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Unimodular lattices with long shadow

Gabriele Nebe[a,*] and Boris Venkov[b,1]

[a] *Abteilung Reine Mathematik, Universität Ulm, 89069 Ulm, Germany*
[b] *St. Petersburg Branch of the Steklov Mathematical Institute, Fontanka 27, 191011 St. Petersburg, Russia*

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Abstract

Let L be an odd unimodular lattice of dimension n with shadow $n-16$. If $\min(L) \geqslant 3$ then $\dim(L) \leqslant 46$ and there is a unique such lattice in dimension 46 and no lattices in dimensions 44 and 45. To prove this, a shadow theory for theta series with spherical coefficients is developed.

1. Introduction

An interesting aspect of odd unimodular lattices is that they come together with their shadows. Let Λ be a unimodular lattice in the bilinear space $(\mathbb{R}^n, (,))$. Then the shadow $S(\Lambda)$ of an odd lattice Λ is

$$S(\Lambda) := \Lambda_0^* - \Lambda.$$

Here Λ_0 denotes the even sublattice of Λ and L^* denotes the dual lattice of the lattice L. (The shadow of an even lattice is $S(\Lambda) = \Lambda$.) The vectors in $S(\Lambda)$ are $1/2$ times the characteristic vectors of Λ. In this note, we only consider positive definite lattices. Define $\sigma(\Lambda) := 4\min(S(\Lambda))$ to be the minimal norm of a characteristic vector in Λ. Then $\sigma(\Lambda) \equiv n \pmod 8$.

Splitting off the vectors of length 1 in Λ one gets a unimodular lattice Γ with $\dim(\Lambda) - \sigma(\Lambda) = \dim(\Gamma) - \sigma(\Gamma)$ in smaller dimension. Therefore, we will always assume that the minimum of Λ is $\geqslant 2$.

*Corresponding author.
E-mail addresses: nebe@mathematik.uni-ulm.de (G. Nebe), bvenkov@pdmi.ras.ru (B. Venkov).
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Elkies [Elk1,Elk2] proved that $\mathbb{Z}^n$ is the only odd unimodular lattice Λ with $\sigma(\Lambda) = n$ and found the short list of lattices Λ with $\sigma(\Lambda) = n - 8$. The largest possible dimension here is $n = 23$ where the lattice Λ is the shorter Leech lattice O_{23}.

The next cases $\sigma(\Lambda) = n - 16$ and $\sigma(\Lambda) = n - 24$ have been considered by Gaulter [Gau]. He shows that the dimension n of a unimodular lattice Λ with $\min(\Lambda) > 1$ and $\sigma(\Lambda) = n - 16$ is bounded by 2907 and $n \leqslant 8\ 388\ 630$ for $\sigma(\Lambda) = n - 24$.

In this paper, we study lattices Λ with $\sigma(\Lambda) = n - 16$. If $\min(\Lambda) \geqslant 3$ then we show that $n \leqslant 46$. This bound is the best possible, because $\Lambda = O_{23} \perp O_{23}$ satisfies $\dim(\Lambda) = 46$ and $\sigma(\Lambda) = 46 - 16$ and this is the only such lattice of dimension 46 (see Theorem 3.5). In dimensions 45 and 44 there are no such lattices of minimum $\geqslant 3$ (Theorems 3.6 and 3.7). To prove these theorems, we adopt the theory of theta series with spherical coefficients to the shadow theory of unimodular lattices. In the last section, we give some examples of lattices Λ with $\sigma(\Lambda) = n - 16$ for dimensions $n \leqslant 35$.

2. Theta series with spherical coefficients

In the whole paper, let Λ be a unimodular lattice of dimension n and $\sigma(\Lambda) = n - 16$. We will also always assume that $\min(\Lambda) > 1$.

Since Λ is a unimodular lattice, its theta series $\theta_\Lambda(z) := \sum_{\lambda \in \Lambda} q^{(\lambda,\lambda)}$, where $q := \exp(\pi i z)$ is a modular form for the theta group

$$\Theta := \left\langle S : z \mapsto -\frac{1}{z}, T^2 : z \mapsto z + 2 \right\rangle$$

of weight $\frac{n}{2}$ and hence a linear combination of $\theta_3^a \Delta_8^b$ with $a + 8b = n$ (cf. [Ran, Theorem 7.1.4]). Here, $\theta_3 = \theta_{\mathbb{Z}}$ is the theta series of the 1-dimensional unimodular lattice $\mathbb{Z}$ and Δ_8 is the cusp form of weight 4. The theta series of the shadow $S(\Lambda)$ can be obtained from θ_Λ by a simple variable substitution:

$$\theta_{S(\Lambda)}(z) = \left(\frac{i}{z}\right)^{n/2} \theta_\Lambda\left(-\frac{1}{z} + 1\right).$$

With this substitution we define the shadow of a modular form ϕ of weight m to be

$$S(\phi)(z) := \left(\frac{i}{z}\right)^{m} \phi\left(-\frac{1}{z} + 1\right).$$

Whereas $\theta_3(-\frac{1}{z} + 1)$ starts with $2(\frac{z}{i})^{1/2} q^{1/4}$ (and hence $S(\theta_3)$ with $2q^{1/4}$), where $q = \exp(\pi i z)$, the shadow of Δ_8 starts with -1. Therefore, the condition on $\min(S(\Lambda))$ shows that

$$\theta_\Lambda = \theta_3^n + A\theta_3^{n-8}\Delta_8 + B\theta_3^{n-16}\Delta_8^2$$

for some A, B. Since $\min(\Lambda) \geqslant 2$, one finds $A = -2n$. Moreover, B is determined if one fixes the number of vectors of length 2 in Λ. Let Λ_j be the set of vectors of length

j in Λ and $a_j := |\Lambda_j|$. Then the above argumentation shows that

$$\text{(U3)} \quad a_3 = \tfrac{4}{3}n(n^2 - 69n + 1208) + 2(n-24)a_2,$$

$$\text{(U4)} \quad a_4 = 2n(n^3 - 94n^2 + 2783n - 24425) + 2(n-21)(n-28)a_2.$$

For details see [Elk1].

We now consider the theta series of Λ with spherical coefficients. Let P_d be a harmonic polynomial of even degree d in n variables. Then

$$\theta_{\Lambda,P_d}(z) := \sum_{\lambda \in \Lambda} P_d(\lambda) q^{(\lambda,\lambda)}$$

is a modular form for the theta group to the character χ with $\chi(S) = i^d$ and $\chi(T^2) = 1$. If d is divisible by 4 then $\theta_{\Lambda,P_d} \in \mathbb{C}[\theta_3, \Delta_8]$ and if $d \equiv 2 \pmod 4$ then by [Ran, Theorem 7.1.6] $\theta_{\Lambda,P_d} \in \Phi\mathbb{C}[\theta_3, \Delta_8]$, where $\Phi = (\theta_2^4 - \theta_4^4)$. One easily sees that

$$\Phi(z) := \theta_{\mathbb{Z}}(1+z)^4 - \theta_{S(\mathbb{Z})}(z)^4.$$

The shadow of Φ starts with 2 and therefore the minimal q-power in the shadow of $\Phi\theta_3^{m-8j}\Delta_8^j$ is $q^{(m-8j)/4}$.

This shows that

$$\theta_{\Lambda,P_2} = c\Phi\theta_3^{n-16}\Delta_8^2$$

for some constant c. Therefore, $\theta_{\Lambda,P_2} = 0$ if $a_2 = 0$. Then the layers of Λ and $S(\Lambda)$ form 2-designs. In general, this equation gives for all $\alpha \in \mathbb{R}^n$

$$\sum_{u \in \Lambda_3} (u,\alpha)^2 - 2(n-36) \sum_{r \in \Lambda_2} (r,\alpha)^2 = (4(n^2 - 69n + 1208) + 2a_2)(\alpha,\alpha) \quad \text{(C2)}$$

and

$$\sum_{v \in \Lambda_4} (v,\alpha)^2 - 2(n-24)(n-49) \sum_{r \in \Lambda_2} (r,\alpha)^2$$
$$= (8(n^3 - 94n^2 + 2783n - 24425) + 4(n-25)a_2)(\alpha,\alpha). \quad \text{(D2)}$$

Similarly one gets

$$\theta_{\Lambda,P_4} = c_1\theta_3^{n-16}\Delta_8^3 + c_2\theta_3^{n-8}\Delta_8^2$$

and

$$\theta_{\Lambda,P_6} = \Phi(c_1'\theta_3^{n-16}\Delta_8^3 + c_2'\theta_3^{n-8}\Delta_8^2)$$

for some constants c_1, c_2, c'_1, c'_2. From this one finds:

$$\sum_{v\in\Lambda_4}(v,\alpha)^4 - 2(n-28)\sum_{u\in\Lambda_3}(u,\alpha)^4 + 2(n^2-55n+636)\sum_{r\in\Lambda_2}(r,\alpha)^4$$
$$= -216\sum_{r\in\Lambda_2}(r,\alpha)^2(\alpha,\alpha) + (24(n-41)(n-46)+12a_2)(\alpha,\alpha)^2 \tag{D4}$$

and

$$\sum_{v\in\Lambda_4}(v,\alpha)^6 - 2(n-40)\sum_{u\in\Lambda_3}(u,\alpha)^6 + 2(n^2-79n+1584)\sum_{r\in\Lambda_2}(r,\alpha)^6$$
$$= 30\sum_{u\in\Lambda_3}(u,\alpha)^4(\alpha,\alpha) - 60(n-39)\sum_{r\in\Lambda_2}(r,\alpha)^4(\alpha,\alpha)$$
$$-\ 180\sum_{r\in\Lambda_2}(r,\alpha)^2(\alpha,\alpha)^2 - 240(n-37)(\alpha,\alpha)^3. \tag{D6}$$

3. Main results

In this section, it is assumed that Λ is a unimodular lattice of dimension n with $\sigma(\Lambda) = n-16$ and $\min(\Lambda)\geqslant 3$ i.e. $a_2 = 0$.

3.1. Bounds for the dimension

Fix $u_0\in\Lambda_3$ and define $m_i := |\{u\in\Lambda_3 | (u,u_0) = i\}|$. Since $m_i\neq 0$ only for $i = 0, 1, -1, 3, -3$ and $m_1 = m_{-1}$, $m_3 = m_{-3} = 1$, Eq. (C2) yields

$$m_1 = 3(2n^2 - 138n + 2413).$$

We keep the following notation:
For $v\in\Lambda_4$ let

$$N_i(v) := \{u\in\Lambda_3 \mid (v,u) = i\} \quad \text{and} \quad n_i(v) := |N_i(v)|.$$

If one writes $N_2(v) = \{u_1,\dots,u_k\}\,\dot\cup\,\{v-u_1,\dots,v-u_k\}$ then the vectors

$$z_i := u_i - \tfrac{1}{2}v \quad (1\leqslant i\leqslant k)$$

are pairwise orthogonal roots in $v^\perp$.

Therefore, $k\leqslant n-1$. For the mean value mv of $n_2(v)$ one finds

$$\mathrm{mv} = \frac{1}{a_4}\sum_{v\in\Lambda_4} n_2(v) = \frac{a_3}{a_4}m_1 = \frac{2n(n^2-69n+1208)(2n^2-138n+2413)}{n(n^3-94n^2+2783n-24425)}. \tag{MV}$$

Since also $\mathrm{mv}\leqslant 2(n-1)$, this shows the following lemma.

Lemma 3.1. *Let Λ be an odd unimodular lattice of dimension n with minimum $\geqslant 3$ and $\sigma(\Lambda) \geqslant n-16$. Then $n < 80$.*

Notation and Strategy 3.2. *Fix $v \in \Lambda_4$ and* let $k := \frac{1}{2}|N_2(v)|$. *As above we define k pairwise orthogonal vectors of norm 2 as $z_i := u_i - \frac{1}{2}v$ $(1 \leqslant i \leqslant k)$. By $L(v)$ we will always denote the lattice generated by v and the vectors in $N_2(v)$:*

$$L(v) := \langle N_2(v), v \rangle_{\mathbb{Z}}.$$

If $u \in N_1(v)$ then $(u, z_i) = (u, u_i) - 1/2 \in 1/2 + \mathbb{Z}$ is nonzero for all $1 \leqslant i \leqslant k$. Therefore

$$u = \frac{1}{4}\sum_{i=1}^{k} \varepsilon_i z_i + \frac{1}{4}v + t$$

with odd integers ε_i and some vector $t \in L(v)^{\perp}$.

Lemma 3.3. *$n_2(v) \leqslant 44$. If $n_2(v) = 44$ then $n_1(v)$ is a power of 2.*

Proof. Let $v \in \Lambda_4$ with $n_2(v) \geqslant 44$. Eq. (D2) together with the bounds $n_2(v) \leqslant 2(n-1)$ and $n \leqslant 80$ imply that

$$n_1(v) > 0$$

is nonzero. Let $u \in N_1(v)$ and write $u = \frac{1}{4}\sum_{i=1}^{k} \varepsilon_i z_i + \frac{1}{4}v + t$ as above. Since $3 = (u, u) \geqslant \frac{2}{16}k + \frac{4}{16}$ this implies $k \leqslant 22$. Moreover, if $k = 22$ then $t = 0$ and $\varepsilon_i = \pm 1$ for all i. Therefore, $n_2(v) \leqslant 44$ and if $n_2(v) = 44$ then any $u \in N_1(v)$ is of the form

$$u = \frac{1}{4}\sum_{i=1}^{k} \varepsilon_i z_i + \frac{1}{4}v$$

with $\varepsilon_i = \pm 1$.

Let

$$\Gamma := \Lambda \cap (\mathbb{Q} \otimes L(v)).$$

Since all vectors in $L(v)$ have even scalar product with v, the parity of (x, v) is constant in a class of $\Gamma/L(v)$. Let $c \in \Gamma/L(v)$ be a class such that (x, v) is odd for all $x \in c$ and choose $x \in c$ of minimal norm. Then $(x, v) = \pm 1$ and replacing x by $-x$ we may assume that $(x, v) = 1$. If $(x, u_i) = -1$ for some $u_i \in N_2(v)$ then $(x, v - u_i) = 2$, contradicting the minimality of x. Therefore, $(x, u_i) = 0$ or 1 for all $u_i \in N_2(v)$ and we may choose $u_1, \ldots, u_k$ such that $(x, u_i) = 1$ for all $1 \leqslant i \leqslant k$. Hence, $x = \frac{1}{4}(z_1 + \cdots + z_k) + \frac{1}{4}v$ has norm $k/8 + 1/4 = 3$ if $k = 22$. Therefore, all odd classes in $\Gamma/L(v) \subset L(v)^*/L(v)$ are represented by vectors in $N_1(v) \cup -N_1(v)$. Moreover, the precise form of the vectors in $N_1(v)$ shows that all these $2n_1(v)$ classes are distinct.

Since the determinant of $L(v)$ is 2^{24} one has that $2n_1(v) = \frac{1}{2}|\Gamma/L(v)|$ is a power of 2. $\square$

Since $n_2(v) \leqslant 44$ for all $v \in \Lambda_4$, also the mean value mv is $\leqslant 44$. Using formula (MV) one gets:

Corollary 3.4. *Let Λ be an odd unimodular lattice of dimension n with minimum $\geqslant 3$ and $\sigma(\Lambda) \geqslant n-16$. Then $23 \leqslant n \leqslant 46$.*

3.2. The case of dimensions 46, 45, and 44

Let O_{23} be the unique unimodular lattice of dimension 23 with no roots.

Theorem 3.5. *Let Λ be an odd unimodular lattice of dimension 46 with minimum 3 and $\sigma(\Lambda) = 30$. Then $\Lambda \cong O_{23} \perp O_{23}$.*

Proof. By formula (MV) the mean value of $n_2(v)$ is mv$=44$. Since $n_2(v) \leqslant 44$ by Lemma 3.3 for all vectors $v \in \Lambda_4$ it follows that $n_2(v) = 44$ for all $v \in \Lambda_4$. From Eq. (D2) one now also gets that $n_1(v) = 1024$ for all $v \in \Lambda_4$. Let $L := L(v) := \langle N_2(v), v \rangle$ be as in 3.2. Then $\det(L) = 2^{24}$ and $\dim(L) = 23$. Let $\Gamma := \langle N_1(v), L \rangle$. As in the proof of Lemma 3.2 one sees that $L \subset \Gamma \subset L^*$ and that the $2 \cdot 2^{10}$ elements in $N_1(v) \cup -N_1(v)$ lie in distinct classes of Γ/L that have odd scalar product with v. Therefore, $|\Gamma/L|$ is divisible by $2 \cdot (2 \cdot 2^{10}) = 2^{12}$ and Γ is a unimodular lattice of dimension 23 and minimum 3. Hence, $\Lambda = \Gamma \perp \Gamma^{\perp} \cong O_{23} \perp O_{23}$. $\square$

Theorem 3.6. *There is no unimodular lattice Λ of dimension 45 with minimum 3, that satisfies $\sigma(\Lambda) = 45 - 16$.*

Proof. By (MV) one gets mv > 40, so there is a vector $v \in \Lambda_4$ such that $n_2(v) \geqslant 42$. If $n_2(v) = 44$ then $n_1(v) = 848$ is not a power of 2. Hence, by Lemma 3.3 there is $v \in \Lambda_4$ such that $n_2(v) = 42$, i.e. $k = 21$. From Eq. (D2) one calculates $n_1(v) = 856$. Choose $u, u' \in N_1(v)$. In the notation of 3.2, we can define the $z_1, \dots, z_k \in N_2(v) - \frac{v}{2}$ such that $u = \frac{1}{4}(z_1 + \cdots + z_k) + \frac{1}{4}v + t$ and $u' = \frac{1}{4}(-z_1 - \cdots - z_l + z_{l+1} + \cdots + z_k) + \frac{1}{4}v + t'$ with $t, t' \in L(v)^{\perp}$ of norm 1/8. If l is even then $2(u - u') = u_1 + \cdots + u_l - \frac{l}{2}v + 2(t - t')$ shows that $2(t - t') \in \Lambda$ and if l is odd then $k - l$ is even and $2(u + u') = u_{l+1} + \cdots + u_k - \frac{k-l+2}{2}v + 2(t + t')$ implies that $2(t + t') \in \Lambda$. Therefore, one of $2(t \pm t') \in \Lambda$ is a vector of norm $\leqslant 4(1/8 + 2/8 + 1/8) = 2$. Since Λ has minimum 3, this shows that $t' = (-1)^l t$. Let

$$L := \langle N_2(v), v, 8t \rangle \quad \text{and} \quad \Gamma := \Lambda \cap (\mathbb{Q} \otimes L).$$

Then $\det(L) = 2^{26}$. Let $\tilde{L} := \{\gamma \in \Gamma \mid (\gamma, t) \in \mathbb{Z}\}$. The vectors $u \in N_1(v) \cup N_{-1}(v)$ satisfy $(u, t) = \pm\frac{1}{8}$. Therefore, $\tilde{L}$ is a sublattice of index 8 in Γ. The elements

$u \in N_1(v) \cup N_{-1}(v)$ lie in distinct classes of Γ/L. Since all these classes have scalar product $\pm\frac{1}{8} + \mathbb{Z}$ with t and $|N_1(v) \cup N_{-1}(v)| > 2^{10}$, the order of Γ/L is divisible by $4 \cdot 2^{11} = 2^{13}$. Since Γ is integral and $\det(L) = 2^{26}$, this shows that Γ is a unimodular lattice and $\Lambda = \Gamma \oplus \Gamma'$, where $\Gamma' = \Gamma^{\perp} \cap \Lambda$ is a unimodular lattice of dimension 22 and with minimum 3. But there is no such lattice Γ', so this is a contradiction. $\square$

Theorem 3.7. *There is no unimodular lattice Λ of dimension* 44 *with minimum* 3 *that satisfies $\sigma(\Lambda) = 28$.*

Proof. Using formula (MV) one gets mv > 37. Therefore, there is a vector $v \in \Lambda_4$ such that $n_2(v) \geqslant 38$.

- If $n_2(v) = 44$ then $n_1(v) = 688$ is not a power of two. Hence, by Lemma 3.3 this case is impossible.
- Assume now that there is $v \in \Lambda_4$ with $n_2(v) = 42$, i.e. $k = 21$. From Eq. (D2) one calculates that then $n_1(v) = 696$. As in the proof of Theorem 3.6 one sees that all vectors $u \in N_1(v)$ can be written as $u = \frac{1}{4}(\varepsilon_1 z_1 + \cdots + \varepsilon_k z_k) + \frac{1}{4}v + \varepsilon t$ with suitable signs $\varepsilon, \varepsilon_i$ and $t \in \langle N_2(v), v\rangle^{\perp}$ with $(t,t) = 1/8$. Let

$$L := \langle N_2(v), v, 8t\rangle \quad \text{and} \quad \Gamma := \Lambda \cap (\mathbb{Q} \otimes L).$$

  Then $\det(L) = 2^{26}$. Since $|N_1(v) \cup N_{-1}(v)| = 2 \cdot 696 > 2^{10}$, the order of Γ/L is divisible by $4 \cdot 2^{11} = 2^{13}$. Hence, Γ is a unimodular lattice and $\Lambda = \Gamma \oplus \Gamma'$, where $\Gamma' = \Gamma^{\perp} \cap \Lambda$ is a unimodular lattice of dimension 21 and with minimum 3. But there are no such lattices Γ', so this is a contradiction.
- Assume now that $n_2(v) = 40$, i.e. $k = 20$. Then $n_1(v) = 704$. Choose $u, u' \in N_1(v)$. Then we can define the $z_1, \ldots, z_k$ such that $u = \frac{1}{4}(z_1 + \cdots + z_k) + \frac{1}{4}v + t$ and $u' = \frac{1}{4}(-z_1 - \cdots - z_l + z_{l+1} + \cdots + z_k) + \frac{1}{4}v + t'$ with $t, t' \in (\mathbb{Q} \otimes L(v))^{\perp}$ of norm $1/4$. Then $4t \in \Lambda$ shows that $(t, u') = (t, t') \in \frac{1}{4}\mathbb{Z}$, hence $(t, t') = 0, \pm\frac{1}{4}$. If $(t, t') = \pm\frac{1}{4}$ then $t = \pm t'$. Let $L_1 := \langle N_2(v), v\rangle$ and $\Gamma_0 := \langle N_1(v), L_1\rangle$. Let $\{\pm t_1, \ldots, \pm t_s\}$ be the different t_i that occur as projection of $N_1(v)$ to $L_1^{\perp}$ and $L_2 := \langle 4t_1, \ldots, 4t_s\rangle$. Then $L_2 \subset \Lambda$. Since L_2^* is the projection $\pi_2(\Gamma_0)$ onto $\mathbb{Q}L_2$ one has $L_2 = \mathbb{Q}L_2 \cap \Lambda$. Let $\Gamma_1 := \Lambda \cap \mathbb{Q}(L_1 \perp L_2)$ and π_1, π_2 be the orthogonal projections of Γ onto $\mathbb{Q}L_1$ and $\mathbb{Q}L_2$. Then $\pi_2(\Gamma_1)/(\Gamma_1 \cap \mathbb{Q}L_2) = L_2^*/L_2 \cong (\mathbb{Z}/4\mathbb{Z})^s$. On the other hand, this factor group is isomorphic to $\pi_1(\Gamma_1)/(\Gamma_1 \cap \mathbb{Q}L_1)$ which is a subquotient of $L_1^*/L_1 \cong (\mathbb{Z}/2\mathbb{Z})^{18} \times (\mathbb{Z}/4\mathbb{Z})^2$. Therefore, $s \leqslant 2$. Let

$$L := \langle N_2(v), v, 4t_1, \ldots, 4t_s\rangle \quad \text{and} \quad \Gamma := \mathbb{Q}L \cap \Lambda.$$

  Then $\det(L) = 2^{18} \cdot 4^2 \cdot 4^s$ and Γ is an integral overlattice of L. The 1408 vectors $u \in N_1(v) \cap N_{-1}(v)$ lie in distinct classes of Γ/L. Since they all have odd scalar product with v, one concludes that the order of Γ/L is divisible by $2^{11} \cdot 2$. Therefore, if $s = 1$ then Γ is a unimodular lattice of dimension 22 with no roots, which is a contradiction, and if $s = 2$ then $\det(\Gamma) = 4$. But then $\Gamma' := \Gamma^{\perp} \cap \Lambda$ is a

21-dimensional lattice of determinant 4 with minimum 3. Therefore, it is contained in a unimodular lattice of dimension 21 with root system lA_1 or $lA_1 \perp \mathbb{Z}$. Since there is no such unimodular lattice (see [SPLAG, Table 16.7]) this is a contradiction.

- Assume now that $n_2(v) = 38$, i.e. $k = 19$. Then $n_1(v) = 712$. Choose $u, u' \in N_1(v)$. Then we can define the $z_1, \ldots, z_k$ such that $u = \frac{1}{4}(z_1 + \cdots + z_k) + \frac{1}{4}v + t$ and $u' = \frac{1}{4}(-z_1 - \cdots - z_l + z_{l+1} + \cdots + z_k) + \frac{1}{4}v + t'$ with $t, t' \in (\mathbb{Q} \otimes L)^{\perp}$ of norm $3/8$. Then $(u, u') = 1/4 + 1/8(19 - 2l) + (t, t')$ which shows that $(t, t') \in \{\pm 1/8, \pm 3/8\}$. Moreover, if l is even then $2(u - u') \in \Lambda$ and hence $2(t - t') \in \Lambda$ which shows that $(t, t') = -1/8$ or $t = t'$. If l is odd then $2(t + t') \in \Lambda$ and $(t, t') = 1/8$ or $t = -t'$. Let $\{\pm t_1, \ldots, \pm t_s\}$ be the different t_i that occur as projection of $N_1(v) \cup N_{-1}(v)$ on $\langle N_2(v), v \rangle^{\perp}$. W.l.o.g. assume that $t = t_1$ and that $2(t - t_i) \in \Lambda$ for all i. Then $(t, t_i) = -1/8$. If $i \neq j$ then $(2(t - t_i), 2(t - t_j)) = 4(5/8 + (t_i, t_j)) \in \mathbb{Z}$ shows that $(t_i, t_j) = -1/8$ for all $i \neq j$. Therefore, the sum of each four vectors $t_1 + t_2 + t_3 + t_4 = 0$ which shows that $s \leqslant 4$. Let

$$L := \langle N_2(v), v, 2(t_1 - t_2), \ldots, 2(t_1 - t_s), 8t_1 \rangle \quad \text{and} \quad \Gamma := \langle N_1(v), L \rangle.$$

Since the elements $u \in N_1(v)$ satisfy $8u \in L$, $|\Gamma/L|$ is a power of 2. Explicit calculation with the gram matrix of L yields

s	$\det(L)$	$\dim(L)$
4	$2^{18} \cdot 8 \cdot 2^2 \cdot 8$	23
3	$2^{18} \cdot 8 \cdot 2^2 \cdot 8$	23
2	$2^{18} \cdot 8 \cdot 4 \cdot 8$	22
1	$2^{18} \cdot 8 \cdot 24$	21

The 1424 vectors $u \in N_1(v) \cap N_{-1}(v)$ lie in distinct classes modulo L. Since they all have odd scalar product with v, one concludes that the order of Γ/L is divisible by $2^{11} \cdot 2$.

If $s = 1$ then Γ is a lattice of dimension 21 of determinant 3 with no roots. Gluing either with a vector of length 3 or with the root lattice A_2, one sees that such a lattice is either the orthogonal complement of a vector of norm 3 in a 22-dimensional unimodular lattice or the orthogonal complement of A_2 in a 23-dimensional unimodular lattice. An inspection of the possible root sublattices of the unimodular lattices of dimension $\leqslant 23$ (see [SPLAG, Table 16.7]) shows that there is no such lattice Γ with minimum 3.

If $s = 3$ or 4 then $\det(\Gamma) = 4$ and $\Gamma^{\perp} \cap \Lambda$ is a 21-dimensional lattice of determinant 4 with minimum 3, which is a contradiction as above.

If $s = 2$, then $\det(\Gamma) = 4$. As above, Γ is contained in a unimodular lattice Δ of dimension 22 such that the root system of Δ is either A_1^k or $A_1^k \perp \mathbb{Z}$. There is a unique such lattice Δ. It has root system A_1^{22}. Then Γ is the unique sublattice of index 2 with no roots. Since $\Gamma' := \Gamma^{\perp} \cap \Lambda$ has the same properties as Γ, the uniqueness implies that $\Gamma \cong \Gamma'$ and Λ contains $\Gamma \perp \Gamma'$ of index 4. The unique unimodular overlattice of $\Gamma \perp \Gamma'$ is isometric to $\Delta \perp \Delta$ and contains vectors of length 2. $\square$

4. Some numerical values

Table 1 displays some values that can be calculated from the formulas in Section 2. We keep the notation from Section 3. In particular, Λ is an odd unimodular lattice of dimension n with $\sigma(\Lambda) = n - 16$ and $\min(\Lambda) \geqslant 3$. Then $a_3 = |\Lambda_3|$, $a_4 = |\Lambda_4|$. Fix $u \in \Lambda_3$. Then we denote

$$m_i := |\{u' \in \Lambda_3 \mid (u, u') = i\}|, \quad (i = 0, \pm 1, \pm 3)$$

and

$$m'_i := |\{v \in \Lambda_4 \mid (u, v) = i\}|, \quad (i = 0, \pm 1, \pm 2).$$

Then one has $m'_2 = m_1$ and $m'_1 = 12(n^3 - 96n^2 + 2921n - 26838)$. Then we get the following values as given in Table 1.

The last column contains the mean value

$$\mathrm{mv} := \frac{1}{a_4} \sum_{v \in \Lambda_4} n_2(v),$$

Table 1

dim	a_3	a_4	m_1	m'_1	mv
23	4600	93150	891	20736	44
24	4096	98256	759	21528	31.64044944
25	3600	101250	639	21744	22.72000000
26	3120	102180	531	21456	16.21374046
27	2664	101142	435	20736	11.45755473
28	2240	98280	351	19656	8
29	1856	93786	279	18288	5.521335807
30	1520	87900	219	16704	3.787030717
31	1240	80910	171	14976	2.620689655
32	1024	73152	135	13176	1.889763780
33	880	65010	111	11376	1.502538071
34	816	56916	99	9648	1.419354839
35	840	49350	99	8064	1.685106383
36	960	42840	111	6696	2.487394958
37	1184	37962	135	5616	4.210526316
38	1520	35340	171	4896	7.354838710
39	1976	35646	219	4608	12.14004376
40	2560	39600	279	4824	18.03636364
41	3280	47970	351	5616	24
42	4144	61572	435	7056	29.27694407
43	5160	81270	531	9216	33.71428571
44	6336	107976	639	12168	37.49633252
45	7680	142650	759	15984	40.86309148
46	9200	186300	891	20736	44

where for $v \in \Lambda_4$,

$$n_2(v) := |\{v' \in \Lambda_4 \mid (v', v) = 2\}|.$$

5. Examples

The unimodular lattices without roots are known up to dimension 28. There are unique such lattices in dimensions 23, 24 and 26, namely the shorter Leech lattice O_{23} (with $\sigma(O_{23}) = 23 - 8$), the Leech lattice Λ_{24} and the unique 26-dimensional unimodular lattice S_{26} found by Borcherds [Bor] which satisfies $\sigma(S_{26}) = 26 - 16$. In dimension 27, there are 3 unimodular lattices without roots (see [Bor,BaV]), two of which have a characteristic vector of norm $11 = 27 - 16$ [BaV, Théorème 1.1]. The 28-dimensional unimodular lattices of minimum 3 are classified [BaV]. There are 38 such lattices, 36 of which have a characteristic vector of norm $12 = 28 - 16$. All these classifications have been verified by King [Kin] who develops methods to calculate a mass formula for unimodular lattices of given dimension and with given root system.

In dimensions n with $29 \leqslant n \leqslant 35$, one finds lattices Λ with $\sigma(\Lambda) = n - 16$ as neighbours

$$\Lambda = N(\mathbb{Z}^n, v, p) := \left\langle \{x \in \mathbb{Z}^n \mid (v, x) \equiv 0 \pmod{p}\} \cup \left\{\frac{1}{p} v\right\} \right\rangle$$

of the $\mathbb{Z}^n$ lattice for some $v \in \mathbb{Z}^n$ and a prime p with p^2 dividing (v, v). For example, one can choose vectors v with $v_k = k$ for $k = 1, \ldots, n - 4$ and the last four components and p as follows:

n	v_{n-3}	v_{n-2}	v_{n-1}	v_n	p
29	26	27	189	2583	73
30	27	28	334	2593	61
31	28	39	323	2233	73
32	29	48	219	293	83
33	30	31	233	2981	67
34	129	130	933	935	97
35	293	1487	4287	4502	109

Note that by [Kin, Proposition 12] the mass of the 31-dimensional unimodular lattices Λ with no roots that satisfy $\sigma(\Lambda) = 15$ is $(146880/2)$ times the mass of all even extremal unimodular lattices in dimension 32, so it is approximately $4.03 \cdot 10^{11}$.

Using codes, Bachoc and Gaborit construct a 40-dimensional unimodular lattice Λ of minimum 3 with $\sigma(\Lambda) = 24$ [BaG].

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