F = E^*$, we say A is *dentable*.

It should be noted that a set is dentable if its closed convex hull is dentable [16]. (The converse is true for bounded sets.)

PROPOSITION 1. *A subset A of E is F-dentable if and only if for each $\epsilon > 0$ there is a slice $S(f, \alpha, A)$ of A of diameter less than ϵ, where $f \in F$.*

The above proposition is an easy consequence of the separation theorem, and we omit the proof.

DEFINITION. (i) A point $x \in A \subset E$ is called an *F-denting point* of A if for each $\epsilon > 0$ there is a slice $S(f, \alpha, A)$ of diameter less than ϵ which contains x and is determined by a functional $f \in F$. If $F = E^*$, we simply call x a *denting point.*

(ii) A point $x \in A \subset E$ is called an *F-strongly exposed point of A* if there exists an $f \in F$ of norm one such that, for each $\epsilon > 0$, there is an $\alpha > 0$ such that the slice $S(f, \alpha, A)$ contains x and has diameter less than ϵ. The functional f is said to *strongly expose x*. If $F = E^*$, we say that x is a *strongly exposed point.*

It will be immediate from Theorem 5 (below) that if E has the Radon–Nikodym property, then every C in E is the closed convex hull of its extreme points. This was first proved, in a much simpler way, by Joram Lindenstrauss and he has kindly given us permission to reproduce his argument here.

THEOREM 2. (Lindenstrauss). *If every bounded subset of the Banach space E is dentable and if C is a bounded closed convex subset of E, then C is the closed convex hull of its extreme points.*

Proof. By a *support face* of C we mean a set of the form

$$F(g, C) = \{x \in C : g(x) = M(g, C)\},$$

where $g \in E^*$ has norm 1. Note that any slice $S(f, \alpha, C)$ of C contains a nonempty support face of C: If $0 < \delta < \alpha[2M(C)]^{-1}$, then it is easily seen that $F(g, C) \subset S(f, \alpha, C)$ whenever $\| g \| = 1$ and $\| f - g \| < \delta$. By [4, Cor. 4], there always exists a functional g satisfying these last two conditions such that $F(g, C)$ is nonempty. Now, let $S(f, \alpha, C)$ be any slice of C and choose a nonempty support face $F_1 = F(g, C) \subset S(f, \alpha, C)$. By hypothesis, there exists a slice of F_1 of diameter less than 1 and the above argument (applied to F_1) yields a nonempty support face $F_2 = F(g_1, F_1)$ of F_1 contained in this slice. Obviously, diam $F_2 < 1$. Continuing by induction, there exists a nested sequence $F_1 \supset F_2 \supset F_3 \supset, \ldots,$ of such sets (each one necessarily a face of C) with diam $F_n \to 0$. By completeness, their intersection is a single point which is contained in $S(f, \alpha, C)$. This fact, in conjunction with the separation theorem, completes the proof.

LEMMA 3. *Let T be an isomorphism (i.e., a linear bicontinuous bijection) of E and let $S(f, \alpha, C)$ be a slice of C of diameter d. Then $T[S(f, \alpha, C)]$ is a slice of TC of diameter at most $d \| T \|$.*

Proof. Let T^* denote the adjoint of T and let $f^* = (T^*)^{-1}f$. For any x in C we have $f(x) = f^*(Tx)$; it follows that $M(f, C) = M(f^*, TC)$ and that

$$T[S(f, \alpha, C)] = S(f^*, \alpha, TC).$$

The assertion about the diameter of the latter slice is easily verified.

LEMMA 4. *Suppose that every bounded subset of E is dentable and that $g \in E^*$, $\| g \| = 1$. If $\epsilon > 0$ and $C \backslash g^{-1}(0)$ is nonempty, then there exists a slice of C of diameter less than ϵ which misses the set $D = C \cap g^{-1}(0)$.*

Proof. We assume, of course, that D is nonempty. Let $z \in C \backslash g^{-1}(0)$ (say $g(z) > 0$) and let $r = g(z)^{-1}$. For any $x \in D$ define T_x by

$$T_x y = y - 2rg(y)(z - x), \qquad y \in E.$$

Each T_x is a reflection of E through the hyperplane $g^{-1}(0)$ along the line through 0 determined by $z - x$; in particular, it is readily verified that for each $x \in D$,

$$T_x^{-1} = T_x, \qquad T_x z = 2x - z, \qquad T_x \text{ is the identity on } g^{-1}(0)$$

and

$$\| T_x \| \leqslant 1 + 2r \| z - x \| \leqslant 1 + 4rM(C) \equiv M.$$

Let $\mathscr{K} = \{C\} \cup \{T_x C : x \in D\}$ and let

$$C_1 = \overline{co} \cup \{K : K \in \mathscr{K}\}.$$

Since $M(T_x C) \leqslant M \cdot M(C)$ for each $x \in D$, the closed convex set C_1 is bounded. If $x \in D$, then

$$x = 1/2\, z + 1/2\, T_x z$$

so x is a midpoint of a segment in C_1 of length

$$\| z - T_x z \| = 2 \| z - x \| \geqslant 2g(z) > 0.$$

By hypothesis, there exists a slice $S(f, \alpha, C_1)$ of C_1 of diameter d, where

$$d < \min\{\epsilon/M, g(z)\}.$$

Now, $M(f, C_1) = M(f, \cup \{K: K \in \mathscr{K}\})$, so $M(f, K_o) > M(f, C_1) - \alpha$ for at least one set $K_o \in \mathscr{K}$. Thus

$$\beta = M(f, K_o) - [M(f, C_1) - \alpha] > 0$$

and $S(f, \beta, K_o) \subset S(f, \alpha, C_1)$, so diam $S(f, \beta, K_o) \leqslant d$. If $S(f, \alpha, C_1)$ were to contain a point $x \in D$, it would also contain at least one endpoint of the segment in C_1 of which x is the midpoint, contradicting the fact that $d < g(z)$. Thus the smaller slice $S(f, \beta, K_o)$ also misses D.

Consider the possible choices for K_o: Either $K_o = C$, in which case the proof is complete, or $K_o = T_x C$ for some $x \in D$. In the latter case, Lemma 3 (applied to $T = T_x^{-1}$) shows that $T_x^{-1} S(f, \beta, T_x C)$ is a slice of C of diameter at most $\| T_x^{-1} \| d = \| T_x \| d \leqslant Md < \epsilon$. Moreover, this slice also misses D: If $y \in D$, then $y \notin S(f, \beta, T_x C)$, hence

$$y = T_x^{-1} y \notin T_x^{-1} S(f, \beta, T_x C).$$

THEOREM 5. *Suppose that every bounded subset of E is dentable and that C is bounded, closed and convex. Then C is the closed convex hull of its denting points.*

Proof. By the separation theorem, it suffices to show that each slice $S(g, \beta, C)$ of C contains a denting point of C. By translation, we can assume that the origin is contained in the hyperplane

$$\{x \in E : g(x) = M(g, C) - \beta\},$$

i.e., that this is the same as $g^{-1}(0)$. Let $C_1 = S(g, \beta, C)$ and apply the previous lemma to get a slice of C_1 which misses $C_1 \cap g^{-1}(0)$ and has diameter less than $1/2$. This slice is necessarily a slice of C and is contained in C_1. We can continue by induction to get a nested sequence of slices of C whose diameters converge to 0; their intersection is necessarily a denting point of C inside C_1.

We now show that with some further effort, we can replace "denting points" by "strongly exposed points" in the above result. For this purpose we prove two lemmas, which are stated in terms of F-denting points and F-strongly exposed points, where F is a point-separating linear subspace of E^*. This added generality complicates the statements of the lemmas, but causes no difficulties whatsoever in their proofs. Moreover, it allows us subsequently to prove a result concerning weak*-strongly exposed points for subsets of E^*, where, e.g., "weak*-dentable" means "F-dentable" when F is taken to be the canonical embedding of E into the dual of E^*. Of course, for their application to subsets of E, we will take $F = E^*$.

LEMMA 6. *Suppose that F is a linear subspace of E^* and that every bounded $\sigma(E, F)$-closed convex subset of E is the $\sigma(E, F)$- closed convex hull of its F-denting points. Suppose, moreover, that $S(f, \alpha, C)$ is a slice of the bounded $\sigma(E, F)$-closed convex set C, with $f \in F$, and that $0 < \epsilon < 1$. Then there exists a slice $S(g, \beta, C)$ of diameter less than ϵ such that $g \in F$, $\|f - g\| < \epsilon$ and*

$$S(g, \beta, C) \subset S(f, \alpha, C).$$

Proof. By translation we can assume that the origin is contained in the hyperplane

$$H = \{x \in E : f(x) = M(f, C) - \alpha\};$$

equivalently, $H = f^{-1}(0)$ and $M(f, C) = \alpha > 0$. Let $M = M(C)$ and define C_1 to be the $\sigma(E, F)$-closed convex hull of $S(f, \alpha, C) \cup (\lambda U \cap H)$, where $\lambda > 2M\epsilon^{-1}$. From the hypotheses it follows there is an F-denting point in $C_1 \backslash H$, hence there is a slice $S(g, \beta, C_1)$ of C_1 of diameter less than ϵ which misses $\lambda U \cap H$ and for which $g \in F$. This implies that

$$M(g, C_1) = M(g, S(f, \alpha, C)) = M(g, C),$$

so

$$S(g, \beta, S(f, \alpha, C)) = S(g, \beta, C)$$

is a slice of C of diameter less than ϵ which is contained in $S(f, \alpha, C)$. It remains to show that $\|f - g\| < \epsilon$.

Choose $z \in S(g, \beta, C)$ with

$$g(z) > M(g, C_1) - \beta \geqslant M(g, \lambda U \cap H) \geqslant 0.$$

Since $\lambda U \cap H$ is symmetric, we conclude that

$$g(\lambda U \cap H) \subset [-g(z), g(z)].$$

Equivalently, $g[U \cap f^{-1}(0)] \subset [-\lambda^{-1}g(z), \lambda^{-1}g(z)]$. By [4, Lemma 3.1], this implies that either

$$\|f - g\| \leqslant 2\lambda^{-1}g(z) = 2\lambda^{-1}M < \epsilon$$

or

$$\|f + g\| \leqslant 2\lambda^{-1}g(z).$$

To see that this second possibility cannot occur, note that $f(z) \geqslant \alpha > 0$ [since $z \in S(g, \beta, C) \subset S(f, \alpha, C)$] and hence

$$\|f + g\| \geqslant (f + g)(z\|z\|^{-1}) > g(z)\|z\|^{-1} \geqslant g(z)\, M^{-1}$$

which would imply that $2M\epsilon^{-1} < \lambda < 2M$, contradicting the fact that $\epsilon < 1$.

The following lemma is due to Errett Bishop (private communication, 1967) and we are grateful to him for his permission to include it here. Since we deal with the same set C throughout the proof, we will use $S(f, \alpha)$ in place of $S(f, \alpha, C)$.

LEMMA 7. (Bishop). *Let E be a Banach space, F a norm closed subspace of the dual space E^* and C a bounded closed convex subset of E. Suppose that for each slice $S(f, \alpha)$ of C, where $f \in F$, and for each $\epsilon > 0$, there exists a slice $S(g, \beta)$ of diameter less than ϵ such that $g \in F$,*

$$S(g, \beta) \subset S(f, \alpha) \qquad \textit{and} \qquad \| f - g \| < \epsilon.$$

Then every slice $S(f, \alpha)$ of C (with $f \in F$) contains a point of C which is strongly exposed by a functional from F.

Proof. We will write $M(f)$ for $M(f, C)$ and we assume without loss of generality that $M(C) = 1$. Let $S(f, \alpha)$ be a slice of C, where $f \in F$. Let $g_o = f$, $\beta_o = \beta$ and use the hypotheses inductively to construct a sequence of functionals $g_1, g_2, \ldots,$ of norm one in F and a sequence of positive numbers $\beta_1, \beta_2, \ldots,$ such that

$$\| g_{k+1} - g_k \| < 2^{-k}\beta_k, \qquad \beta_{k+1} < 2^{-1}\beta_k,$$

$$\text{diam } S(g_{k+1}, \beta_{k+1}) < 2^{-1}\beta_k \qquad \text{and} \qquad S(g_{k+1}, \beta_{k+1}) \subset S(g_k, \beta_k).$$

By the triangle inequality and a standard estimate we have

$$\| g_{k+j} - g_k \| < 2^{-k+1}\beta_k \qquad \text{for each} \quad k \text{ and } j, \tag{1}$$

so (since $\beta_k \to 0$) the sequence $\{g_k\}$ is Cauchy and hence converges to a functional g in F of norm one. Furthermore, the nested sequence of closed sets $\{S(g_k, \beta_k)\}$ has a nonempty intersection consisting of a single point $x_o \in S(f, \alpha)$. We will show that x_o is strongly exposed by g. From (1) it follows that (by taking the limit as $j \to \infty$)

$$\| g - g_k \| \leqslant 2^{-k+1}\beta_k$$

for each k. From this it follows that

$$| g(x) - g_k(x) | \leqslant 2^{-k+1}\beta_k \qquad \text{for} \quad x \in C, \qquad k = 1, 2, \ldots, . \tag{2}$$

Thus, for $x \in C$ we have $M(g) \geqslant g(x) \geqslant g_k(x) - 2^{-k+1}\beta_k$ and hence

$$M(g) \geqslant M(g_k) - 2^{-2}\beta_k \qquad \text{if} \quad k \geqslant 3. \tag{3}$$

Furthermore, if $x \in S(g, \beta_k/4)$, then by (2) and (3) we have (for $k \geqslant 3$)

$$g_k(x) \geqslant g(x) - 2^{-2}\beta_k \geqslant [M(g) - \beta_k/4] - 2^{-2}\beta_k \geqslant M(g_k) - \beta_k$$

so $S(g, \beta_k/4) \subset S(g_k, \beta_k)$ if $k \geqslant 3$. It follows that $\cap S(g, \beta_k/4) = \{x_o\}$ and diam $S(g, \beta_k/4) \to 0$, so x_0 is strongly exposed by g.

The last two lemmas, together with the separation theorem, provide a proof of the following result.

PROPOSITION 8. *Suppose that E is a Banach space and that F is a norm closed subspace of E^* which separates the points of E. Assume, moreover, that every bounded $\sigma(E, F)$-closed subset of E is the $\sigma(E, F)$-closed convex hull of its F-denting points. Then every such set is the $\sigma(E, F)$-closed convex hull of its F-strongly exposed points.*

THEOREM 9. *Let E be a Banach space. Then every bounded subset of E is dentable if and only if every bounded closed convex subset of E is the closed convex hull of its strongly exposed points.*

The proof of this theorem needs little comment; we simply combine Theorem 5 with the case $F = E^*$ in the above proposition.

THEOREM 10. *Suppose that E is a Banach space and that every bounded weak*-closed subset of E^* is the weak*-closed convex hull of its weak*-denting points. Then the same conclusion holds for its weak*-strongly exposed points.*

Again, the proof is immediate from Proposition 8, by taking $F \subset E^{**}$ to be the natural embedding of E.

In order to apply this theorem, we first recall a definition.

DEFINITION. A Banach space E is said to be *weakly compactly generated* (WCG) if there exists a weakly compact subset of E whose linear span is dense in E.

(Note that to say that a dual Banach space E^* is WCG means that it is generated by a $\sigma(E^*, E^{**})$-compact set.)

Isaac Namioka [13] has recently shown, using topological methods, that *if E^* is* WCG, *then every weak*-compact convex subset of E^* is the weak*-closed convex hull of its weak*-denting points.* Namioka did not use the term "weak*-denting point", but the above assertion follows trivially from [13, Theor 4.8] and the methods of his earlier paper [14].

COROLLARY 11. *If E is a Banach space and if E^* is* WCG, *then every weak*-compact convex subset of E^* is the weak*-closed convex hull of its weak*-strongly exposed points.*

Proof. This is immediate from Namioka's result and Theorem 10.

The above result suggests an interesting question. It is relatively easy to prove that if E^* is WCG, then every bounded subset of E^* is dentable (cf. [7]). Now the operators T_x in Lemma 4 are, when defined in a dual space, the adjoints of operators of the same form. This fact makes it easy to see that Lemma 4 can be applied to dual spaces and weak* dentability, as can Theorem 5, so that (in conjunction with Theorem 10) it follows that for any dual space in which every bounded weak*-closed convex set is weak*-dentable, every such set is the weak*-closed convex hull of its weak*-strongly exposed points. Thus, an alternative proof of Corollary 11 above would follow from an affirmative answer to the following question. More generally, the conclusion to Corollary 11 would hold in any dual space having the Radon–Nikodym property.

Question. Suppose that E is a Banach space and suppose that every bounded subset of E^* is dentable. Is every bounded weak* closed convex subset of E^* weak*-dentable?

It is not difficult, using the methods of [6], to see that it suffices to answer the question for every unit ball of E^* which is defined by an equivalent dual norm in E^*. (i.e., for every bounded symmetric weak*-closed convex subset of E^* having nonempty norm-interior). Note that Bourgin's example (below) shows that a strongly exposed point need not be weak*-strongly exposed.

DEFINITION. A Banach space E is said to be a *strong differentiability space* (SDS) if every convex continuous function defined on an open convex subset of E is Fréchet differentiable on a dense G_δ subset of its domain.

The SDS spaces were introduced by Asplund [1], who proved that E has this property if it admits an equivalent norm whose dual norm in E^* is locally uniformly convex (rotund) (cf. [1] for the definition). He also showed that the conclusion to Corollary 11 holds (in E^*) whenever E is an SDS. This raises the following question.

Question. If E^* is WCG, is E an SDS?

The converse is known to be false, since Asplund [2] has also shown that for any set Γ, the space $c_0(\Gamma)$ is an SDS, although its dual space, which is isometric to $\ell_1(\Gamma)$, is WCG only when Γ is countable [11].

COROLLARY 12. *Suppose that E is a Banach space and that every weak*-compact convex set (i.e., every bounded weak*-closed convex set) in E^* is weak*-dentable. Then every bounded closed convex subset C of E^* is the norm-closed convex hull of its strongly exposed points.*

Proof. If $C \subset E^*$, then the fact that its weak*-closure is weak*-dentable implies that C itself is dentable, and Theorem 9 applies.

There is another approach, due to Namioka [13], to the above results of Asplund.

DEFINITION. A dual Banach space E^* is said to satisfy property (**) provided

(**) Whenever $\{f_\alpha\}$ is a net in E^*, $f \in E^*$, are such that $\|f_\alpha\| \to \|f\|$ and $f_\alpha \to f$ weak*, then $\|f_\alpha - f\| \to 0$.

It is evident that E^* has property (**) if and only if the weak* and norm topologies agree on the unit sphere $\{f \in E^* : \|f\| = 1\}$ of E^*. It is not difficult to show that $\ell_1(\Gamma)$ has property (**), as does any E^* in which the norm is locally uniformly convex. From Namioka [13, Theor. 4.11] it follows that the conclusion to Corollary 11 is valid in any dual space which admits an equivalent dual norm having property (**). Another consequence of Namioka's Theorem 4.11 and Corollary 12 is contained in the third part of the next corollary. The first part (concerning strong differentiability spaces) was first proved by Collier and Edelstein [5], by applying the methods of Asplund [1, 2] and Asplund and Rockafellar [3]. The second part of the corollary (concerning WCG spaces) is a generalization (via Namioka's Theor. 4.8 [13]) of a result by John and Zizler [9]. It should be noted that in their Corollary 3, John and Zizler assume that both E and E^* are WCG, while Namioka [13] gets the same conclusion assuming that only E^* is WCG. Lindenstrauss has informed us that he has an example of a Banach space E which is not WCG but for which E^* *is* WCG.

COROLLARY 13. *Suppose that*

(i) *E is an SDS,*

(ii) *E^* is WCG or*

(iii) *E^* has an equivalent dual norm satisfying condition* (**).

Then every bounded closed convex subset of E^* is the norm-closed convex hull of its strongly-exposed points.

Proof. Each part follows from Corollary 12 in conjunction with Asplund [1, 2] for part (i), or with Namioka [13] for parts (ii) and (iii).

If we replace E and E^* by E^* and E^{**} (resp.) in the above corollary, then we can conclude that every C in $E \subset E^{**}$ is dentable, hence is the closed convex hull of its strongly exposed points, by Theorem 9. In case (ii), this generalizes another result of John and Zizler [9], who assumed that both E^* and E^{**} were WCG. This remark also follows from the fact that any WCG dual space (in this case, E^{**}) has the Radon–Nikodym property [7], together with Theorem 9.

It is natural to ask whether the conclusion to Corollary 13 can be strengthened to the assertion that C is the norm-closed convex hull of its weak*-strongly exposed points. (By considering the weak* closure of C, it can be seen that such points will exist.) That the answer is negative is shown by the following example, which was provided to us by Richard Bourgin.

EXAMPLE. Let $E = c_0$ and let $C_1 \subset E^* = \ell_1$ be those real sequences $x = (x_n)$ such that $x_n \geqslant 0$ for each n and $\Sigma x_n = 1$. Let $y = (1/2, 0, 0, \ldots,)$ and let C be the norm-closed convex hull of $C_1 \cup \{y\}$. Then C is the norm-closed convex hull of its strongly exposed points, but not of its weak* strongly exposed points, nor is it the weak*-closed convex hull of the latter. Furthermore, the point y is strongly exposed but not weak*-strongly exposed.

Proof. The first assertion follows either by direct verification or from at least two different general arguments, since ℓ_1 is both WCG (being separable) and the dual of an SDS. The second assertion follows from the fact that all of the weak*-strongly exposed points of C are in C_1. Indeed, the only other candidate is the point y. But, since y is in the weak*-closure of C_1, no weak*-closed hyperplane can separate it from C_1, hence no weak*-slice of C contains y and misses C_1. The fact that the origin is in the weak*-closure of C but not in C proves the assertion about the weak*-closed convex hull. Finally, the functional defined by the element $(1, 1, 1, \ldots,) \in \ell_\infty$ strongly exposes y.

Lindenstrauss [10] has shown that if a Banach space E has the property that every bounded closed nonempty convex set has an extreme point, then E has the Krein–Milman property. (A very simple proof has been given by Richard Bourgin, cf. [15, Lemma 1].) We can add the following related (and obvious) corollary to Theorem 9.

COROLLARY 14. *If every bounded closed nonempty convex subset of the Banach space E has a denting point, then every such set is the closed convex hull of its strongly exposed points.*

We conclude with some related results and open questions.

One of the main results in this area is Troyanski's renorming theorem [18] which, when combined with a result of Lindenstrauss [11], shows that every weakly compact convex subset of a Banach space is the closed convex hull of its strongly exposed points. We see no way of obtaining this result by our methods; although such a set generates a WCG Banach space, not every such space (e.g., the sequence space c_0) has the Radon–Nikodym property.

Since we have considered dual spaces having the Radon–Nikodym property, we should mention Stegall's [17] characterization of such spaces: E^* has the Radon–Nikodym property if and only if every separable subspace of E has a separable dual space.

As we mentioned in the introduction, the question still remains open as to whether the Krein–Milman property implies the Radon–Nikodym property. This can now be formulated in the following manner. If every bounded closed convex subset of the Banach space E is the closed convex hull of its extreme points, is every such set the closed convex hull of its strongly exposed points?

Note added in proof. R. Huff and P. D. Morris ("Dual spaces with the Krein–Milman property have the Radon–Nikadym property," to appear in *Proc. Amer. Math. Soc.*) have ingeneously applied Stegall's construction [17] to give a partial answer to the above question. The example mentioned before Corollary 13 will appear in W. B. Johnson and J. Lindenstrauss, "Some remarks on weakly compactly generated Banach spaces." G. A. Edgar ("A noncompact Choquet theorem," to appear in *Proc. Amer. Math. Soc.*) has shown that the elements of a bounded closed convex separable subset of a Banach space with the Radon–Nikodym property admit integral representations by Borel probability measures on the extreme points. G. Choquet has informed me that this result can also be deduced from a theorem of his announced in "Représentations intégrales dans les cônes convexes sans base compacte" *C. R. Acad. Sci.* (Paris) **253** (1961), 1901–1903.

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