On the recursive sequence \( x_{n+1} = (\alpha - \beta x_{n-k})/g(x_n, x_{n-1}, \ldots, x_{n-k+1}) \)

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**Article Info**

Article history:
Received 19 March 2009
Accepted 3 August 2010

Keywords:
Difference equations
Stability
Attractivity

**Abstract**

This paper is devoted to investigating the asymptotic behavior of the recursive sequence

\[ x_{n+1} = \frac{\alpha - \beta x_{n-k}}{g(x_n, x_{n-1}, \ldots, x_{n-k+1})}, \quad n = 0, 1, \ldots \]

where \( \alpha \geq 0 \) and \( \beta > 0 \) and \( g \) is continuous on \( \mathbb{R}^k \). We show that under certain conditions this equation has a unique positive (negative) equilibrium point which is a global attractor with some basin \( S \subset \mathbb{R}^{k+1} \). Also we establish the oscillation of all solutions with initial conditions \( \{x_{-i}\}_{i=0}^k \) such that \( (x_0, x_{-1}, \ldots, x_{-k}) \in S \). We apply these results to the recursive sequence

\[ x_{n+1} = \frac{\alpha - \beta x_{n-k}}{\gamma + \sum_{i=0}^{k-1} (a_i x_{n-i} \pm b_i x_n^2)}, \quad n = 0, 1, \ldots \]

where \( \alpha, \gamma, a_i, b_i \geq 0, i = 0, \ldots, k-1 \), and \( \beta > 0 \).  

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1. Introduction

Studying the asymptotic behavior of the rational sequence

\[ x_{n+1} = \frac{\alpha + \beta x_n}{\gamma + \sum_{i=1}^{k} \gamma x_{n-i}}, \quad n = 0, 1, \ldots \]  

(1.1)

when some of the coefficients are negative was suggested by Kocić and Ladas in [1]. The difficulty in such problems is finding the good set, that is, the largest domain \( D \) in which solutions exist for any set of initial conditions \( \{x_{-i}\}_{i=0}^{k} \) such that \( (x_0, \ldots, x_{-k}) \in D \). Problems of finding good sets are still open; for example, the good set for the rational recursive sequence

\[ x_{n+1} = \alpha + \frac{x_{n-k}}{x_n}, \quad n = 0, 1, \ldots \]  

(1.2)

when \( \alpha \) is negative is unknown. See [2,3]. Aboutaleb et al. [4] studied the asymptotic stability of the rational recursive sequence

\[ x_{n+1} = \frac{\alpha - \beta x_n}{\gamma + x_{n-1}}, \quad n = 0, 1, \ldots \]  

(1.3)

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where \( \alpha, \beta \) and \( \gamma \) are non-negative with arbitrary initial conditions \( x_{-1} \) and \( x_0 \). An interesting (Lyness-type) special case of Eq. (1.3) was investigated in [5]. Li and Sun [6] extended the results of [4] to the \( k + 1 \)-order rational recursive sequence
\[
x_{n+1} = \frac{\alpha - \beta x_n}{\gamma + x_{n-k}}, \quad n = 0, 1, \ldots
\]
(1.4)
The global asymptotic stability of the rational recursive sequence (1.1) was investigated for when the coefficients \( \alpha, \beta, \gamma \) and \( \gamma_1 \) are non-negative (see [7,18,9]). For other related results see [10,11,6,12–14]. For the terminology used here, we refer the reader to [15,8].

In this paper we extend the results of [16] concerning the equation
\[
x_{n+1} = \frac{\alpha - \beta x_{n-k}}{\gamma + x_n}, \quad n = 0, 1, \ldots
\]
(1.5)
to the more general equation of the form
\[
x_{n+1} = \frac{\alpha - \beta x_{n-k}}{g(x_n, x_{n-1}, \ldots, x_{n-k+1})}, \quad n = 0, 1, \ldots
\]
(1.6)
where \( \alpha, \gamma \geq 0, \beta \geq 0 \) and \( g(u_1, \ldots, u_k) \) is continuous. We investigate sufficient conditions for the unique positive (negative) equilibrium point to be a global attractor with some basin. Also the oscillation of all solutions with initial conditions \( \{x_n\}_{n=0}^\infty \) such that \( (x_0, x_{-1}, \ldots, x_{-k}) \) lies in that basin will be obtained.

The special case of Eq. (1.6) when \( \alpha = 0 \), that is the equation
\[
x_{n+1} = \frac{-x_{n-k}}{g(x_n, x_{n-1}, \ldots, x_{n-k+1})}, \quad n = 0, 1, \ldots
\]
(1.7)
will be studied in Section 2. We show that if there exists \( a > 0 \) such that either \( g(u_1, \ldots, u_k) \geq 1 \) or \( g(u_1, \ldots, u_k) \leq -1, u_i \in [-a, a], i = 1, \ldots, k \), then the zero equilibrium point is a global attractor with basin \([-a, a]^{k+1}\). In Section 3 we apply the results of Section 2 to the rational recursive sequence
\[
x_{n+1} = \frac{-x_{n-k}}{\gamma + \sum_{i=0}^{k-1} (a_i x_{n-i} + b_i x_{n-i}^2)}, \quad n = 0, 1, \ldots
\]
(1.8)
where \( \gamma, a_i, b_i \in \mathbb{R} \), such that \( a_i^2 + b_i^2 \neq 0 \) for some \( i \in \{0, 1, \ldots, k-1\} \). In Section 4, we establish sufficient conditions for an equilibrium point of the general equation
\[
x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots
\]
(1.9)
to be a global attractor with basin \( I^{k+1} \), where \( I \) is an invariant interval of Eq. (1.9) in the sense that \( \{x_n\}_{n=0}^\infty \subset I \) for any set of initial conditions \( \{x_{-k}\}_{k=0}^\infty \subset I \). Here \( f \) is continuous and non-increasing in each argument. Also we obtain sufficient conditions for the oscillation of solutions of Eq. (1.9). In Section 5, we use the general results of Section 4 to investigate the asymptotic behavior of solutions of Eq. (1.6), for when \( \alpha, \beta > 0 \).

In Section 6 we apply the results of Section 5 to the rational recursive sequence
\[
x_{n+1} = \frac{\alpha - \beta x_{n-k}}{\gamma + \sum_{i=0}^{k-1} (a_i x_{n-i} \pm b_i x_{n-i}^2)}, \quad n = 0, 1, \ldots
\]
(1.10)
where \( \alpha, \beta > 0 \) and \( \gamma, a_i, b_i \geq 0, i = 0, 1, \ldots, k, k \), such that \( a_i^2 + b_i^2 \neq 0 \) for some \( i \), and prove the global attractivity of the positive (negative) equilibrium point of Eq. (1.10) with some basin that depends on the coefficients.

2. The recursive sequence \( x_{n+1} = -x_{n-k}/g(x_{n}, x_{n-1}, \ldots, x_{n-k+1}) \)

This section is devoted to investigating the attractivity of the zero equilibrium point of the rational recursive sequence
\[
x_{n+1} = \frac{-x_{n-k}}{g(x_n, x_{n-1}, \ldots, x_{n-k+1})}, \quad n = 0, 1, \ldots
\]
(2.1)
where \( g \) is a continuous function. For \( a > 0 \), define
\[
I^a_0 = [-a, 0], \quad I^+_a = [0, a] \quad \text{and} \quad I^-_a = [-a, a].
\]
We assume that one of the following conditions:
\[
g(u_1, \ldots, u_k) > 1, \quad u_i \in I_a, i = 1, \ldots, k,
\]
and
\[
g(u_1, \ldots, u_k) < -1, \quad u_i \in I_a, i = 1, \ldots, k
\]
holds. We need the following lemma in proving the attractivity of the zero equilibrium.
Lemma 2.1. If $|g(u_1, \ldots, u_k)| \geq 1$, $u_i \in I_a$, $i = 1, \ldots, k$, then $I_a$ is an invariant of Eq. (2.1).

Proof. The proof is straightforward and will be omitted. □

Theorem 2.2. Assume there exists $a > 0$ such that condition (2.2) holds. Let $\{x_n\}_{n \geq -k}$ be a solution of Eq. (2.1) with initial conditions in the interval $I_a$. If $x_i \in I_a^0$ (respectively $I_a^1$) for some $i \in \{0, \ldots, k\}$, then $\{x_{i+n(k+1)}\} \subset I_a^0$ (respectively $I_a^1$) and increases (respectively decreases) when $n$ is even (respectively odd) to 0.

Proof. Assume that $x_{-i} \in I_a^q$ for some $i \in \{0, \ldots, k\}$ and $r \in \{0, 1\}$. First, we prove by induction that $x_{-i+n(k+1)} \in I_a^r$ (respectively $I_a^{-r}$) when $n$ is even (respectively odd). At $n = 0$, the statement is true. Assume that $x_{-i+2n(k+1)} \in I_a^r$. We have

$$x_{-i+(2n+2)(k+1)} = \frac{-x_{i-(2n+2)(k+1)} - x_{-i+2n(k+1)+1}}{g(x_{i-(2n+2)(k+1)-1}, \ldots, x_{-i+2n(k+1)+1})} \in I_a^r.$$

This implies that $x_{-i+n(k+1)} \in I_a^r$ when $n$ is even. We can show similarly that $x_{-i+n(k+1)} \in I_a^{1-r}$ when $n$ is odd. Assume now that $x_{-i} \in I_a^0$ for some $i \geq -k$. For $n$ even, $x_{-i+n(k+1)} \in I_a^0$ and by Eq. (2.4) $x_{-i+(2n+2)(k+1)} \geq x_{-i+2n(k+1)}$, $n \in \mathbb{N}$. This implies that $\{x_{-i+2n(k+1)}\}_{n \geq -k}$ is increasing to a non-positive number, say $a_i \in I_a^0$. When $n$ is odd, $x_{-i+n(k+1)} \in I_a^1$ and we can show that $x_{-i+(2n+1)(k+1)} \leq x_{-i+2n(k+1)}$, $n \in \mathbb{N}$. Then $\{x_{-i+(2n-1)(k+1)}\}_{n \geq -k}$ is decreasing to a non-negative number, say $b_i \in I_a^1$. Similarly, if $x_{-i} \in I_a^1$ for some $i \geq -k$, then $x_{-i+n(k+1)}$ is decreasing to a non-negative number, say $c_i \in I_a^1$, and $\{x_{-i+(2n-1)(k+1)}\}_{n \geq -k}$ is increasing to a non-positive number, say $d_i \in I_a^0$. Condition (2.2), relation (2.4) and the continuity of $g$ imply that $a_i = c_i = d_i = 0$.

By the same argument we can show the following result.

Theorem 2.3. Assume there exists $a > 0$ such that condition (2.3) holds. Let $\{x_n\}_{n \geq -k}$ be a solution of Eq. (2.1) with initial conditions in the interval $I_a$. If $x_i \in I_a^0$ (respectively $I_a^1$) for some $i \in \{0, \ldots, k\}$, then $\{x_{i+n(k+1)}\} \subset I_a^0$ (respectively $I_a^1$) and decreases (respectively increases) when $n$ is even (respectively odd) to 0.

As a direct consequence of Theorems 2.2 and 2.3, we get the following results:

Corollary 2.4. Assume that there is $a > 0$ such that either condition (2.2) or (2.3) holds. Then the zero equilibrium point of Eq. (2.1) is a global attractor with basin $I_a^{k+1}$.

Corollary 2.5. If one of the following conditions:

$$g(u_1, \ldots, u_k) > 1, \quad u_i \in \mathbb{R}, \quad i = 1, \ldots, k,$$

and

$$g(u_1, \ldots, u_k) < -1, \quad u_i \in \mathbb{R}, \quad i = 1, \ldots, k,$$

holds, then the zero equilibrium point of Eq. (2.1) is a global attractor.

Proof. Let $\{x_n\}_{n \geq -k}$ be a solution of Eq. (2.1) with initial conditions $\{x_{-i}\}_{i=0}^k$. There is $a > 0$ such that $x_{-i} \in I_a$, $i = 0, \ldots, k$. By the previous corollary, the zero equilibrium point is a global attractor. □

3. The recursive sequence $x_{n+1} = \frac{-x_{n-k} - \gamma}{\gamma + \sum_{i=0}^{k-1} (a_i x_{n-i} + b_i x_{n-2i})}$

In this section we apply the results of Section 2 to the rational difference equation

$$x_{n+1} = \frac{-x_{n-k}}{\gamma + \sum_{i=0}^{k-1} (a_i x_{n-i} + b_i x_{n-2i})}, \quad n = 0, 1, \ldots$$

where $\gamma$, $a_i$, $b_i \in \mathbb{R}$. Eq. (2.1) yields Eq. (3.1) on setting

$$g(x_n, x_{n-1}, \ldots, x_{n-k+1}) = \gamma + \sum_{i=0}^{k-1} (a_i x_{n-i} + b_i x_{n-2i}).$$

Assume that there exists $a > 0$ such that either

$$\gamma > 1 + a \sum_{i=0}^{k-1} |a_i| + a^2 \sum_{i=0}^{k-1} |b_i|$$

(3.2)
or
\[ \gamma < -1 - a \sum_{i=0}^{k-1} |a_i| - a^2 \sum_{i=0}^{k-1} |b_i|. \]  \hfill (3.3)

Since conditions (3.2) and (3.3) imply conditions (2.2) and (2.3) respectively, by Theorems 2.2 and 2.3, we obtain the following results.

**Theorem 3.1.** Assume that there exists \( a > 0 \) such that condition (3.2) holds. Let \( \{x_n\}_{n \geq -k} \) be a solution of Eq. (3.1) with initial conditions in the interval \( I_a \). If \( x_{-i} \in I_0 \) (respectively \( I_1 \)) for some \( i \in \{0, \ldots, k\} \), then \( x_{-i+n(k+1)} \in I_0 \) (respectively \( I_1 \)) and decreases (respectively increases) when \( n \) is even (respectively odd) to 0.

**Theorem 3.2.** Assume that there exists \( a > 0 \) such that condition (3.3) holds. Let \( \{x_n\}_{n \geq -k} \) be a solution of Eq. (3.1) with initial conditions in the interval \( I_a \). If \( x_{-i} \in I_0 \) (respectively \( I_1 \)) for some \( i \in \{0, \ldots, k\} \), then \( x_{-i+n(k+1)} \in I_0 \) (respectively \( I_1 \)) and increases (respectively decreases) when \( n \) is odd (respectively even) to 0.

As a direct consequence of Theorems 3.1 and 3.2, we get the following results.

**Corollary 3.3.** Assume that there is a \( a > 0 \) such that either condition (3.2) or (3.3) holds. Then the zero equilibrium point of Eq. (3.1) is a global attractor with basin \( I_0 \).

### 4. General results

In this section, we suppose that \([a, b]\) is an invariant interval for the general difference equation
\[ x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots \]  \hfill (4.1)
where \( f \) is continuous. In the following theorem we obtain sufficient conditions for an equilibrium point to be a global attractor with basin \([a, b]\).

**Theorem 4.1.** Let \( f \) be non-increasing in each of its arguments. Assume that \( G \) has a fixed point \( \bar{x} \in (a, b) \) such that
\[ a < \liminf_{n \to \infty} x_n \leq \bar{x} \leq \limsup_{n \to \infty} x_n < b, \]
for every solution \( \{x_n\} \) with initial conditions \( \{x_{-i}\} \subset [a, b] \). Then the following conditions are equivalent and each of them is sufficient for \( \bar{x} \) to be a global attractor of Eq. (4.1) with basin \([a, b] \):

(a) \( \bar{x} \) is the unique fixed point of \( G^2 \) in \([a, b]\).

(b) \( G^2(\bar{x}) > \bar{x}, \forall \bar{x} \in (a, \bar{x}) \).

(c) If \( \lambda, \Lambda \in [a, b] \) are such that
\[ G(\Lambda) \leq \lambda \leq \bar{x} \leq \Lambda \leq G(\lambda), \]  \hfill (4.2)
then
\[ \lambda = \bar{x} = \Lambda. \]  \hfill (4.3)

(d) The system
\[ y = G(x) \quad \text{and} \quad x = G(y) \]  \hfill (4.4)
has exactly one solution \((\bar{x}, \bar{x}) \in [a, b]^2\).

**Proof.** We prove that \((a \Rightarrow b \Rightarrow c \Rightarrow a \Rightarrow d \Rightarrow c)\).

\( (a \Rightarrow b) \) Assume on the contrary that there exists \( x \in (a, \bar{x}) \) such that \( G^2(\bar{x}) \leq x \). Since \( G^2(a) > a \), then \( G^2 \) has a fixed point in \((a, \bar{x})\) which is a contradiction.

\( (b \Rightarrow c) \) Assume that \( \lambda, \Lambda \in [a, b] \) are such that
\[ G(\Lambda) \leq \lambda \leq \bar{x} \leq \Lambda \leq G(\lambda). \]  \hfill (4.2)
Since \( G \) is non-increasing on \([a, b] \), then \( \lambda \geq G(\Lambda) \geq G^2(\lambda) \). Clearly \( \lambda = \bar{x} \), because if \( \lambda < \bar{x} \), then by \( (b) \), \( G^2(\lambda) > \lambda \) which is impossible.

\( (c \Rightarrow a) \) Assume towards a contradiction that \( x_0 \neq \bar{x} \) is another fixed point of \( G^2 \) in \([a, b]\). If \( x_0 < \bar{x} \), take \( \lambda = x_0 \) and \( \Lambda = G(x_0) \). Then \( (4.2) \) holds but not \( (4.3) \). If \( x_0 > \bar{x} \), take \( \lambda = G(x_0) \) and \( \Lambda = x_0 \). Then \( (4.2) \) holds but not \( (4.3) \).

\( (a \Rightarrow d) \) If system \((4.4) \) has a solution \((x, y) \neq (\bar{x}, \bar{x}) \in [a, b]^2 \), then \( G^2 \) has a fixed point different from \( \bar{x} \), which contradicts \((a) \).

\( (d \Rightarrow c) \) Let \( \lambda, \Lambda \in [a, b] \) be such that \((4.2) \) holds. Set
\[ U_1 = G(\lambda) \quad \text{and} \quad L_1 = G(\Lambda), \]
and for \( n = 1, 2, \ldots \) set
\[ U_{n+1} = G(L_n) \quad \text{and} \quad L_{n+1} = G(U_n). \]
We can see by induction that
\[ a \leq \cdots \leq L_n \leq \cdots \leq L_2 \leq L_1 \leq \bar{\lambda} \leq U_1 \leq U_2 \leq \cdots \leq U_n \leq \cdots \leq b. \]

Hence each of \( \{L_n\} \), \( \{U_n\} \) converges to a number, say \( L, U \in [a, b] \) respectively. Then \( (L, U) \) is a solution of system \((4.4)\) and \( L = U = \bar{\lambda} \). Clearly \( U \geq \Lambda \geq \bar{\lambda} \geq \lambda \geq L \). Therefore \( \Lambda = \bar{\lambda} = \lambda \).

Let \( \{x_n\} \) be a solution of Eq. (4.1) with initial conditions \( x_{-i} \in [a, b] \), \( i = 0, \ldots, k \). Set
\[ \lambda = \liminf_{n \to \infty} x_n \quad \text{and} \quad \Lambda = \limsup_{n \to \infty} x_n. \]

Let \( \epsilon > 0 \) be such that \( \epsilon < \min[b - \Lambda, \lambda - a] \). There exists \( n_0 \in \mathbb{N} \) such that
\[ \lambda - \epsilon < x_n < \lambda + \epsilon, \quad \forall n > n_0. \]

Hence
\[ f(\lambda + \epsilon, \ldots, \lambda + \epsilon) < x_{n+1} < f(\lambda - \epsilon, \ldots, \lambda - \epsilon) \quad \forall n > n_0 + k. \]

By continuity of \( G \), we get the following inequality:
\[ G(\Lambda) \leq \lambda \leq \bar{\lambda} \leq \Lambda \leq G(\lambda). \]

By (c), \( \lambda = \Lambda = \bar{\lambda} \). □

The next theorem presents a detailed description of the semicycles of any solution of Eq. (4.1) about an equilibrium point \( \bar{\lambda} \) and also establishes the strict oscillation of solutions. For the definition of positive and negative semicycles we refer the reader to [5].

**Theorem 4.2.** Let \( f \) be decreasing in each of its arguments. Assume that \( G \) has a fixed point \( \bar{x} \in [a, b] \). Every non-trivial solution of Eq. (4.1) with initial conditions \( \{x_{-i}\} \subset [a, b] \) satisfies the following statements:

1. \( \{x_n\} \) cannot have \( k + 1 \) consecutive terms equal to \( \bar{x} \).
2. Every semicycle of \( \{x_n\} \) has at most \( k + 1 \) terms.
3. \( \{x_n\} \) is strictly oscillatory.

**Proof.** (1) If \( x_m = x_{m+1} = \cdots = x_{m+k} = \bar{x} \) for some \( m \geq -k \), then \( x_n = \bar{x} \), \( n \geq -k \) which contradicts the hypothesis.

(2) Assume that a semicycle \( S \) starts with \( x_m, x_{m+1}, \ldots, x_{m+k} \). When \( S \) is a negative semicycle, then \( x_m, x_{m+1}, \ldots, x_{m+k} < \bar{x} \), whence
\[ x_{m+k+1} = f(x_{m+k}, \ldots, x_m) > f(\bar{x}, \ldots, \bar{x}) = \bar{x}. \]

When \( S \) is a positive semicycle, then at least one term of \( \{x_m, x_{m+1}, \ldots, x_{m+k}\} \) is greater than \( \bar{x} \), and so \( x_{m+k+1} < \bar{x} \).

(3) By (1) and (2), the strict oscillation follows. □

**5. The recursive sequence** \( x_{n+1} = (\alpha - \beta x_{n-k})/g(x_n, \ldots, x_{n-k+1}) \)

In this section we study the asymptotic behavior of the difference equation
\[ x_{n+1} = \frac{\alpha - \beta x_{n-k}}{g(x_n, \ldots, x_{n-k+1})}, \quad n = 0, 1, \ldots \quad (5.1) \]

where \( \alpha, \beta > 0 \) and \( g(u_0, \ldots, u_{k-1}) \) is a continuous function. We define
\[ g(x) = g(x, x, \ldots, x). \]

We assume that \( g \) satisfies one of the following conditions:

(C1) \( g(u_0, \ldots, u_{k-1}) \) is non-decreasing in each argument \( u_i \in \mathbb{R}^+ \) and \( g(0) > \beta \).

(C2) \( g(u_0, \ldots, u_{k-1}) \) is non-decreasing in each argument \( u_i \in \mathbb{R}^+ \) and \( g(0) < -2\beta \).

We define
\[ C = \frac{\alpha(g(0) - \beta)}{g(0)g(\alpha/\beta)} \quad \text{and} \quad D = \frac{\alpha}{g(0)}, \quad \text{when condition (C1) holds.} \quad (5.2) \]

and
\[ C = \frac{2\alpha}{g(-\alpha/\beta)} \quad \text{and} \quad D = \frac{\alpha(g(0) + 2\beta)}{g(0)g(-\alpha/\beta)}, \quad \text{when condition (C2) holds.} \quad (5.3) \]

Clearly, in the two cases, \( C < D \). We need the following lemmas in proving the main result.

**Lemma 5.1.** If condition (C1) (respectively (C2)) holds, then Eq. (5.1) has a unique equilibrium point \( \bar{x} \in (0, \alpha/\beta) \) (respectively \( \bar{x} \in (-\alpha/\beta, 0) \)).
Proof. We can see that $\bar{x}$ is an equilibrium point of Eq. (5.1) iff $\bar{x}$ is a zero of the function

$$h(x) = x - \frac{\alpha}{\beta + g(x)}.$$ 

Assume that condition (C1) holds. Since $h(x)$ is an increasing continuous function on $\mathbb{R}_{>0}$, $h(0) < 0$, and $h(\alpha/\beta) > 0$, then $h(x)$ has a unique positive zero in $(0, \alpha/\beta)$; hence Eq. (5.1) has a unique positive equilibrium point in this interval. Now, assume that condition (C2) holds. The function $h$ is increasing continuous on $\mathbb{R}_{>0}$, $h(0) > 0$ and $h(-\alpha/\beta) < 0$. Hence $h(x)$ has a unique negative zero in $(-\alpha/\beta, 0)$.

Lemma 5.2. Assume that either condition (C1) or (C2) holds. Then the interval $[C, D]$ is invariant for Eq. (5.1).

Proof. Let $x_{-i} \in [C, D], i = 0, 1, \ldots, k$. First, assume that condition (C1) holds. We have

$$\alpha \left(1 - \frac{\beta}{g(0)} \right) \leq \alpha - \beta x_{-k} \leq \alpha$$

and

$$1 \geq \frac{1}{g(\alpha/\beta)} \geq \frac{1}{g(\alpha/\beta)} \geq \frac{1}{g(\alpha/\beta)}.$$ 

Then $x_i \in [C, D]$. The result follows by induction. Now assume that condition (C2) holds. We have

$$\alpha - \beta D \leq \alpha - \beta x_{-k} \leq \alpha - \beta C$$

and

$$1 \geq \frac{1}{g(0)} \geq \frac{1}{g(\alpha/\beta)}.$$ 

This implies that

$$\frac{\alpha - \beta C}{g(0)} \leq x_1 \leq \frac{\alpha - \beta D}{g(\alpha/\beta)}.$$ 

Simple calculations show that $(\alpha - \beta D)/g(\alpha/\beta) < D$ and $(\alpha - \beta C)/g(0) \geq C$. Consequently, $x_1 \in [C, D]$. The result follows by induction.

Theorem 5.3. Assume that $\alpha, \beta > 0$ are such that condition (C1) (resp. (C2)) holds. Let $\{x_n\}$ be a solution of Eq. (5.1) with initial conditions $x_{-i} \in [0, \alpha/\beta]$ (resp. $x_{-i} \in [-\alpha/\beta, 0]$), $i = 0, \ldots, k$. Then

$$C \leq x_{n+k+1} \leq D, \quad n \in \mathbb{N}, \quad (5.4)$$

and

$$\liminf_{n \to \infty} x_n \leq \bar{x} \leq \limsup_{n \to \infty} x_n, \quad (5.5)$$

Proof. Assume that condition (C1) holds. Let $\{x_n\}$ be a solution of Eq. (5.1) with initial conditions $\{x_{-i}\} \subset [0, \alpha/\beta]$. Since

$$0 \leq \alpha - \beta x_{-k} \leq \alpha \quad \text{and} \quad \frac{1}{g(x_0, \ldots, x_{k+1})} \leq \frac{1}{g(0)},$$

then $0 \leq x_1 \leq D < \alpha/\beta$. By induction we get $0 \leq x_i \leq D, i = 1, \ldots, k+1$. One can check that $C \leq x_{k+1+i} \leq D, i = 1, \ldots, k+1$. Inequality (5.4) follows by Lemma 5.2. Similarly we can get inequality (5.4) when condition (C2) holds. Set

$$\lambda = \liminf_{n \to \infty} x_n \quad \text{and} \quad \Lambda = \limsup_{n \to \infty} x_n.$$ 

For every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\lambda - \epsilon \leq x_n \leq \lambda + \epsilon, n \geq n_0$. If $\lambda > \bar{x}$, take $\epsilon = \lambda - \bar{x}$. There exists $n_0 \in \mathbb{N}$ such that $\bar{x} < x_n, n \geq n_0$. Hence $\bar{x} > x_{n_0+k}, n \geq n_0$ which is a contradiction. Therefore $\lambda \leq \bar{x}$, Similarly we can show that $\bar{x} \leq \Lambda$.

We combine Theorem 4.1, Lemma 5.1 and Theorem 5.3 to obtain the following result which establishes sufficient conditions for the positive (resp. negative) equilibrium point $\bar{x}$ to be a global attractor for Eq. (5.1) with basin $[0, \alpha/\beta]^{k+1}$ (resp. $[-\alpha/\beta, 0]^{k+1}$). Set

$$G(x) = \frac{\alpha - \beta x}{g(x)}.$$ 

Theorem 5.4. Assume that $\alpha, \beta > 0$ are such that condition (C1) (resp. (C2)) holds. Then the following conditions are equivalent and each of them is a sufficient condition for $\bar{x}$ to be a global attractor of Eq. (5.1) with basin $[0, \alpha/\beta]^{k+1}$ (resp. $[-\alpha/\beta, 0]^{k+1}$):

(a) $\bar{x}$ is the unique fixed point of $G^2$ in $[0, \alpha/\beta]$ (resp. $[-\alpha/\beta, 0]$).
Assume that $x_1, \ldots, x_k$ satisfy the following statements:

If condition $x_i$ is a global attractor for Eq. (6.1) then

$$G(\lambda) \leq x_i \leq \lambda \leq G(\lambda),$$

Theorem 4.2

Proof. (resp. Theorem 6.1.) One can see that the function $G(x)$ has exactly one solution: $(x, \lambda) \in [0, \alpha/\beta]^k$ (resp. $[-\alpha/\beta, 0]$).

The following result is a direct consequence of Theorem 4.2. It presents a detailed description of the semicycles of any solution $\{x_n\}$ of Eq. (5.1) about the positive (resp. negative) equilibrium point $\lambda$ with initial conditions $x_{-i} \in [0, \alpha/\beta]$ (resp. $[-\alpha/\beta, 0]$), $i = 0, \ldots, k$. Also we establish the strict oscillation of such solutions.

Theorem 5.5. Assume that $\alpha, \beta > 0$ are such that condition (C1) (resp. (C2)) holds. Then every solution $\{x_n\}$ of Eq. (5.1) with initial conditions $x_{-i} \in [0, \alpha/\beta], i = 0, \ldots, k$ (resp. $[-\alpha/\beta, 0]$), which are not all equal to $\lambda$, satisfies the following statements:

(1) $\{x_n\}$ cannot have $k + 1$ consecutive terms equal to $\lambda$.
(2) Every semicycle of $\{x_n\}$ has at most $k + 1$ terms.
(3) $\{x_n\}$ is strictly oscillatory.

6. The recursive sequence $x_{n+1} = (\alpha - \beta x_{n-k})/(\gamma + \sum_{i=0}^{k-1} (a_i x_{n-i} + b_i x_{n-i}^2))$

In this section we investigate the attractivity of the rational recursive sequences

$$x_{n+1} = \frac{\alpha - \beta x_{n-k}}{\gamma + \sum_{i=0}^{k-1} (a_i x_{n-i} + b_i x_{n-i}^2)}; \quad n = 0, 1, \ldots$$

and

$$x_{n+1} = \frac{\alpha - \beta x_{n-k}}{\gamma + \sum_{i=0}^{k-1} (a_i x_{n-i} - b_i x_{n-i}^2)}; \quad n = 0, 1, \ldots$$

where $\alpha, \beta, \gamma > 0$; $a_i, b_i \geq 0, i = 0, 1, \ldots, k - 1, \gamma + \sum_{i=0}^{k-1} (a_i + b_i) x_{n-i}^2 \neq 0$ for some $i$. Suppose that one of the following conditions holds:

$$\gamma > \alpha + \sum_{i=0}^{k-1} b_i \frac{\alpha^2}{\beta^2},$$

$$\gamma < -2 \beta - \sum_{i=0}^{k-1} b_i \frac{\alpha^2}{\beta^2}.$$  

Eq. (6.1) (resp. (6.2)) can be written in the form (5.1) where

$$g(x_1, \ldots, x_{n-k+1}) = \gamma + \sum_{i=0}^{k-1} (a_i x_{n-i} + b_i x_{n-i}^2)$$  \text{ for Eq. (6.1).}

and

$$g(x_1, \ldots, x_{n-k+1}) = \gamma + \sum_{i=0}^{k-1} (a_i x_{n-i} - b_i x_{n-i}^2)$$  \text{ for Eq. (6.2).}

One can see that the function $g(u_1, \ldots, u_k)$ is increasing in every positive (negative) variable $u_i, i = 1, \ldots, k$. By Lemma 5.1, Eq. (6.1) (resp. (6.2)) has a unique positive (resp. negative) equilibrium point $\lambda \in [0, \alpha/\beta]$ (resp. $(-\alpha/\beta, 0)$).

Theorem 6.1. If condition (6.3) (resp. (6.4)) holds, then $\lambda$ is a global attractor for Eq. (6.1) (resp. (6.2)) with basin $[0, \alpha/\beta]^{k+1}$ (resp. $(-\alpha/\beta, 0)$).

Proof. Assume that condition (6.3) (resp. (6.4)) holds.

Set

$$G(x) = \frac{\alpha - \beta x}{\gamma + \sum_{i=0}^{k-1} (a_i x + b_i x^2)} \quad \text{resp.} \quad G(x) = \frac{\alpha - \beta x}{\gamma + \sum_{i=0}^{k-1} (a_i x - b_i x^2)}.$$
Let $\lambda$ and $\Lambda$ be non-negative numbers in $[0, \alpha/\beta]$ (resp. $[-\alpha/\beta, 0]$) such that (5.6) holds. Then
\[
\beta(\Lambda - \lambda) - \gamma(\Lambda - \lambda) + \sum_{i=0}^{k-1} b_i \Lambda \lambda (\Lambda - \lambda) \geq 0 \quad \text{resp.} \quad \beta(\Lambda - \lambda) - \gamma(\Lambda - \lambda) + \sum_{i=0}^{k-1} b_i \Lambda \lambda (\Lambda - \lambda) \leq 0.
\] (6.5)

If $\Lambda > \lambda$, then (6.5) yields
\[
\gamma - \beta \leq \sum_{i=0}^{k-1} b_i \Lambda \lambda \leq \sum_{i=0}^{k-1} b_i \alpha^2/\beta^2.
\]
which contradicts condition (6.3) (resp. (6.4)). Therefore $\lambda = \Lambda = \bar{x}$. \[\Box\]

We use Theorem 5.5 to get the following result.

**Theorem 6.2.** If condition (C1) (resp. (C2)) holds, then every non-trivial solution $\{x_n\}$ of Eq. (6.1) (resp. (6.2)) with initial conditions in $[0, \alpha/\beta]$ (resp. $[-\alpha/\beta, 0]$) is strictly oscillatory.

**References**


