Free Ljusternik–Schnirelman Theory and the Bifurcation Diagrams of Certain Singular Nonlinear Problems

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1. INTRODUCTION

This paper may be regarded as a sequel to the author's recent paper [9], in which it was announced. However, it contains more material than what was announced in [9], as will be explained below.

The discussion is centered around the singular nonlinear Dirichlet problem

\[-u'' + w(t) |u|^{\sigma}u = \lambda u,\]

\[u(0) = 0, \quad u \in L^2[0, \infty[,\]

where \(\sigma > 0\) is a constant and \(w\) is a positive continuous function satisfying the minimal growth condition

\[\int_0^\infty w^{-2/\sigma} \, dt < \infty.\]

This problem exhibits bifurcation from the essential spectrum (cf. [2, 3, 8, 9] and the references therein), and its bifurcation diagram as well as nodal properties of its solutions have been studied in [9] with the help of an elementary shooting method and in [3, 8] with the help of critical point theory on level surfaces generated by prescribing the \(L^2\)-norm of the solutions. Benci and Fortunado [2] treated a related problem using Ljusternik–Schnirelman theory for the unconstrained case, so that \(\lambda > 0\) is a parameter which may be prescribed rather than an eigenvalue to be sought.

It is this approach that we take up here again, with the version of Ljusternik–Schnirelman theory given by Clark [4] as our point of departure. Combining the theory of Clark [4] with various results and methods
from [3, 8, 9], we obtain most of the information needed to complete the
global description of the bifurcation diagram of (1.1), (1.2) which was
begun in [9].

However, most of these techniques can easily be applied to more general
problems, thus yielding various generalizations and supplements of results
from [2–4, 6, 8–11]. Therefore only Sections 6 and 7 deal with problem
(1.1), (1.2) as such, while situations of varying degree of generality are con-
sidered in Sections 2 to 5.

To be specific, Section 2 contains a brief account of the main result of
Clark [4], and we prove that the Ljusternik–Schnirelman critical values $c_j$
tend to zero as $j \to \infty$. This result was proved by Benci and Fortunado [2]
for the special boundary value problem they considered, utilizing various
particularities of their problem for the proof. However, we would like to
emphasize that this limiting relation is an intrinsic feature of Ljuster-
nik–Schnirelman theory and can be proved in full generality by standard
arguments of that theory (see, e.g., [13]).

In Section 3 these results are applied to the operator-theoretic nonlinear
eigenvalue problem

$$Lu + F(u) = \lambda u$$  \hspace{1cm} (1.4)

in a Hilbert space $H$, where $L$ is a self-adjoint linear operator in $H$ which is
bounded below, and where $F$ is a nonlinear, strictly monotone odd
gradient operator. Under assumptions similar to those used in [3] for an
analogous constrained problem, it turns out that (1.4) enjoys all the
properties which were proved by Benci and Fortunado [2] for a particular
boundary value problem. In particular, if the g.l.b. $\lambda^*$ of $L$ is assumed to
belong to the essential spectrum of $L$, then, for every $\lambda > \lambda^*$ there exists a
sequence $(u_{j,\lambda},)$ of solutions to (1.4) such that $u_{j,\lambda} \to 0$ as $\lambda \to \lambda^*$ and such
that the $u_{j,\lambda}$ admit variational characterizations of Ljusternik–Schnirelman
type.

The results of Section 3 hold, in particular, for a wide class of boundary
value problems on unbounded domains $G \subseteq \mathbb{R}^n$, as is briefly discussed in
Section 4. This class is given by almost the same assumptions as have been
used in Section 5 of Bongers, Heinz, and Küpper [3], and it is a natural
generalization of problem (1.1), (1.2) under the growth condition (1.3)
(and also of the problem treated in [2]). In Section 5, special attention is
given to the case $n = 1$, in which the assumptions can be relaxed roughly as
was done for the constrained case in Section 6 of [3], and in which the
nodal structure of the solutions becomes an important object of study.
Concerning nodal properties, the methods of Heinz [8] carry over to the
unconstrained case, and we obtain the precise analogues of the main results
of [8], thereby generalizing earlier results of Coffman [6], Hempel [10],
and Jones and Küpper [11].
At this point, a word should be said about the relationships between nodal properties and the variational method. Presumably, Nehari [12] was the first to use a variational principle involving the variation of systems of prescribed zeroes to construct solutions with a given number of zeroes. His method was extended to problems on an unbounded interval by Ryder [14]. Coffman [5] first established a relationship between Nehari's method and Ljusternik–Schnirelman theory by proving that the characteristic numbers defined by Nehari coincide with the Ljusternik–Schnirelman critical levels of a suitable auxiliary functional. The differential equations treated in [5, 12, 14] are typified, for example, by the equation

\[ u'' + \lambda u + w(t) |u|^r u = 0, \]

and they are termed "superlinear" by Coffman and many other authors, whereas Eq. (1.1) belongs to a class called "sublinear" by these authors, because it is thought of as being written in the form

\[ u'' + \lambda u - w(t) |u|^s u = 0. \]

For such "sublinear" problems on a compact interval Hempel [10] established the existence of solutions with prescribed number of zeroes using a suitable variational principle, and his critical levels were subsequently identified with Ljusternik–Schnirelman levels by Coffman [6]. Independently of Coffman and Hempel, the author [8] recently gave a proof of the existence of eigenfunctions with prescribed number of zeroes and prescribed \(L^2\)-norm for sublinear problems both on bounded and unbounded intervals. In this proof Ljusternik–Schnirelman theory on level surfaces is combined with a dual variational principle again involving movable systems of prescribed zeroes. Certain complications arise due to the need to keep the \(L^2\)-norm fixed while the prescribed zeroes are being shifted, but when the constraint is removed the dual variational principle from [8] actually reduces to the one considered in [10, 6]. (cf. Section 5 for details.) Moreover, the papers [6, 8] share a fundamental construction in which systems of prescribed locations of zeroes are used to generate a set of known genus in the underlying function space. Thus the treatment of nodal properties in Section 5 may be viewed either as a generalization of that of Coffman [6] or as an analogue of that of Heinz [8]. However, both the existence proof in [10] and the proof of equivalence of the two variational principles in [6] break down in singular cases such as (1.1), (1.2), and so it is necessary to follow the arguments from [8] in order to establish the results for singular sublinear problems in Section 5.

Finally, in Section 6, problem (1.1), (1.2) is considered under the assumptions
(L) \( w \in C^1 \) and \( w'/w \) is nondecreasing.

and

(E) \( w'(t_0) > 0 \) for some \( t_0 \geq 0 \),

which have already been used in [9]. These assumptions imply (1.3), and it is a trivial matter to infer the improvements announced in [9] from the material of Sections 3 and 5. Beyond this, we relate the shooting method from [9] and the present variational method by studying the dependence of the potential functional

\[
J_\lambda(u) := \int_0^\infty \left[ \frac{1}{2} (u'^2 - \lambda u^2) + \frac{1}{p} w |u|^p \right] dt
\]

(where \( p := \sigma + 2 \)) on the initial slope \( u'(0) \) of a solution \( u \) of (1.1), (1.2). The resulting monotonicity theorem (Theorem 6.2) leads to interesting new uniqueness properties of (1.1), (1.2), and in fact it turns out that the description of the bifurcation diagram given in [9] under additional hypotheses is already valid when only (E) and (L) are assumed. Furthermore, the "preferred solutions" \( u_{n,\lambda} \) which have been constructed in [9] and which make up the continuous branches of the bifurcation diagram, are seen to enjoy three equivalent characterizations. To describe them, we fix \( \lambda > 0 \) and \( n \in \mathbb{N} \), and we remove the obvious sign ambiguity by considering the set \( S \) of all solutions of (1.1), (1.2) having positive initial slope and precisely \( n - 1 \) distinct interior zeroes. Then \( u = u_{n,\lambda} \) is equivalent to each of the following statements:

(i) \( u \in S \), and the initial slope is minimal for \( u \) among all functions in \( S \).

(ii) \( u \in S \), and \( J_\lambda(u) \) is maximal among the \( J_\lambda(v), \forall \in S \).

(iii) \( u \in S \), and \( J_\lambda(u) \) is the \( n \)th Ljusternik–Schnirelman level of \( J_\lambda \).

Thus it becomes clear that both the shooting method and the variational method lead to the same distinguished solutions \( u_{n,\lambda} \).

It is still an open question whether or not (1.1), (1.2) has other nontrivial solutions besides the \( \pm u_{n,\lambda} \). However, the limiting behavior of the Ljusternik–Schnirelman levels as established in Section 2 has the surprising consequence that Eq. (1.1) has no oscillating solutions, as shall be seen in Section 7. Hypotheses (E) and (L) are sufficient for this result, but not necessary. Instead, it essentially suffices to assume (1.3) and the uniqueness (up to sign) of solutions with given number of zeroes on a given compact interval, i.e., the uniqueness property that was derived from condition (L) in [9]. The paper is then concluded with a counterexample in which (1.3)
is satisfied, yet oscillating solutions do exist, so that the uniqueness property just mentioned must be violated. The example thus shows that additional requirements such as (L) are indispensible for the results of [9] and of the two final sections of the present paper.

For the sake of conciseness the author has refrained from reiterating any commentary or examples concerning the various sets of assumptions. However, the assumptions themselves have been carefully restated for the convenience of the reader.

2. REMARKS ON FREE LJUSTERNIK–SCHNIRELMAN THEORY

Let us briefly recall the version of Ljusternik–Schnirelman theory given by Clark [4]. Consider a Banach space $X$ and a functional $J : C^1(X, \mathbb{R})$, let $X^*$ be the topological dual of $X$, $\langle \cdot, \cdot \rangle : X^* \times X \to \mathbb{R}$ the canonical pairing, and $dJ : X \to X^*$ the gradient of $J$. Moreover, let $\Sigma$ be the system of closed subsets of $X \setminus \{0\}$ which are symmetric with respect to reflection at the origin (i.e., which satisfy $-A = A$) and, for every $A \in \Sigma$, let $\gamma(A)$ denote the genus of $A$. (For definition and properties of the genus see, e.g., [4], [5], or [13]). The Ljusternik–Schnirelman levels $c_j$ of $J$ are then defined by

$$c_j := \inf_{A \in \Sigma} \sup_{\gamma(A) \geq j} J(u) \quad (j \geq 1).$$

We assume

(i) $J$ is bounded below on $X$, $J$ is even, and $J(0) = 0$;

(ii) $J$ satisfies the Palais–Smale condition.

This means that if $(u_n)_{n \geq 1}$ is a sequence in $X$ such that $(J(u_n))_n$ is bounded and $dJ(u_n) \to 0$ strongly in $X^*$ as $n \to \infty$, then $(u_n)$ has a subsequence which converges strongly in $X$. Finally, for some $m \in \mathbb{N}$ we assume:

(iii)$_m$ There exists $K \in \Sigma$ such that $\gamma(K) = m$ and $\sup_{u \in K} J(u) < 0$.

Under these assumptions it is clear that $-\infty < c_1 \leq c_2 \leq \cdots \leq c_m < 0$.

The main result of Clark [4] can now be stated as follows:

**Theorem 2.1.** If $J \in C^1(X, \mathbb{R})$ satisfies hypotheses (i), (ii), (iii)$_m$ for some $m \in \mathbb{N}$, then there exist antipodal pairs $(u_j, -u_j)$ of critical points of $J$ such that $J(u_j) = c_j$ for $1 \leq j \leq m$.

In particular, if (iii)$_m$ is satisfied for all $m \in \mathbb{N}$, then there exists a non-decreasing sequence $(c_j)_{j \geq 1}$ of negative critical values of $J$.

With respect to this sequence we prove:
Proposition 2.2. If $J$ satisfies hypotheses (i), (ii), and (iii), for every $m \in \mathbb{N}$, then we have

$$\lim_{j \to \infty} c_j = 0. \quad (2.1)$$

Proof. Obviously the nondecreasing sequence $(c_j)$ converges to a number $c^* \leq 0$. Suppose $c^* < 0$ and put $C := \{ u \mid J(u) = c^*, \ dJ(u) = 0 \}$. Then $C \subseteq \Sigma$, and it follows from Theorem 2.1 and the Palais–Smale condition that $C$ is nonempty and compact. Hence $k := \gamma(C)$ is finite; and there exists an open symmetric neighborhood $U$ of $C$ such that $\bar{U} \subseteq X - \{0\}$ and $\gamma(\bar{U}) = k$.

As usual, let us write

$$J^c := J^{-1}(\mathbb{R} - c^*) \quad (c \in \mathbb{R}).$$

Note that $J^c \in \Sigma$ for $c < 0$ and that $c > c_j$ implies $\gamma(J^c) > j$, while $c < c_j$ implies $\gamma(J^c) < j$ by definition of the $c_j$ ($j \geq 1$). Now by the standard deformation lemma of Ljusternik–Schnirelman theory (cf. [4, 13], for example) there is $\varepsilon \in ]0, -c^*[\$ such that $J^{c^* + \varepsilon} = U$ can be mapped into $J^{c^* - \varepsilon}$ by an odd continuous operator, which implies

$$\gamma(J^{c^* + \varepsilon} - U) \leq \gamma(J^{c^* - \varepsilon}).$$

Choose $j$ such that $c^* - \varepsilon < c_j$. Then we obtain

$$\gamma(J^{c^* + \varepsilon}) \leq \gamma(J^{c^* + \varepsilon} - U) + \gamma(\bar{U}) \leq \gamma(J^{c^* - \varepsilon}) + k < j + k < \infty.$$

But since $c^* + \varepsilon > c_j$ for all $j \geq 1$, $\gamma(J^{c^* + \varepsilon})$ must be infinite. This contradiction proves (2.1).

3. Application to Abstract Nonlinear Eigenvalue Problems

In this section we present an abstract setting for the boundary value problems to be treated in the sequel. Roughly speaking, we consider the nonlinear eigenvalue problem

$$T^*Tu + F(u) = \lambda u, \quad (3.1)$$

where $T$ is a closed, densely defined linear operator in a real Hilbert space $H$, and $F$ is a nonlinearity. However, since the variational approach leads to weak solutions (3.1) is not to be considered as an equation in $H$. Instead, we seek solutions of (3.1) in a certain dense subspace $X$ of $H$ which is equipped with an appropriate stronger norm $\| \cdot \|_X$, and both sides
of (3.1) are to be viewed as elements of the dual $X^*$ of the normed space $(X, \| \cdot \|_X)$. The self-adjoint Hilbert space operator $L := T^*T$ is needed only for its spectral properties.

To describe the space $X$ precisely, we need an auxiliary Banach space $(Y, \| \cdot \|_Y)$, of which we require

(S1) $Y$ is reflexive and continuously embedded in $H$.

Denoting the norm of $H$ by $\| \cdot \|$, we therefore have

$$\|u\| \leq c_0 \|u\|_Y$$

(3.2)

for all $u \in Y$, where $c_0 > 0$ denotes the norm of the embedding. Moreover, the domain $D_T$ is a Hilbert space with respect to the graph norm $\| \cdot \|_T$ of $T$, which is defined by

$$\|u\|_T^2 = \|u\|^2 + \|T u\|^2$$

($u \in D_T$). We then put $X := Y \cap D_T$ and define $\| \cdot \|_X$ by

$$\|u\|_X^2 = \|u\|_T^2 + \|u\|_T^2 \quad (u \in X).$$

It follows from (S1) and the definitions that $(X, \| \cdot \|_X)$ is a reflexive Banach space because $X$ can be mapped isometrically onto a closed subspace of $Y \times D_T$. We further assume:

(S2) The embedding $X \rightarrow H$ is compact.

(S3) $D_L \cap X$ is dense in $D_L$ with respect to the graph norm of $L = T^*T$.

Clearly (S3) implies that $X$ is dense in $H$. Hence the conjugate operator of the embedding $X \rightarrow H$ is an embedding $H \rightarrow X^*$, so that we may consider all our spaces as linear subspaces of $X^*$. The canonical pairing $X^* \times X \rightarrow \mathbb{R}$ then agrees with the scalar product of the real Hilbert space $H$ on $H \times X$, and we shall denote both pairings by $\langle \cdot, \cdot \rangle$. About the non-linearity $F$ we assume:

(N1) $F$ is a continuous odd monotone gradient operator $X \rightarrow X^*$

(N2) $\|F(u)\|_{X^*} = o(\|u\|_X)$ as $\|u\|_X \rightarrow 0$

(N3) If $u_n \rightarrow^x u$ and

$$\langle F(u_n) - F(u), u_n - u \rangle \rightarrow 0.$$

then $u_n \rightarrow^y u$.

(Here "$\rightarrow^x$" denotes weak convergence and "$\rightarrow$" denotes strong convergence in the space indicated by the superscript.)
(N4) There are constants $\delta > 0$ and $p > 2$ such that

$$\langle F(u), u \rangle \geq \delta \|u\|_p^p$$

for all $u \in X$.

These hypotheses are almost identical to those used in Sections 3 and 4 of [3] (with the exception of (N4), which is formally stronger than the corresponding hypothesis (II, 3) from [3], but it is trivially verified in all relevant cases. Note also that $X$ and $Y$ are not required to be uniformly convex and that we do not need any counterpart of condition (III, 2) of [3]).

Denote by $\Phi$ the potential functional of $F$ which satisfies $\Phi(0) = 0$. We then have

$$\Phi(u) = \int_0^1 \langle F(tu), u \rangle \, dt$$

(3.3)

for all $u \in X$, and it clearly follows from (N1) that $\Phi$ is convex, even, non-negative and of class $C^1$ on $X$. For every $\lambda > 0$ we now define a functional $J_\lambda \in C^1(X, \mathbb{R})$ by

$$J_\lambda(u) := \frac{1}{2} \|Tu\|^2 + \Phi(u) - \frac{\lambda}{2} \|u\|^2 \quad (u \in X).$$

Then the equation $dJ_\lambda(u) = 0$ is obviously equivalent to

$$T_1 T_1 u + F(u) = \lambda u,$$

(3.4)

where $T_1$ denotes the bounded linear operator $X \to H$ induced by $T$, and $T_1^*: H \to X^*$ denotes its conjugate operator. Thus (3.4) is a precise formulation for the problem of finding the "weak solutions" of (3.1) in $X$.

The assumptions (S1)–(S3), (N1)–(N4) entail some simple properties of $J_\lambda$ which we now list for future reference. Here and in the sequel we use the notation

$$h_\delta(t) := \frac{\delta}{p} t^p - \frac{\lambda}{2} \frac{c_0^2}{t^2} \quad (t > 0).$$

**Lemma 3.1.** For every $\lambda > 0$ we have

(a) $J_\lambda(u) \geq h_\delta(\|u\|_p)$ for every $u \in X$.

(b) If $J_\lambda$ is bounded above on a subset $B$ of $X$, then $B$ is bounded in the normed space $X$.

(c) If $(u_k)_{k \geq 1}$ is a sequence in $X$ such that $(J_\lambda(u_k))_k$ is bounded above, then $(u_k)_{k \geq 1}$ possesses a subsequence which is weakly convergent in $X$. 
Proof. (a) Let \( u \in X \). From (3.3) and (N4) we get
\[
\Phi(u) = \int_0^1 t^{-1} \langle F(tu), tu \rangle \, dt \geq \int_0^1 \delta t^{-1} \|tu\|_\delta^2 \, dt = \frac{\delta}{p} \|u\|_\phi^2,
\]
and hence (3.2) yields
\[
J_\lambda(u) \geq \Phi(u) - \frac{\lambda}{2} \|u\|^2 \geq \frac{\delta}{p} \|u\|_\phi^2 - \frac{1}{2} \lambda c_0^2 \|u\|_\gamma^2
\]
for every \( \lambda > 0 \), which is the desired result.

(b) Suppose \( J_\lambda(u) \leq b \) for every \( u \in B \) (\( \lambda > 0 \) fixed). Since \( p > 2 \), we have
\[
\lim_{t \to -\infty} h_\lambda(t) = \infty.
\]
Thus, \( h_\lambda(\|u\|_Y) \leq J_\lambda(u) \leq b \) implies that \( B \) is bounded in \( Y \) and hence bounded in \( H \) by (S1). Moreover, since \( \Phi \geq 0 \) we have
\[
\frac{1}{2} \|Tu\|^2 \leq J_\lambda(u) + \frac{\lambda}{2} \|u\|^2 \leq b + \frac{\lambda}{2} \|u\|^2
\]
for every \( u \in B \), which implies that \( \|Tu\| \) remains bounded on \( B \). Hence the result follows by definition of \( \| \cdot \|_X \).

Part (c) is an immediate consequence of (b) and a well-known theorem on reflexive Banach spaces.

Let \( \lambda^* \) be the lowest point of the spectrum of the self-adjoint nonnegative operator \( L \). To facilitate statement and proof of the main result of this section, we assume
\[
\lambda^* = 0,
\]
which is no loss of generality since we may always replace \( T \) by the operator \( \overline{T} := (L - \lambda^* I)^{1/2} \). We then have:

**Theorem 3.2.** Suppose hypotheses (S1)–(S3), (N1)–(N4) as well as (3.5) are satisfied and let \( c_j(\lambda) \) be the Ljusternik–Schnirelman levels of \( J_\lambda \) as defined in Section 2 (\( j \in \mathbb{N}, \lambda > 0 \)).

(a) If \( \lambda^* \) is an eigenvalue of \( L \) of multiplicity \( m \), then for every \( \lambda > 0 \), Eq. (3.4) has at least \( m \) distinct antipodal pairs \((u_{j, \lambda}, -u_{j, \lambda})\) of solutions such that
\[
c_j(\lambda) = J_\lambda(u_{j, \lambda}) < 0
\]
(\( 1 \leq j \leq m \)), and these solutions bifurcate from \( \lambda^* = 0 \) in the sense that
\[
\lim_{\lambda \to 0^+} u_{j, \lambda} = 0
\]
strongly in \( X \) for every \( j \).
If \( \lambda^* \) belongs to the essential spectrum of \( L \), then, for every \( \lambda > 0 \), Eq. (3.4) has an infinite sequence \((u_{j,\lambda}, -u_{j,\lambda})_{j \geq 1}\) of distinct solutions satisfying (3.6) and (3.7) for all \( j \). Moreover, we have

\[
\lim_{j \to \infty} c_j(\lambda) = 0
\]

for every \( \lambda > 0 \).

**Proof.** Fix \( \lambda > 0 \), and let us verify conditions (i), (ii), (iii) from Section 2 for \( J_\lambda \). Since \( h_\lambda \) is bounded below on \([0, \infty[\), it follows from Lemma 3.1(a) that (i) is satisfied. Next, in order to verify the Palais–Smale condition, consider a sequence \((u_k)\) in \( X \) such that \((J_\lambda(u_k))\) is bounded and \(dJ_\lambda(u_k) \to 0\) strongly in \( X^* \) as \( k \to \infty \). Then by Lemma 3.1(c), \((u_k)\) has a subsequence (again denoted by \((u_k)\)) which tends to an element \( u \in X \) weakly in \( X \). Put

\[
h_k := u_k - u, \quad w_k := dJ_\lambda(u_k)
\]

for \( k \in \mathbb{N} \). Then \( h_k \to X 0 \), \( w_k \to X * 0 \), and hence \( \langle w_k, h_k \rangle \to 0 \) and \( \langle dJ_\lambda(u), h_k \rangle \to 0 \). Moreover, (S2) implies that \( h_k \to Y 0 \), i.e., \( \|h_k\| \to 0 \). Therefore, noting that \( w_k = T^\prime_1 T^\prime_1 u_k + F(u_k) - \lambda u_k \), we obtain

\[
0 = \|Tu_k - Tu\|^2 + \langle F(u_k) - F(u), u_k - u \rangle = \langle w_k, h_k \rangle - \langle dJ_\lambda(u), h_k \rangle + \lambda \|h_k\|^2 \to 0
\]

and hence \( \|Th_k\| \to 0 \) as well as

\[
\langle F(u_k) - F(u), u_k - u \rangle \to 0.
\]

Thus we may apply (N3), which yields \( u_k \to Y u \). The desired relation \( u_k \to X u \) now follows from the definition of \( \| \|_{X} \), and (ii) is established. Finally, note that because of assumption (S3) we have the result of Lemma 4.1 of [3]. Therefore we may choose an \( m \)-dimensional linear subspace \( Z \) of \( X \) such that

\[
\|Tu\|^2 \leq \frac{\lambda}{3} \|u\|^2
\]

for every \( u \in Z \), where \( m \) is the multiplicity of \( \lambda^* \) in case (a) and \( m \) is arbitrary in case (b). On the finite-dimensional space \( Z \) the norms induced by \( H \) and \( X \) are equivalent, hence (N2) implies \( \|F(u)\| = o(\|u\|) \) and further \( \Phi(u) = o(\|u\|^2) \) for \( u \to 0 \) in \( Z \). Putting \( S_r := \{u \in Z \mid \|u\| = r\} \), we may therefore choose \( r > 0 \) so small that

\[
\Phi(u) \leq \frac{\lambda}{6} \|u\|^2
\]

(3.10)
for \( u \in S_r \). Combining (3.9) and (3.10), we find

\[
J_\lambda(u) \leq \frac{\lambda}{6} \|u\|^2 + \frac{\lambda}{6} \|u\|^2 - \frac{\lambda}{2} \|u\|^2 = -\frac{1}{6} \lambda u^2 < 0
\]

for all \( u \in S_r \). On the other hand, \( \gamma(S_r) = \dim Z = m \), as is well known. Hence (iii) is valid for all \( m \) in case (b), and in case (a) it is valid for the special \( m \) indicated there. This implies \( c_j(\lambda) < 0 \) for \( j = 1, \ldots, m \), and the existence assertions follow from Theorem 2.1, while (3.8) follows from Proposition 2.2.

It remains to prove (3.7). Writing \( h_j(t) = t^2((\delta/p)t^\sigma - \lambda e_0^2/2) \), where \( \gamma := p - 2 > 0 \), we see that \( h_j(t) < 0 \) implies \( t < (p\lambda e_0^2/2\delta)^{1/\sigma} \). Now fix \( j \geq 1 \) such that solutions \( u_{j,i} \in X \) satisfying (3.6) exist for every \( \lambda > 0 \). Then it follows from \( c_j(\lambda) < 0 \) and Lemma 3.1(a) that \( \|u_{j,i}\|_Y < C_1 \lambda^{1/\sigma} \), and hence \( u_{j,i} \to_\gamma 0 \) as \( \lambda \to 0^+ \). By (S1) this implies \( \lim_{\lambda \to 0^+} \|u_{j,i}\| = 0 \).

To check the behaviour of \( \|Tu_{j,i}\| \), we estimate

\[
\|Tu_{j,i}\|^2 \leq 2J_\lambda(u_{j,i}) + \lambda \|u_{j,i}\|^2 = 2c_j(\lambda) + \lambda \|u_{j,i}\|^2 < \lambda \|u_{j,i}\|^2,
\]

and hence \( \|Tu_{j,i}\| \to 0 \) as \( \lambda \to 0^+ \), which yields (3.7). Thus the proof of Theorem 3.2 is complete.

Of course one cannot expect that the solutions \( u_{j,i} \) can always be selected so as to form continuous branches in \( X \). However, in Section 6 we shall meet a situation where this can actually be done, and for this the following two observations will be helpful:

**Proposition 3.3.** With the assumptions and notations of Theorem 3.2, consider a fixed \( j \in \mathbb{N} \). We have

(a) \( c_j(\lambda) \) is a continuous function of \( \lambda \), and

(b) if we put \( u_{j,0} := 0 \) and suppose that for every \( \lambda > 0 \) we have selected a solution \( u_{j,i} \) satisfying (3.6) in such a way that \( \{(\lambda, u_{j,i}) \mid \lambda \geq 0\} \) is closed in \( [0, \infty \times X \), then the map \( \lambda \to u_{j,i} \) is continuous from \( [0, \infty \) to \( X \).

**Proof.** Putting \( \sigma := p - 2 \) again, we have seen in the proof of (3.7) above that \( J_\lambda(u) < 0 \) implies \( \|u\|_Y^\sigma \leq p\lambda e_0^2/2\delta \), whence (3.2) yields

\[
\|u\|_Y \leq C_1 \lambda^{1/\sigma}
\] (3.11)

with a constant \( C > 0 \). Now, to prove part (a), consider \( 0 < \lambda_1, \lambda_2 \leq \mu < \infty \). We know that \( c_j(\lambda) < 0 \) for all \( \lambda \). Choosing \( \varepsilon > 0 \) such that \( c_j(\lambda_2) + \varepsilon < 0 \), we find a set \( K \in \Sigma \) such that \( \gamma(K) \geq j \) and \( J_{\lambda_2}(u) \leq c_j(\lambda_2) + \varepsilon \) for every \( u \in K \) (notations as in Section 2). Thus (3.11) yields

\[
\|u\| \leq C_1 \lambda_2^{1/\sigma} \leq C \mu^{1/\sigma}
\]
for all $u \in K$, and hence, observing that $J_{\lambda_1}(u) = J_{\lambda_2}(u) + \frac{1}{2}(\lambda_2 - \lambda_1) \|u\|^2$, we obtain:

$$c_f(\lambda_1) \leq \sup_{u \in K} J_{\lambda_1}(u) \leq \sup_{u \in K} J_{\lambda_2}(u) + \frac{1}{2} |\lambda_2 - \lambda_1| C^2 \mu^{2/\sigma}$$

$$\leq c_f(\lambda_2) + \epsilon + \frac{1}{2} |\lambda_2 - \lambda_1| C^2 \mu^{2/\sigma}.$$

Since $\epsilon$ can be chosen arbitrarily small, this implies

$$c_f(\lambda_1) - c_f(\lambda_2) \leq \frac{1}{2} C^2 \mu^{2/\sigma} |\lambda_1 - \lambda_2|.$$

But the roles of $\lambda_1$ and $\lambda_2$ may be interchanged in this argument, and hence we obtain a similar estimate for $|c_f(\lambda_1) - c_f(\lambda_2)|$, from which assertion (a) clearly follows.

To prove (b) consider a sequence $(\lambda_k)_{k \geq 1} \subseteq [0, \infty]$ which tends to a limit $\lambda_0$. If $\lambda_0 = 0$, the desired relation is just (3.7). Thus suppose that $\lambda_0 > 0$. Then $J_{\lambda_0}$ satisfies the Palais–Smale condition, as we have seen in the proof of Theorem 3.2. Moreover, since $J_{\lambda_k}(u_{j,\lambda_k}) = c_f(\lambda_k) < 0$, we can infer from (3.11) that

$$J_{\lambda_0}(u_{j,\lambda_k}) \leq c_f(\lambda_k) + \frac{1}{2} C^2 \mu^{2/\sigma} |\lambda_k - \lambda_0|$$

for all $k$, where $\mu := \sup_{k \geq 0} \lambda_k < \infty$. Hence the sequence $(J_{\lambda_0}(u_{j,\lambda_k}))_k$ has the upper bound $C^2 \mu^{2/\sigma} + 1$ and it is bounded below by virtue of Lemma 3.1(a). Furthermore, $dJ_{\lambda_0}(u_{j,\lambda_k}) = dJ_{\lambda_k}(u_{j,\lambda_k}) + (\lambda_k - \lambda_0)u_{j,\lambda_k} = (\lambda_k - \lambda_0)u_{j,\lambda_k}$, and from (3.11) and the continuous embedding $H \hookrightarrow X^*$ it follows that

$$\|u_{j,\lambda_k}\|_{X^*} \leq C^* \mu^{1/\sigma} \quad (k \geq 0)$$

with a constant $C^* > 0$. This implies $\|dJ_{\lambda_0}(u_{j,\lambda_k})\|_{X^*} \leq C^* \mu^{1/\sigma} |\lambda_k - \lambda_0| \to 0$ as $k \to \infty$. Hence the Palais–Smale condition tells us that $(u_{j,\lambda_k})_{k \geq 1}$ possesses a strongly convergent subsequence. But since the graph of the map $\lambda \to u_{j,\lambda}$ is closed by assumption, the only possible limit for such a convergent subsequence is $u_{j,\lambda_0}$. Hence we have $u_{j,\lambda_0} = \lim_{k \to \infty} u_{j,\lambda_k}$ strongly in $X$ by a standard lemma, and the proof is complete.

**Remark 3.4.** For the particular boundary value problem considered by Benci and Fortunado in [2], they also proved

$$\lim_{j \to \infty} \|u_{j,\lambda}\|_X = 0 \quad (3.12)$$

for every $\lambda > 0$. This result can be recovered in the present abstract setting under the additional assumption:

$$(N5) \quad \langle F(u), u \rangle = 2\Phi(u) \implies u = 0.$$
To see this, note first that for fixed $\lambda > 0$ the set of solutions $u_{j,\lambda}$ of (3.4) satisfying (3.6) is relatively compact by the Palais–Smale condition. But if $u$ is the limit of any sequence of the form $(u_{j,\cdot})_{k \geq 1}$, then $u$ satisfies (3.4) and also

$$J_{\lambda}(u) = 0$$

by (3.8). By definition of $J_{\lambda}$ this implies $\langle F(u), u \rangle = 2\Phi(u)$ and hence $u = 0$ by (N5). This clearly proves (3.12).

A sufficient condition for (N5) is, e.g.,

$$\langle F(tu), u \rangle < t \langle F(u), u \rangle$$

for $u \neq 0$, $0 < t < 1$. In particular, (N5) is satisfied if $F$ is homogeneous of order $q > 1$ and satisfies (N4). This is the case for the problems treated in [2, 9], and Sections 6 and 7 of the present paper.

### 4. Application to Semilinear Elliptic Boundary Value Problems

In this section we apply Theorem 3.2 to essentially the same class of boundary value problems as that considered in [3], thus generalizing the existence and bifurcation results of Benci and Fortunado [2]. We consider the Dirichlet problem

$$\mathcal{L}u + f(x, u) = \lambda u$$

$$u |_{\partial G} = 0, \quad u \in L^2(G)$$

on a domain $G \subseteq \mathbb{R}^n$ (unbounded in general!), where $\mathcal{L}$ is a second-order real linear elliptic differential operator of the form

$$\mathcal{L}u := -\nabla(P(x) \nabla u) + Q(x)u,$$

and $\lambda > 0$ is a parameter. About the coefficients of $\mathcal{L}$ we assume that $Q \in L^{\infty}(G)$ and that $P = (P_{jk})_{1 \leq j, k \leq n}$ is a symmetric matrix of bounded real $C^1$-functions $P_{jk}$ such that, for some $\varepsilon > 0$, we have

$$\sum_{j,k=1}^{n} P_{jk}(x) \xi_j \xi_k \geq \varepsilon \sum_{k=1}^{n} \xi_k^2$$

for every $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and every $x \in G$. Standard Hilbert space operator theory then tells us that in the Hilbert space $H := L^2(G)$ the operator $\mathcal{L}$ has one and only one self-adjoint extension whose domain is contained in the Sobolev space $W_0^{1,2}(G)$. This extension is taken to be the operator $L$ from Section 3. By adding a suitable constant to $Q$ we arrange for (3.5) to hold. Then $L \geq 0$, and hence $L = T^*T$ with $T := L^{1/2}$. Moreover,
it follows from well-known operator theory that $W^{1,2}_0(G)$ is then precisely the domain of $T$.

Next, the nonlinear term is assumed to be of the form

$$f(x, \eta) = \int_0^\eta g(x, \xi) \, d\xi \quad (x \in G, \eta \in \mathbb{R})$$  \hspace{1cm} (4.3)

where $g: G \times \mathbb{R} \to \mathbb{R}$ satisfies:

(G1) $g$ is continuous on $G \times \mathbb{R}$, $g(x, -\eta) = g(x, \eta)$ for every $x \in G$, $\eta \in \mathbb{R}$ and there exist constants $\sigma > 0$, $\delta > 0$ and a continuous function $w: G \to [0, \infty]$ such that

$$g(x, \eta) \geq \delta w(x) |\eta|^\sigma$$  \hspace{1cm} (4.4)

for all $x \in G$, $\eta \in \mathbb{R}$, and

$$\int_G w^{-2/\sigma} \, dx < \infty$$  \hspace{1cm} (4.5)

We put $Y := L^p(G, w \, dx)$ with $p := \sigma + 2$ (i.e., $Y$ is the space of all measurable functions $u$ on $G$ such that $\|u\|_Y^p := \int_G |u|^p w \, dx < \infty$).

Following the procedure of Section 3, we then put $X := Y \cap W^{1,2}_0(G)$. As a general type of growth restriction of $f$ we assume:

(G2) There exist continuous functions $g_1: G \times \mathbb{R} \to [0, \infty]$ and $\Omega: [0, \infty] \to [0, \infty]$ such that $\Omega(0) = 0$ and

$$g(x, \eta) \leq w(x) |\eta|^\sigma + g_1(x, \eta)$$  \hspace{1cm} (4.6)

for every $x \in G$, $\eta \in \mathbb{R}$ as well as

$$\int_G g_1(x, h(x)) |u(x) v(x)| \, dx \leq \Omega(\|h\|_X) \|u\|_X \|v\|_X$$

for any $h, u, v \in X$. (Examples and sufficient criteria for (G2) are given in [3]).

Under these assumptions the nonlinear term in (4.1) defines an operator $F: X \to X^*$ in the obvious way, and conditions (S1), (S2), (N1)–(N4) are satisfied, as follows from arguments given in Section 5 of [3]. However, additional regularity assumptions are needed to ensure (S3), and various possibilities for this have been discussed in [3]. Let us single out the following hypotheses:

(R1) For some integer $m > n/4$ the domain $G$ is of class $C^{2m}$ (in the sense of Agmon [1, Sect. 9]), and the functions $P_{jk}$ ($j, k = 1, \ldots, n$) have bounded continuous derivatives in $G$ up to order $2m - 1$, and
(R2) the function equal to \( w \) on \( G \) and to 0 on \( \mathbb{R}^n - G \) is locally integrable. Thus it is clear from the material in [3] that Theorem 3.2 is applicable to problem (4.1), (4.2) if conditions (G1), (G2), (R1), (R2) are satisfied, and hence we have

**Theorem 4.1.** Consider problem (4.1), (4.2), and suppose the data satisfy (G1), (G2), (R1) and (R2). Then all assertions of Theorem 3.2 hold true for problem (4.1), (4.2) in place of (3.4).

Note that the solutions \( u_{ij} \) of (4.1), (4.2) given by this theorem lie in \( W^{1,2}(G) \) by construction. As indicated in [3], Hölder continuity of the data and an additional growth restriction on \( f \) permit the application of elliptic regularity theory, so that the \( u_{ij} \) turn out to be classical solutions in this case.

5. **The One-Dimensional Case**

Let us continue our discussion of boundary value problems by considering some special features of the case \( n = 1 \). The domain \( G \) is then an open interval, and we restrict our attention to \( G := ]0, \infty[ \), the generalizations to other intervals being obvious. The problem to be treated thus reads:

\[-(P(t)u')' + Q(t)u + f(t, u(t)) = \lambda u \quad (5.1)\]

for \( t > 0 \), and

\[u(0) = 0, \quad u \in L^2(0, \infty),\quad (5.2)\]

where \( Q \in C^0(\mathbb{R}) \), \( P \in C^1(\mathbb{R}) \) are bounded, \( P(t) \geq P_0 > 0 \) for all \( t \geq 0 \), and \( f \) can be written in the form (4.3) with a continuous \( g: G \times \mathbb{R} \rightarrow \mathbb{R} \) such that \( g(t, \cdot) \) is even for every \( t \geq 0 \).

Roughly speaking, the first main result of this section says that the assertions of Theorem 4.1 remain valid when the estimate (4.6) is required only locally near \( \eta = 0 \). To be specific, we retain assumption (G1), construct the spaces \( Y \) and \( X \) as in Section 4, and replace (G2) by the weaker assumption

\[(G2') \quad g(t, \eta) \leq w(t) \Omega_0(\eta|t|) + g_1(t, \eta) \text{ for every } (t, \eta) \in G \times \mathbb{R}, \text{ where } \Omega_0: [0, \infty[ \rightarrow [0, \infty[ \text{ is nondecreasing and such that } \limsup_{y \rightarrow 0^+} \Omega_0(y)y^{-\sigma} < \infty, \text{ and } g_1 \text{ is as in (G2)}.\]

Moreover, assumption (R1) is dropped, and (R2) boils down to

\[(R2') \quad \int_0^b w \, dt < \infty \text{ for } b > 0.\]

Using [3, Sect. 6], it is an easy matter to verify that conditions
(S1)-(S3), (N1)-(N4) hold for the problem under consideration, provided (G1), (G2') and (R2') are satisfied. Moreover, the solutions guaranteed by Theorem 3.2 are classical by a standard theorem of the Calculus of Variations. Thus we obtain

**Theorem 5.1.** Consider problem (5.1), (5.2), and suppose conditions (G1), (G2') and (R2') are satisfied. Then all assertions of Theorem 3.2 hold, and the \( u_{j'} \) are classical solutions of (5.1), (5.2).

**Remarks 5.2.** (a) Note that relation (3.7) implies \( u_{j'}(t) \to 0 \) as \( \lambda \to 0^+ \) uniformly on \([0, \infty[\) for every fixed \( j \). This follows from Proposition 6.1 of [3].

(b) In general condition (N4) fails when (G1) is replaced by a “local version” as was done in Section 6 of [3] and Section 5 of [8]. It is an interesting open question to what extent (G1) resp. (N4) can be relaxed in the unconstrained case.

Another special feature of the one-dimensional case is the possibility of investigating the nodal structure of the solutions, as was done for the constrained problem in [8]. To avoid new technicalities, we specialize to the equation considered in [8], i.e., we take \( P = 1 \) and assume that \( f \) and \( g \) are defined and continuous on all \( \bar{G} \times \mathbb{R} \), so that we are dealing with the differential equation

\[
-u'' + Q(t)u + f(t, u) = \lambda u
\]

on \( I := \bar{G} = [0, \infty[\). Accordingly, we replace (R2') by

\( (R2'') \) \( w \) has a continuous extension to \( I \).

Finally, we add the crucial hypothesis of “superlinearity”:

\( (SL) \) For every \( t \in I \) the function \( \eta \to \eta^{-1}f(t, \eta) \) is strictly increasing on \([0, \infty[\).

In the remainder of this section we consider a fixed \( \lambda > 0 \), and so we drop \( \lambda \) from the notations. Thus the functional corresponding to (5.3), (5.2) is given by

\[
J(u) := \frac{1}{2} \int_0^\infty (u'^2 + \lambda u^2) \, dt + \int_0^\infty Qu^2 \, dt + \int_0^\infty \int_0^\infty f(t, \eta) \, d\eta \, dt
\]

for \( u \in X \), and \((c_j)_{j \geq 1}\) denotes the sequence of its Ljusternik–Schnirelman levels. As in [8], we also consider a “dual” variational principle and define

\[
b_j := \sup_{Z \in \mathcal{J}_j} \inf_{u \in B(Z)} J(u) \quad (j \geq 1),
\]
where $\mathcal{Z}_j$ denotes the set of all finite increasing sequences

$$0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = \infty$$

with $m \leq j$, and, for every $Z \in \mathcal{Z}_j$, the set $B(Z) \subseteq X$ is given by

$$B(Z) := \{ u \in X \mid u(t) = 0 \text{ for all } t \in Z \}.$$ 

Note, however, that due to the absence of the constraint $\|u\| = \text{const}$ the minimization problem pertaining to a fixed partition $Z = \{t_0, t_1, \ldots, t_m \}$ can now be rewritten in the simpler form used by Coffman [6] and Hempel [10]. Specifically, if we define functionals $J^a_\nu$ for $0 \leq a < b \leq \infty$ by (5.4) with the integrals over $[0, \infty[$ replaced by integrals over $[a, b[$, then we evidently have

$$\inf_{u \in B(Z)} J(u) = \sum_{k=1}^m \inf_{u \in X_k} J^a_{\nu_k}(u),$$

where $X_k := W^{1,2}_0([t_{k-1}, t_k[) \cap Y (k = 1, \ldots, m)$, and the existence and uniqueness (up to sign) of the solutions of the minimization problem on each of the $[t_{k-1}, t_k[$ is well known (cf. [6, 9, 10] and the references therein). Also we know (e.g., from Theorem 5.1) that the minimizing solution on the unbounded interval $[t_{m-1}, t_m]$ is not the trivial one, and hence the minimum of $J$ on $B(Z)$ is certainly not attained at $u \equiv 0$.

Our main result on nodal properties is a complete analogue of Theorem 4.2 of [8]. It reads:

**Theorem 5.3.** Consider problem (5.3), (5.2), and suppose that hypotheses (G1), (R2") and (SL) are satisfied. Then, for every $j \in \mathbb{N}$ we have

(a) $b_j = c_j,$

(b) $c_{j+1} > c_j,$

(c) there exists a solution $u_j \in X - \{0\}$ of (5.3), (5.2) such that $J(u_j) = c_j$ and $u_j$ has precisely $j - 1$ different interior zeroes,

(d) if $u \in X - \{0\}$ is a solution of (5.3), (5.2) such that $J(u) > c_j$, then $u$ has at least $j$ different interior zeroes.

We do not give a detailed proof of this theorem, for the arguments leading up to the main result of [8] carry over to the present situation after trivial modifications. The essential point is to study the maps $U_x$, where, for every sign distribution $\alpha = (\alpha_1, \ldots, \alpha_j) \in \{ +1, -1 \}^j$ and every nondecreasing sequence $0 = t_0 \leq t_1 \leq \cdots \leq t_j = \infty$, the function $U_x(t_1, \ldots, t_{j-1}) \in X - \{0\}$ is defined by requiring that

$$U_x(t_1, \ldots, t_{j-1})|_{[t_{k-1}, t_k[} = x_k v_k \quad (1 \leq k \leq j)$$
whenever \( t_{k-1}, t_k \neq \emptyset \), with \( v_k \) being the unique nonnegative function in \( W^{1,2}_0([t_{k-1}, t_k]) \cap Y \) that minimizes \( J^v_{u_{k-1}} \). The most severe technical problem concerning the maps \( U_s \) is to prove their continuity with respect to the norm topology of \( X \). Such a continuity proof was given by Coffman in [6] for the case of a compact interval \( I \). For our present singular case, one may either generalize Coffman's proof or follow the procedure of [8], where similar maps corresponding to the constrained problem have been treated. (Note that Lemma 3.6 of [8] immediately yields an analogous lemma for \( J \) in place of \( \Psi \), and the proofs of Lemma 3.7 and Proposition 3.8 of [8] can be mimicked, using Lemma 3.1(b), (c) of the present paper along with the compact embedding \( X \rightarrow L^2(I) \) whenever the constraint \( \|u\| = R \) was employed in [8].) Since we know that Clark's version of Ljusternik–Schnirelman theory is available for the functional \( J \) on \( X \), it is also clear that the arguments from Section 4 of [8] may be repeated in the present situation, and hence Theorem 5.3 can be established.

**Remark 5.4.** For part (b) of Theorem 5.3 there is a very simple and elegant alternative proof based on an idea of Coffman [5]. One merely observes that the map \( u \rightarrow u'(0) \) is odd and continuous on the set \( K_j \) of critical points in \( J^{-1}(c_j) \), and that it has no zeroes on \( K_j \) due to the well-known uniqueness of solutions of the initial value problem for (5.3) with initial values \( u(0) = u'(0) = 0 \). Hence \( \gamma(K_j) = 1 \), and \( c_{j+1} = c_j \) would contradict the standard multiplicity theorem of Ljusternik–Schnirelman theory.

6. **APPLICATION TO BIFURCATION THEORY**

Let us now return to our original problem, i.e., the equation

\[-u'' + w(t)|u|^\sigma u = \dot{u}u\]  
(6.1)

on the interval \( I := [0, \infty[ \) together with the boundary conditions

\[u(0) = 0, u \in L^2(I).\]  
(6.2)

As in [9], we assume \( \sigma > 0 \), \( w \in C^1(I) \), \( w(t) > 0 \) for every \( t \in I \), and finally

\[(L) \quad w^{-1}w' \text{ is nondecreasing on } I, \text{ and}\]

\[(E) \quad w'(t_0) > 0 \text{ for some } t_0 \in I.\]

These hypotheses will be in force throughout this section.

It was already observed in [9] that these assumptions imply (4.5) (for \( G = \emptyset, \infty \)), and hence our problem satisfies the assumptions of Theorems 5.1 and 5.3.

In order to exploit these theorems for the study of the bifurcation
diagram of (6.1), (6.2) we have to introduce some new notation. In [9] Eq. (6.1) was considered together with the initial conditions

\[ u(0) = 0, \quad u'(0) = \xi \]  

(6.3)

for arbitrary \( \xi > 0 \), and the unique inextensible solution of this initial value problem was denoted by \( u(\cdot, \xi, \lambda) \). Let \( N(\xi, \lambda) \) be the number of distinct interior zeroes of \( u(\cdot, \xi, \lambda) \) and put

\[ \Omega_n(\lambda) := \sup \{ \xi > 0 \mid N(\xi, \lambda) \geq n \} \]

for \( n \geq 0 \). Finally, let \( Q(\lambda) \) be the set of \( \xi > 0 \) such that \( u(\cdot, \xi, \lambda) \) is defined on all of \( I \) and belongs to \( L^2(I) \) (and hence to \( X \), as was proved in [9]), and write

\[ Q_n(\lambda) := \{ \xi \in Q(\lambda) \mid N(\xi, \lambda) = n - 1 \} \]

for \( n \geq 1 \). Then it is clear from [9] that \( N(\xi, \lambda) \) is monotonically non-increasing and continuous from the right as a function of \( \xi > 0 \). Thus

\[ Q_n(\lambda) = Q(\lambda) \cap [\Omega_n(\lambda), \Omega_{n-1}(\lambda)] \]

and it was already remarked in [9] that \( Q_n(\lambda) \neq \emptyset \) iff \( \Omega_n(\lambda) < \Omega_{n-1}(\lambda) \) iff \( \Omega_n(\lambda) \in Q_n(\lambda) \). Hence Theorem 5.3 yields the following improvement of the results of [9, Sect. 3].

**Corollary 6.1.** For every \( \lambda > 0 \) and every \( n \in \mathbb{N} \) we have

(a) \( \Omega_n(\lambda) \in Q_n(\lambda) \),

(b) \( \Omega_n(\lambda) < \Omega_{n-1}(\lambda) \),

(c) \( Q_n(\lambda) \) has no cluster point in \([\Omega_n(\lambda), \Omega_{n-1}(\lambda)]\).

In particular the "preferred solution"

\[ u_{n,\lambda} := u(\cdot, \Omega_n(\lambda), \lambda) \]  

(6.4)

has precisely \( n - 1 \) distinct interior zeroes, and consequently the additional assumptions mentioned in Remark 3.9(b) and Theorem 4.2(b) of [9] are not actually needed.

The elementary shooting method from [9] yields a special system of solutions, namely the \( u_{n,\lambda} \) given by (6.4). On the other hand, the variational method yields the special solutions described by Theorem 5.3(c). Better insight in the bifurcation diagram of (6.1), (6.2) is gained by interrelating the two methods, and this can be done by studying \( J_{\lambda}(u(\cdot, \xi, \lambda)) \) as \( \xi \) ranges through \( Q(\lambda) \). In this direction we have the following result:
THEOREM 6.2. Let $n \in \mathbb{N}$, $\lambda > 0$, $\alpha, \beta \in Q(\lambda)$, and $\Omega_n(\lambda) \leq \alpha < \beta < \Omega_{n-1}(\lambda)$. Then $J_\lambda(u(\cdot, \alpha, \lambda)) > J_\lambda(u(\cdot, \beta, \lambda))$.

Before we embark on the rather lengthy proof of this theorem, let us explore its benefits. First of all, it yields a uniqueness result which shows that the two types of special solutions mentioned above are actually identical:

COROLLARY 6.3. Let $n \in \mathbb{N}$ and $\lambda > 0$ be arbitrary. Then the set of functions $v \in X$ such that $J_\lambda(v) = c_n(\lambda)$ and $v$ is a solution of (6.1), (6.2) having precisely $n - 1$ distinct interior zeroes consists exactly of $v = u_{n,\alpha}$ and $v = -u_{n,\alpha}$.

Proof. Let $S$ denote the set in question. We know from Theorem 5.3(c) that $S \neq \emptyset$. Now consider an arbitrary $v \in S$. Passing to $-v$ if necessary, we may assume that $\xi := v'(0) > 0$. Then $\xi \in Q_n(\lambda)$ by definition, and hence $\xi > \Omega_n(\lambda)$. Thus Theorem 6.2 implies

$$c_n(\lambda) = J_\lambda(v) \leq J_\lambda(u_{n,\alpha}) \leq c_n(\lambda),$$

the last inequality being due to Theorem 5.3(d) and the known fact that $N(\Omega_n(\lambda), \lambda) = n - 1$ (cf. Corollary 6.1). Hence $J_\lambda(u_{n,\alpha}) = c_n(\lambda)$, i.e., $u_{n,\alpha} \in S$.

Applying Theorem 6.2 once more, we see that $\xi = \Omega_n(\lambda)$ and hence $v = u_{n,\alpha}$. Thus we have $S = \{u_{n,\alpha}, -u_{n,\alpha}\}$, as asserted.

Since in many situations uniqueness implies continuous dependence on parameters, we may expect to obtain a continuous dependence result from Corollary 6.3. Such a result was proved in [9] (Theorem 4.2) under the additional assumption that $w' \geq 0$ on $I$. However, this assumption is not needed, as we shall now see:

COROLLARY 6.4. For every $n \in \mathbb{N}$, the map $\lambda \mapsto u_{n,\alpha}$ (with $u_{n,0} := 0$) is continuous from $[0, \infty[$ to $X$, and the map $\Omega_n : [0, \infty[ \to \Omega_n(\lambda)$ is also continuous.

Proof. The second assertion follows from the first. For, if $v$ is a solution of (6.1), (6.3) and $h \in C^1(I)$ is such that $h \equiv 1$ on $[0, \frac{1}{2}]$ and $h \equiv 0$ on $[1, \infty[$, then

$$\xi = -(v'(1) h(1) - v'(0) h(0)) = \int_0^1 (\lambda v - w |v|^q v) h \, dt - \int_0^1 v' h' \, dt,$$

from which our claim becomes evident by taking $v = u_{n,\alpha}$ and remembering that $X$ is continuously embedded in $W_0^{1,2}(I)$.

To prove the first assertion we only have to verify the assumptions of Proposition 3.3(b). Let us therefore consider a sequence $(\lambda_k)_k \geq 1$ such that
the limits \( \lambda_0 := \lim_{k \to \infty} \lambda_k \) and \( v := \lim_{k \to \infty} u_{n, \lambda_k} \) exist. In case \( \lambda_0 = 0 \) there is nothing to prove by (3.7). Hence suppose \( \lambda_0 > 0 \). Since the left-hand side (resp. the right-hand side) of (6.1) defines a continuous operator \( X \to X^* \) (resp. \( \mathbb{R} \times X \to X^* \)), it is clear that \( v \) is a weak solution of (6.1), (6.2) with \( \lambda = \lambda_0 \). But then it is also a classical solution by a standard regularity theorem (see, e.g., [7]), and hence \( v'(0) \neq 0 \). On the other hand, the convergence in the norm of \( X \) implies uniform convergence on \( I \) (cf. [3], Proposition 6.1). These two facts imply that \( v'(0) > 0 \) and that \( v \) has at most \( n - 1 \) distinct interior zeroes. However, since \( J_{\lambda_0}(u) \) depends continuously on \( (\lambda, u) \in \mathbb{R} \times X \), we have

\[
J_{\lambda_0}(v) = \lim_{k \to \infty} J_{\lambda_k}(u_{n, \lambda_k}) = \lim_{k \to \infty} c_n(\lambda_k) = c_n(\lambda_0),
\]

where we have also invoked Corollary 6.3 and Proposition 3.3(a), and therefore it follows from Theorem 5.3(b), (d) that \( v \) has at least \( n - 1 \) distinct interior zeroes. Thus it turns out that \( J_{\lambda_0}(v) = c_n(\lambda_0) \), that \( v'(0) > 0 \), and that the number of zeroes of the solution \( v \) is exactly \( n - 1 \). But then Corollary 6.3 tells us that necessarily \( v = u_{n, \lambda_0} \), which completes the proof.

The remainder of this section is devoted to proving Theorem 6.2. We begin with a purely integration-theoretic lemma, in which \( I := [0, \infty[ \), as before.

**Lemma 6.5.** Let \( L := \]a, b[ \) be a nonempty open interval, \( \tau: L \to I \) a function of class \( C^1 \), \( G := \{(\tau(\xi), \xi) \mid \xi \in L\} \), \( f: G \to \]0, \infty[ \) continuous, and suppose

(i) \( \lim_{\xi \to b^-} \tau(\xi) = +\infty \),

(ii) \( A := \int_a^b c(\xi) f(\tau(\xi), \xi) \, d\xi \) exists as an improper integral, and

(iii) if \( a < \xi_1 < \xi_2 < b \) and \( \tau(\xi_1) = \tau(\xi_2) = : t \), then \( f(t, \xi_1) < f(t, \xi_2) \).

Then it follows that \( A > 0 \).

**Proof.** The proof, although based on a very simple idea, is rendered quite technical by complications due to the fact that the set of critical points of \( \tau \) may be an arbitrary closed subset of \( L \). We divide the proof into several steps.

**Step 1.** Put \( E^\pm := \{\xi \in L \mid \pm \tau'(\xi) > 0\} \), \( N := \{\xi \in L \mid \tau'(\xi) = 0\} \) and \( C^\pm := E^\pm \cap \tau^{-1}(\tau(N)) \). We claim that the \( C^\pm \) are sets of measure zero. To see this, note first that \( \tau(C^+) \subseteq \tau(N) \) and that \( \tau(N) \) is a set of measure zero by Sard's theorem (see, e.g., [151]). Denoting the characteristic function of a set \( B \) by \( \chi_B \), as usual, we therefore have

\[
0 \leq \int \chi_{C^\pm} |\tau'| \, d\xi \leq \int \chi_{\tau(C^+)}(\tau(\xi)) |\tau'(\xi)| \, d\xi = \int \chi_{\tau(C^+)} \, dt = 0
\]
and hence \( \int \chi_{C^+} |\tau'| \, d\zeta = 0 \), which implies that the integrand vanishes almost everywhere. But since \( C^+ \subseteq E^\pm \), we have \( \tau'(\zeta) \neq 0 \) for every \( \zeta \in C^+ \). Hence \( \chi_{C^+} \) vanishes almost everywhere, which proves our claim.

**Step 2.** Now consider a compact subinterval \( L_0 = [x_0, \beta_0] \) of \( L \) such that

\[
\tau(\beta_0) = \max_{\zeta \in L_0} \tau(\zeta) \tag{6.5}
\]

and let us introduce the sets

\[
N_0 := L_0 \cap N, \quad E_0^+ := E^+ \cap L_0 \quad \text{and} \quad D^+ := E^\pm \cap [x_0, \beta_0[ - \tau^{-1}(\tau(N_0))].
\]

Since \( \tau' \) is continuous, \( N_0 \) is obviously compact, and the \( E^\pm \) are open. Moreover, it follows that \( \tau(N_0) \) is compact, hence \( \tau^{-1}(\tau(N_0)) \) is closed, and we finally see that the \( D^\pm \) are open in \( \mathbb{R} \). Furthermore, \( E_0^+ - D^+ \) is of measure zero, for \( E_0^+ - D^+ = E_0^+ \cap (\tau^{-1}(\tau(N_0)) \cup \{x_0, \beta_0\}) \subseteq C^\pm \cup \{x_0, \beta_0\} \), so that the result follows from Step 1.

**Step 3.** We construct an injective \( C^1 \)-map \( \Theta : D^- \to D^+ \) having the additional properties

\[
\tau(\Theta(\zeta)) = \tau(\zeta) \tag{6.6}
\]

and

\[
\Theta(\zeta) > \zeta \tag{6.7}
\]

for every \( \zeta \in D^- \). To this end, consider \( \zeta \in D^- \), put \( t := \tau(\zeta) \), \( t_1 := \tau(\beta_0) \), and note that \( t < t_1 \) by (6.5), because we have \( \zeta > x_0 \) and \( \tau'(\zeta) < 0 \) by definition of \( D^- \). Hence, again using \( \tau'(\zeta) < 0 \), we see that the set

\[
T(\zeta) := \tau^{-1}(t) \cap ]\zeta, \beta_0[.
\]

is nonempty and, in fact, has a minimum. Define

\[
\Theta(\zeta) := \min T(\zeta).
\]

From this definition (6.6) and (6.7) clearly follow, and moreover we obtain

\[
]\zeta, \Theta(\zeta)[ \cap \tau^{-1}(t) = \emptyset. \tag{6.8}
\]

This implies that \( \Theta \) is one-to-one. For, if \( \eta \in D^- \) is such that \( \Theta(\eta) = \Theta(\zeta) \), we may assume \( \eta > \zeta \) without loss of generality, and then we have \( \zeta < \eta < \Theta(\zeta) \) by (6.7) and \( \tau(\eta) = t \) by (6.6), which contradicts (6.8).

Next, we show that \( \zeta := \Theta(\zeta) \in D^+ \). We have \( x_0 < \zeta < \beta_0 \) by construction and \( \zeta \notin \tau^{-1}(\tau(N_0)) \) by (6.6). In particular, \( \zeta \notin N_0 \) and hence \( \tau'(\zeta) \neq 0 \). In
case \( \tau'(z) < 0 \), the continuity of \( \tau \) would imply that \( \tau \) attains the value \( t \) in \( [\xi, z[ \), which again contradicts (6.8). Therefore \( \tau'(z) > 0 \), hence \( z \in D^+ \).

Finally, we have to show that \( \Theta \) is continuously differentiable. Consider \( \xi_0 \in D^- \) and \( \zeta_0 := \Theta(\xi_0) \in D^+ \). Then we know that

\[
\tau(\xi_0) = \tau(\zeta_0), \quad \tau'(\zeta_0) \neq 0,
\]

and that \( D^+, D^- \) are open (cf. Step 2). Thus, by the implicit function theorem there exists an open connected neighborhood \( U \subseteq D \) of \( \xi_0 \) and a \( C^1 \)-function \( v \) which maps \( U \) onto an open connected neighborhood \( V \subseteq D^+ \) of \( \zeta_0 \) in such a way that \( v(\xi_0) = \zeta_0 \) and, for every \( \xi \in U \), the equation \( \tau(\xi) = \tau(\eta) \) has the unique solution \( \eta = v(\xi) \) in \( V \). Clearly we are done if we can exhibit an open neighborhood \( U_0 \subseteq U \) of \( \xi_0 \) on which \( \Theta \) agrees with the \( C^1 \)-function \( v \). For this, put

\[
s_0 := \tau(\xi_0),
\]

choose \( \xi_1 \in U, \zeta_1 \in V \) such that \( \xi_0 < \xi_1 \leq \zeta_1 < \zeta_0, \tau(\xi_1) < s_0 \) and \( \tau(\zeta_1) < s_0 \), and put

\[
s_1 := \max_{\zeta_1 < \zeta < \zeta_1} \tau(\zeta).
\]

Then \( s_1 < s_0 \) by (6.8), and hence

\[
U_0 := U \cap \tau^{-1}(\lbracket s_1, \infty \rbracket)
\]

is an open neighborhood of \( \xi_0 \). Now, for \( \xi \in U_0 \) and \( t := \tau(\xi) \), the equation \( \tau(\eta) = t \) has the unique solution \( \eta = \xi \) in \( U \) since \( t \) is strictly decreasing there, it has the unique solution \( \eta := v(\xi) \) in \( V \) by definition of \( v \), and it has no solution in \( [\xi_1, \zeta_1] \) since \( t > s_1 \).

On the other hand, the intervals \( U, [\xi_1, \zeta_1], V \) cover \( [\xi_0, \zeta_0] \). This implies that indeed

\[
v(\xi) - \min T(\xi) - \Theta(\xi)
\]

for every \( \xi \in U_0 \), as desired. Thus \( \Theta \) has in fact all the desired properties.

**Step 4.** The function \( \Theta \) just constructed will now be used to estimate integrals. We set

\[
g(\xi) := \tau'(\xi) f(\tau(\xi), \xi)
\]

for \( \xi \in L \). Now consider a closed subinterval \( L_1 = [\alpha_1, \beta_1] \) of \( L_0 \) such that \( \alpha_0 < \alpha_1 < \beta_1 < \beta_0 \) and

\[
\tau(\beta_0) > \tau(\beta_1) = \tau(\alpha_1) = \max_{\xi \in L_1} \tau(\xi).
\]

We claim that in this case we have

\[
\int_{\alpha_0}^{\beta_0} g \, d\xi > \int_{\alpha_1}^{\beta_1} g \, d\xi.
\]
In particular, if there is \( \zeta_0 \in L_0 \) such that \( \tau(\zeta_0) < \tau(\beta_0) \), then we may choose 
\[ \alpha_1 = \beta_1 = \zeta_0 \]
and obtain

\[ \int_{\zeta_0}^{\beta_0} g \, d\zeta > 0. \]  

(6.11)

To prove (6.10), consider the sets 
\[ M := L_0 - L_1, \quad D_1^+ := D^+ \cap M, \]
\[ E_1^+ := E^+ \cap M, \]
and the integrals

\[ A^+ := \int_{D_1^+} |g| \, d\zeta. \]

As is readily checked using (6.6) and (6.9), we have \( \Theta(D_1^-) \subseteq D_1^+ \). 
Moreover, \( E_1^+ - D_1^+ = (E_0^+ - D_0^+) \cap M \) is a set of measure zero (cf. Step 2). 
Finally, note that \( g \) vanishes on \( N \) by the definitions. Thus we have

\[ \int_{\zeta_0}^{\beta_1} g \, d\zeta - \int_{\zeta_1}^{\beta_1} g \, d\zeta = \int_M g \, d\zeta - \int_{E_1^+} g \, d\zeta = \int_{E_1^+} g \, d\zeta - \int_{E_1} g \, d\zeta = A^+ - A^- \]

and hence (6.10) is equivalent to \( A^+ > A^- \).

Since \( \int_{\beta_0}^{\beta_1} \tau'(\zeta) \, d\zeta = (\tau(\beta_0) - \tau(\beta_1)) > 0 \) by (6.9), it follows from Step 2 that 
\( D_1^+ \) has positive measure and hence \( A^+ > 0 \). This remark establishes (6.10) 
in the case that \( D_1^- = \emptyset \). However, if \( D_1^- \neq \emptyset \), then it has positive 
measure, because it is open. Moreover, (6.7) yields

\[ \tau'(\zeta) = \tau'(\Theta(\zeta)) \Theta'(\zeta) \]
on \( D \), and (6.6), (6.7) together with assumption (iii) imply

\[ f(\tau(\Theta(\zeta)), \Theta(\zeta)) > f(\tau(\zeta), \zeta) \]
on \( D \). Using these facts along with \( \Theta(D_1^-) \subseteq D_1^+ \) and the substitution rule, 
we obtain the following crucial estimate:

\[ A^+ \geq \int_{\Theta(D_1^-)} |\tau'(\zeta)| \, f(\tau(\zeta), \zeta) \, d\zeta \]

\[ = \int_{D_1^-} |\tau'(\Theta(\zeta))| \, f(\tau(\Theta(\zeta)), \Theta(\zeta)) \, |\Theta'(\zeta)| \, d\zeta \]

\[ = \int_{D_1^-} |\tau'(\zeta)| \, f(\tau(\Theta(\zeta)), \Theta(\zeta)) \, d\zeta \]

\[ > \int_{D_1^-} |\tau'(\zeta)| \, f(\tau(\zeta), \zeta) \, d\zeta = A^-, \]

which implies (6.10).
Step 5. To complete the proof, we have to consider two cases. First, suppose that $\tau$ remains bounded from above near $z$. Then pick $\gamma \in L$ and put
\[ t_k := k + \sup_{x < \xi \leq \gamma} \tau(\xi) \]
for all integers $k \geq 0$. By assumption (i) it makes sense to define
\[ \beta_k := \min\{\xi \geq \gamma | \tau(\xi) = t_k\} \]
($k \geq 0$), and the sequence $(\beta_k)$ is strictly increasing and tends to $\beta$ as $k \to \infty$, since $\tau$ remains bounded on compact subsets of $L$. Moreover, each interval $L_k := [\beta_{k-1}, \beta_k]$ ($k \geq 1$) satisfies (6.5) as well as
\[ \min_{\xi \in L_k} \tau(\xi) < \tau(\beta_k), \]
and hence Step 4 tells us that (6.11) is valid for every $L_k$. This clearly implies
\[ \int_{\beta_0}^{\infty} g \, d\xi = \sum_{k=1}^{\infty} \int_{\beta_{k-1}}^{\beta_k} g \, d\xi > 0 \]
by assumption (ii). Moreover, for $0 < \delta < \beta_0 - \alpha$ the interval $L_\delta := [\alpha + \delta, \beta_0]$ clearly satisfies (6.5), so either $\tau$ is constant on $L_\delta$ or (6.11) holds for $L_\delta$. In any case assumption (ii) implies
\[ \int_{\alpha}^{\beta_0} g \, d\xi = \lim_{\delta \to 0^+} \int_{\alpha + \delta}^{\beta_0} g \, d\xi \geq 0, \]
whence the assertion $A > 0$.

Secondly, suppose that $\tau$ is not bounded from above in any right-sided neighborhood of $z$. Again pick $\gamma \in L$ and define the sequence $(\beta_k)$ as before, but with $t_k := k + \tau(\gamma)$. Moreover, it now makes sense to define
\[ \alpha_k := \max\{\xi \leq \gamma | \tau(\xi) = t_k\}, \]
and the sequence $(\alpha_k)$ is strictly decreasing and tends to $\alpha$ as $k \to \infty$. Consider the increasing sequence of compact intervals $L_k := [\alpha_k, \beta_k]$ and the integrals
\[ A_k := \int_{\alpha_k}^{\beta_k} g \, d\xi \]
for $k \geq 1$. It is readily checked that
\[ t_k = \max_{\xi \in L_k} \tau(\xi) \]
for every $k$, and hence (6.9) is satisfied if we take $(L_k, L_{k+1})$ in place of $(L_0, L_1)$. Thus $A_{k+1} > A_k$ by (6.10). Moreover, for $k = 1$ we have (6.5), and $\tau(\gamma) < t_1 = \tau(\beta_1)$. Hence $A_1 > 0$ by (6.11), and now assumption (ii) yields

$$A = \lim_{k \to \infty} A_k > A_1 > 0,$$

as asserted. This completes the proof of Lemma 6.5.

Proof of Theorem 6.2. The parameter $\lambda > 0$ will remain fixed throughout the proof, and we shall henceforth omit it from all notations. In particular, we write $\omega_n := \Omega_n(\lambda)$. We give the proof for odd $n$, the case of even $n$ being completely analogous except for trivial sign changes. Also, we assume $n \geq 2$ since for $n = 1$ there is nothing to prove because of the well-known uniqueness of the positive solution of (6.1), (6.2) (cf. [9] and the references therein). Finally, $Q \cap [x, \beta]$ is finite by Corollary 6.1(c), and hence it is no loss of generality to assume that $x, \beta$ are two consecutive points of $Q$ in $[\omega_n, \omega_{n-1}]$, i.e.,

$$Q \cap [x, \beta] = \emptyset$$

(6.12)

which will be assumed in the sequel. Recall from [9] that the zeroes of $u(\cdot, \xi)$ can be written as an increasing sequence

$$0 < \rho_1(\xi) < \rho_2(\xi) < \cdots$$

with strictly increasing $C^1$-functions $\rho_k : ]0, \omega_k[ \to I$. Put $u_0 := u(\cdot, x)$ and $v_0 := u(\cdot, \beta)$.

We shall now construct a curve in $X$ which joins $u_0$ to $v_0$ and on which the behavior of the functional $J$ can be analysed. To this end, note first that $u(t, \xi) > 0$ for $\omega_n < \xi < \omega_{n-1}$ and $t > x_{n-1}(\xi)$ since $n$ is assumed to be odd. Moreover, we have $u_0(t) > v_0(t)$ for $t > x_{n-1}(\beta)$ (cf. [9, Lemma 3.8]). Hence, if $x < \rho < \beta$, it is impossible that $u(t, \xi) < u_0(t)$ for every $t > x_{n-1}(\xi)$, since $\xi \notin Q$ by (6.12): Thus there exists $t_0 > x_{n-1}(\xi)$ such that

$$u(t_0, \xi) = u_0(t_0)$$

(6.13)

and $t_0$ is uniquely determined by an elementary comparison theorem for Eq. (6.1) (cf. [8, Prop. 2.1]). Write $t_0 = \tau(\xi)$ and put

$$h(t, \xi) := \begin{cases} u(t, \xi) & \text{for } \tau(\xi) > t \geq 0, \\ u_0(t) & \text{for } t \geq \tau(\xi). \end{cases}$$

Since $u_0 \in X$, it is clear that $h(\cdot, \xi) \in X$ for every $\xi \in ]a, \beta[$, and hence we can define a function $G : ]a, \beta[ \to \mathbb{R}$ by

$$G(\xi) := J(h(\cdot, \xi)).$$
Next, let us try to differentiate $G$. We write

$$ u' := \frac{\partial u}{\partial t}, \quad z := \frac{\partial u}{\partial \xi}. $$

Recall from [9] that $z$ and $z' = \partial z/\partial t$ exist, that

$$ \frac{\partial u'}{\partial \xi} = z' \quad (6.14) $$

and that $z(\cdot, \xi)$ satisfies the linearized problem corresponding to (6.1), (6.3), so that in particular

$$ z(0, \xi) = 0 \quad (6.15) $$

for every $\xi > 0$. Now for $\alpha < \xi < \beta$ the implicit function theorem is evidently applicable to Eq. (6.13), because $u'(\tau(\xi), \xi) - u'(\tau(\xi)) \neq 0$ by the uniqueness of the solution of the initial value problem for Eq. (6.1). Hence $\tau \in C^1$, and we have the relation

$$ z(\tau(\xi), \xi) = (u'(\tau(\xi)) - u'(\tau(\xi), \xi)) \frac{dt}{d\xi} (\xi). \quad (6.16) $$

Furthermore, introducing the special data of Eq. (6.1) into (5.4) and taking into account the definition of $h$, we see that we can express $G$ in the form

$$ G(\xi) = F(\xi, \xi) + J(u_0) - F(\xi, \xi), \quad (6.17) $$

where $F$ is defined by

$$ F(t, \xi) := \int_0^t \left[ \frac{1}{2} (u'(s, \xi)^2 - \lambda u(s, \xi)^2) + \frac{w(s)}{p} |u(s, \xi)|^p \right] ds $$

for $\xi > 0, \ t \geq 0$. Clearly $F$ is of class $C^1$, and hence so is $G$. To compute the derivatives, note first that

$$ \frac{\partial}{\partial t} (F(t, \xi) - F(t, \alpha))\big|_{t = \tau(\xi)} = \frac{1}{2} (u'(\tau(\xi), \xi)^2 - u'(\tau(\xi))^2) \quad (6.18) $$

for $\alpha < \xi < \beta$ by (6.13). The differentiation with respect to $\xi$ can be performed under the integral sign using (6.14). We then eliminate $z'$ using integration by parts and (6.15). Noting that the remaining integral vanishes by (6.1), we finally obtain the simple result

$$ \frac{\partial F}{\partial \xi}(t, \xi) = u'(t, \xi) z(t, \xi), $$
and hence (6.16) implies
\[
\frac{\partial F}{\partial \xi} (\tau(\xi), \xi) = (u'u_0' - u'^2) \frac{dt}{d\xi} \quad (6.19)
\]
the arguments being the obvious ones. Combining (6.17), (6.18) and (6.19) (and again omitting the arguments \( \xi \) resp. \( t = \tau(\xi) \)), we obtain
\[
\frac{dG}{d\xi} = \frac{1}{2} (u'^2 - u_0'^2) \frac{dt}{d\xi} + (u'u_0' - u'^2) \frac{dt}{d\xi} = -\frac{1}{2} (u' - u_0')^2 \frac{dt}{d\xi}.
\]
Putting
\[
f(t, \xi) := (u'(t, \xi) - u_0(t))^2,
\]
we can rewrite this result in the form
\[
G(\xi_2) - G(\xi_1) = -\frac{1}{2} \int_{\xi_1}^{\xi_2} \tau'(\xi) f(\tau(\xi), \xi) \, d\xi \quad (6.20)
\]
for \( \alpha < \xi_1 < \xi_2 < \beta \).

It will be proved below that
\[
\lim_{\xi \to \beta^-} G(\xi) = J(u_0) \quad (6.21)
\]
and
\[
\lim_{\xi \to \alpha^+} G(\xi) = J(v_0) \quad (6.22)
\]
and hence, combining (6.20), (6.21) and (6.22), we see that Theorem 6.2 follows from Lemma 6.5 provided the assumptions of that lemma can be verified for the present functions \( f \) and \( \tau \).

In order to prove (6.21), (6.22) we consider the functional \( \Phi \) (notation as in Section 3) corresponding to the nonlinear term in (6.1), i.e.,
\[
\Phi(y) = \frac{1}{p} \int_{\xi_0}^{\xi} w |y|^p \, dt
\]
for \( y \in X \). If \( y \) is an \( L^2 \)-solution of (6.1), (6.3), then integration by part yields
\[
\int_{\xi_0}^{\xi} y'^2 \, dt + \int_{\xi_0}^{\xi} w |y|^p \, dt = \xi \int_{\xi_0}^{\xi} y^2 \, dt
\]
(cf. [9, Lemma 3.5]), and from this one infers
\[
J(y) = -\frac{\sigma}{2} \Phi(y) \quad (6.23)
\]
For \( h(\cdot, \xi) \) we can proceed in the same way, carrying out the integration by parts separately on the intervals \([0, \tau(\xi)]\) and \([\tau(\xi), \infty[\). The result is

\[
G(\xi) = \frac{1}{2} u_0(\tau(\xi))(u'(\tau(\xi), \xi) - u'_0(\tau(\xi))) - \frac{\sigma}{2} \Phi(h(\cdot, \xi)).
\] (6.24)

Now let \( \xi \) tend to \( x \). Then \( h(t, \xi) \to u_0(t) \) uniformly on compact subsets of \( I \) by the elementary theory of initial value problems, and 
\( 0 < h(t, \xi) \leq u_0(t) \) for every \( t > x_{n-1}(\beta), \xi \in ]x, \beta[, \) as we have seen earlier in the proof. Hence the Dominated Convergence Theorem yields \( \Phi(h(\cdot, \xi)) \to \Phi(u_0) \).

Clearly, then, (6.21) will follow from (6.23) and (6.24) once we have shown that

\[
\lim_{\xi \to x^+} (u'(\tau(\xi), \xi) - u'_0(\tau(\xi))) = 0,
\] (6.25)

because we know that \( u_0 \) is bounded.

To prove (6.25), consider the energy function \( E \) defined by

\[
E(t, \xi) := \frac{1}{2} u'(t, \xi)^2 + \frac{2}{p} u(t, \xi)^2 - \frac{w(t)}{p} |u(t, \xi)|^p.
\]

This function has already been studied in Section 3 of [9], and it was proved there that

\[
\frac{\partial E}{\partial t} = - \frac{1}{p} w' |u|^p.
\]

Thus, if \( y \) is an \( L^2 \)-solution of (6.1), (6.3), we have

\[
\frac{1}{2} y'(0)^2 = \int_0^\infty \frac{w'(t)}{p} |y(t)|^p dt,
\] (6.26)

since we know that \( E(t, \xi) \to 0 \), as \( t \to \infty \) whenever \( \xi \in Q \) (cf. [9, Lemma 3.5]). By definition the right-hand side of (6.26) is an improper integral, but assumptions (L) and (E) imply that \( w'(t) \geq 0 \) for all large \( t \), and hence the integral actually exists in the Lebesgue sense. A similar reasoning shows that we have

\[
\xi^2 - R(\xi) = \frac{2}{p} \int_0^\infty w'(t) |h(t, \xi)|^p dt
\] (6.27)

for \( x < \xi < \beta \), where

\[
R(\xi) := 2(E(\tau(\xi), \xi) - E(\tau(\xi), x)),
\]
and where the right-hand side is again a Lebesgue integral. Letting $\xi \to \alpha$ and invoking the Dominated Convergence Theorem as before, we thus conclude from (6.26), (6.27) that

$$\lim_{\xi \to \alpha} R(\xi) = 0. \quad (6.28)$$

Now let $\varepsilon > 0$ be given. Since $u_0'(t) \to 0$ as $t \to \infty$ ([9, Lemma 3.5]), we can choose $T > 0$ so large that $|u_0'(t)| < \varepsilon/3$ for every $t \geq T$. Moreover, by the standard theory of initial value problems $u'(\cdot, \xi)$ tends to $u_0'$ uniformly on $[0, T]$ as $\xi \to \alpha^+$. Thus choose $\delta_1 > 0$ such that

$$|u'(t, \xi) - u_0'(t)| < \varepsilon$$

for $\alpha < \xi < \alpha + \delta_1$ and $0 \leq t \leq T$. By (6.28) we can choose $\delta_2 > 0$ such that $|R(\xi)| < \varepsilon^2/8$ for $\alpha < \xi < \alpha + \delta_1$. Consider $\xi \in ]\alpha, \alpha + \min(\delta_1, \delta_2)[$. Then, if $\tau(\xi) \leq T$, the choice of $\delta_1$ shows that

$$|u'(\tau(\xi), \xi) - u_0'(\tau(\xi))| < \varepsilon.$$  \hspace{1cm} (6.29)

However, if $\tau(\xi) > T$, note that $R(\xi) = u'(\tau(\xi), \xi)^2 - u_0'((\tau(\xi))^2$ by (6.13), and hence the choices of $T$ and $\delta_2$ imply $|u_0'(\tau(\xi))| < \varepsilon/3$ as well as $|u'(\tau(\xi), \xi)| < \varepsilon/2$, because

$$u'(\tau(\xi), \xi)^2 \leq u_0'(\tau(\xi))^2 + |R(\xi)| < \frac{\varepsilon^2}{9} + \frac{\varepsilon^2}{8} < \frac{\varepsilon^2}{4}.$$  \hspace{1cm} (6.29)

Thus we again obtain (6.29), and (6.25) is established. This completes the proof of (6.21). The proof of (6.22) is similar. Note that $h(\cdot, \xi) \to v_0$ uniformly on compact subsets of $I$ as $\xi \to \beta^-$, and that $0 < v_0(t) < h(t, \xi) < u_0(t)$ for all large $t$, $\alpha < \xi < \beta$, so that (6.23), (6.24), (6.26), (6.27) and the Dominated Convergence Theorem can again be invoked. Clearly (6.20), (6.21) and (6.22) imply that the improper integral $A$ from Lemma 6.5 exists. The fact that

$$\lim_{\xi \to \beta^-} \sup_{0 < t < T} |h(t, \xi) - v_0(t)| = 0$$

for every $T > 0$ together with $v_0(t) < u_0(t)$ for $t \geq x_{n-1}(\beta)$ evidently implies $\tau(\xi) \to \infty$ as $\xi \to \beta^-$. Thus conditions (i) and (ii) from Lemma 6.5 hold true, and it remains to verify condition (iii).

Thus consider $\xi_1, \xi_2$ such that $\alpha < \xi_1 < \xi_2 < \beta$ and $\tau(\xi_1) = \tau(\xi_2) = t_1$. Clearly $u'(t_1, \xi_1) > u'_0(t_1)$ ($i = 1, 2$), and hence it follows from the definition of $f$ that we have to prove

$$u'(t_1, \xi_1) < u'(t_1, \xi_2). \quad (6.30)$$
Since both \( u(\cdot, \xi_1) \) and \( u(\cdot, \xi_2) \) are solutions of (6.1), it is clear that \( u'(t_1, \xi_1) \neq u'(t_1, \xi_2) \). Now suppose \( u'(t_1, \xi_1) > u'(t_1, \xi_2) \). Then \( u(t, \xi_1) < u(t, \xi_2) \) for \( t < t_1 \) close to \( t_1 \). On the other hand, \( x_{n-1}(\xi_1) < x_{n-1}(\xi_2) < \tau(\xi_2) = t_1 \) implies \( u(x_{n-1}(\xi_2), \xi_2) = 0 < u(x_{n-1}(\xi_2), \xi_1) \), and hence there exists \( t_0 \in \]x_{n-1}(\xi_2), \tau(\xi_2) [ \) such that \( u(t_0, \xi_1) = u(t_0, \xi_2) \). Thus on the interval \([t_0, t_1]\) the functions \( u(\cdot, \xi_i) \) \((i = 1, 2)\) are two positive solutions of (6.1) which agree at both endpoints. Hence they must agree on all of \([t_0, t_1]\) (cf. [8, Prop. 2.1]), which is absurd. Thus we have (6.30), and the proof of Theorem 6.2 is complete.

7. ON OSCILLATION AND UNIQUENESS

It was seen in [9] that condition (L) implies the following uniqueness property (holding for any \( \lambda > 0 \)):

(U) For every \( a \geq 0, b > a \), and every \( n \in \mathbb{N} \) Eq. (6.1) has at most one solution \( u \) such that \( u(a) = u(b) = 0, u'(a) > 0 \) and such that \( u \) has exactly \( n - 1 \) zeroes in \( ]a, b[ \).

Property (U) can be used to study the bifurcation diagrams corresponding to boundary value problems associated with the differential equation, and this has been done in several cases (cf. [9] for details and references).

In the present section we still consider (6.1), but we drop assumptions (E) and (L). Instead, we assume conditions that ensure the applicability of the variational techniques from Sections 3 and 5 in a natural way. We first derive a nonoscillation result directly from condition (U). This result clearly applies when (L) and (E) are satisfied.

**Theorem 7.1.** Consider Eq. (6.1) with \( \sigma > 0, \lambda > 0, w \) positive, continuous and satisfying (4.5) (with \( G = ]0, \infty[) \). Moreover, suppose \( w(t) \geq w_0 > 0 \) for every \( t \geq 0 \). If in addition (U) holds, then there is no nontrivial solution of (6.1) having infinitely many zeroes.

**Proof.** Since \( \lambda \) is fixed, we again suppress it in the notations. Let us first examine some consequences of condition (U). We again denote the solution of (6.1), (6.3) by \( u(\cdot, \xi) \), and we denote by \( A_n \) the set of \( \xi > 0 \) such that \( u(t, \xi) \) has at least \( n \) zeroes for \( t > 0 \). Then for every \( n \in \mathbb{N} \) we have \( A_n \neq \emptyset \), and there exists a unique \( C^1 \)-function \( x_n: A_n \rightarrow ]0, \infty[ \) such that \( x_n(\xi) \) is just the \( n \)th interior zero of \( u(\cdot, \xi) \) \((\xi \in A_n)\). These facts are proved in [9] without using hypothesis (L) (cf. Remark 2.10c there). Now condition (U) implies that the \( x_n \) are one-to-one, and hence \( x_n \) must be either strictly increasing or strictly decreasing on each connected component of
the open set $A_n$. However, the latter is impossible. For, suppose that for some $n \in \mathbb{N}$ the function $x_n$ is decreasing on some connected component of $A_n$. Then we can continue $x_1(\xi), x_2(\xi), \ldots, x_n(\xi)$ to arbitrarily large values of $\xi$, which yields a contradiction to the a priori estimate for $u'$ given in Lemma 2.3 of [9]. (A detailed account of this argument is contained in the proof of Theorem 2.1 in [9].) Hence $x_n$ is strictly increasing on all of $A_n$, and it follows (again as in [9]) that the $x_k$ ($1 \leq k \leq n$) can be extended down to $\xi = 0$, i.e. there exists $\omega_n > 0$ such that

$$A_n = ]0, \omega_n[.$$ 

Moreover, $\omega_n < \infty$ because it is assumed that $w$ is bounded away from zero on $I$ (cf. [9], Remark 2.10b)).

In addition, our assumptions imply that Theorems 5.1 and 5.3 apply to the present problem, and hence we again obtain assertions (a) and (b) of Corollary 6.1. In particular, if $N(\xi)$ denotes the number of distinct interior zeroes of $u(\cdot, \xi)$, then $N$ is decreasing, right continuous, and $N(\Omega) = \mathbb{N}$, as before (notation from Section 6).

Now observe that in order to prove Theorem 7.1 it suffices to show that every nontrivial solution of (6.1), (6.3) has only a finite number of zeroes. For, if $y \neq 0$ is an arbitrary solution of (6.1) having an infinite number of zeroes and if $t_0$ is its first zero, then $y_0(t) := y(t - t_0)$ satisfies a differential equation of the form (6.1), with $w$ replaced by a translate of $w$, and the assumptions of Theorem 7.1 evidently hold for this new equation. Passing to $-y_0$ if necessary, we can also arrange $y'_0(0) > 0$, so that $y_0$ indeed satisfies (6.3).

Assume therefore that there is $\xi_0 > 0$ such that $N(\xi_0) = \infty$. Let $\omega_x \geq 0$ be the limit of the decreasing sequence $(\omega_n)_{n \geq 1}$. Since the function $N$ is decreasing on $]0, \infty[$, we must have $\omega_n > \xi_0$ for every $n$ and hence $\omega_x \geq \xi_0 > 0$. Moreover, $N(\omega_x) = \infty$ since $N$ is decreasing and attains arbitrarily large values on $]0, \infty[$. Put $v := u(\cdot, \omega_x)$. Then $v \in X$, for we know that any solution of (6.1), (6.3) having infinitely many zeroes must belong to $X$ (cf. [9], Remark 3.9(c) and Lemma 3.5).

By Theorem 5.3(c) there exist solutions $v_n$ of (6.1), (6.2) such that $J(v_n) = c_n$ and $N(\xi_n) = n - 1$ for $\xi_n := v_n'(0)$ ($n \in \mathbb{N}$). Thus $\xi_n \in \Omega_n$ and in particular, $\omega_n \leq \xi_n < \omega_{n-1}$ for all $n$. Hence $\xi_n \to \omega_x$ as $n \to \infty$. Now we can apply Lemma 4.3 from [9], which tells us that $v = \lim_{n \to \infty} v_n$ strongly in $X$. Using (6.23) and the continuity of $\Phi$ on $X$ (cf. Section 3), we infer

$$0 < \frac{\sigma}{2} \Phi(v) = \lim_{n \to \infty} \frac{\sigma}{2} \Phi(v_n) = - \lim_{n \to \infty} J(v_n) = - \lim_{n \to \infty} c_n.$$ 

On the other hand, Theorem 5.1 tells us that relation (3.8) is valid. This contradiction proves the theorem.
Remarks 7.2. (a) If we modify condition (U), requiring the uniqueness of solutions with prescribed number of zeroes not for every interval \([a, b] \subseteq I\), but only for intervals of the form \([0, b]\), then the proof of Theorem 7.1 still goes through except for the argument involving the translated equation. The result then is that every nontrivial solution \(y\) of (6.1) such that \(y(0) = 0\) has only a finite number of zeroes, if (6.1) satisfies the modified condition (U) along with the other assumptions of Theorem 7.1.

(b) The proof of the nonoscillation result 7.1 is undesirably involved, considering that results from abstract critical point theory such as (3.8) and Theorem 5.3(c) are combined with the elementary shooting method from [9]. It is thus to be hoped that a simpler and more direct proof exists. However, it should be noted that neither the requirement (4.5)—which ensures the applicability of critical point theory—nor condition (U)—which ensures the applicability of the shooting method—can be deleted from the assumptions of Theorem 7.1. As to (4.5), this is seen from the case where \(w\) is a positive constant. This case has been discussed in [8] (see Corollary 3.5 therein) and [9] (see Remarks 2.10(c)–(e) therein). An example in which oscillating solutions occur even though (4.5) is satisfied is given below.

We need a technical lemma for the example mentioned in the preceding remark.

**Lemma 7.3.** Let \(a < b\), \(\sigma > 0\), \(\lambda > 0\), and let \(\hat{w} : [a, \infty[ \to \mathbb{R}\) be continuous. Suppose the Dirichlet problem

\[
\begin{align*}
    u'' &= (\hat{w}(t) |u|^\sigma - \lambda)u, \\
    u(a) &= u(b) = 0
\end{align*}
\]

has a positive solution \(\tilde{u}\) on \([a, b]\). Moreover, let \(\varepsilon > 0\) and \(r > 0\) be given. Then there exists a continuous function \(v : [a, \infty[ \to \mathbb{R}\) such that we have

1. \(w(a) = r\) and \(\min_{a \leq t \leq c} w(t) \geq \min(r, \min_{a \leq t \leq c} \hat{w}(t))\) for every \(c \geq a\).
2. \(
\int_a^B \left| w^{-2/\sigma} - \hat{w}^{-2/\sigma} \right| dt < \varepsilon
\)
   for every \(B \geq b\).
3. The solution \(u\) of the initial value problem

\[
\begin{align*}
    u'' &= (w |u|^\sigma - \lambda)u, \\
    u(a) &= 0, \quad u'(a) = \tilde{v}(a)
\end{align*}
\]

is defined on \([a, b + \varepsilon']\) for some \(\varepsilon' \in \]0, \varepsilon]\), and there is a zero \(b_1 > a\) of \(u\) in \([b - \varepsilon, b + \varepsilon][\).
(iv) \( u(t) > 0 \) for \( a < t < b_1 \), and

\[
|u'(b_1) - \hat{u}'(b)| < \varepsilon.
\]

**Proof.** Decreasing \( \varepsilon \) if necessary, we may assume that the solution \( \hat{u} \) of (7.1) can be continued to a half-open interval containing \([a, b + \varepsilon]\). Let \( M \geq r \) (resp. \( K \)) be a bound for \( \hat{w} \) (resp. for \( \hat{u} \)) on \([a, b]\). It follows from standard arguments (e.g., successive approximations) that there exists \( \delta_0 > 0 \) depending only on a bound for \( w \), such that the local solution \( x \) of (7.3), (7.4) exists and satisfies

\[
|x| \leq 1, \quad |x'| \leq |\hat{u}'(a)| + 1
\]

on all of \([a, a + \delta_0]\). Choose \( \delta_0 \in ]0, b - a]\) accordingly (with respect to the bound \( M \)), and define \( w_\delta : [a, \infty[ \rightarrow \mathbb{R} \) for \( 0 < \delta \leq \delta_0 \) by requiring

\[
w_\delta(t) := \left(1 - \frac{t - a}{\delta}\right)r + \frac{t - a}{\delta} \hat{w}(a + \delta)
\]

for \( a \leq t \leq a + \delta \) and \( w_\delta \equiv \hat{w} \) on \([a, \infty[ \). Then \( w_\delta \) is continuous and bounded by \( M \) for every \( \delta \). Let \( x_\delta : [a, a + \delta_0] \rightarrow \mathbb{R} \) be the local solution of (7.3), (7.4) corresponding to \( w = w_\delta \). From (7.1), (7.2) and Taylor's formula we get

\[
\hat{u}(t) = \hat{u}'(a)(t - a) + \int_a^t \left[ (\hat{w}(s) |\hat{u}'(s)|^\alpha - \lambda) \hat{u}(s) \right](t - s) \, ds
\]

and (7.3), (7.4) yield a similar relation for \( x_\delta \). Hence, using our bounds for \( \hat{w}, w_\delta, \hat{u} \) and \( x_\delta \) in \([a, a + \delta_0]\), we obtain

\[
|x_\delta(a + \delta) - \hat{u}(a + \delta)| \leq \frac{\delta^2}{2} (MK^\alpha + \lambda K + M + \lambda) = \frac{\delta^2}{2} M_1.
\]

Similarly it is shown that

\[
|x_\delta'(a + \delta) - \hat{u}'(a + \delta)| \leq \delta M_1.
\]

Thus we have

\[
\lim_{\delta \to 0^+} x_\delta(a + \delta) = \hat{u}(a) \quad (7.6)
\]

\[
\lim_{\delta \to 0^+} x_\delta'(a + \delta) = \hat{u}'(a). \quad (7.7)
\]

Since \( w_\delta(t) = \hat{w}(t) \) for \( t \geq a + \delta \), we can extend \( x_\delta \) to a larger interval by solving the initial value problem for (7.1) with initial conditions \( x_\delta(a + \delta) \), \( x_\delta'(a + \delta) \) at \( t = a + \delta \). In this way we obtain a solution \( u_\delta \) of (7.3), (7.4) for
Now clearly (i) is satisfied for all \( \delta \), and (7.6), (7.7) and the standard theorem on continuous dependence on the initial conditions ensure that (iii), (iv) are satisfied for all sufficiently small \( \delta \). Finally, for any \( B \geq h \) we have

\[
\int_a^B |\tilde{w}^{-2/\alpha} - w_\delta^{-2/\alpha}| \, dt = \int_a^\alpha |\cdots| \, dt \leq 2\delta m^{-2/\alpha},
\]

where \( 0 < m \leq \min(r, \min_{a < t < a + \delta_0} \tilde{w}(t)) \).

This shows that (ii) is also satisfied for every sufficiently small \( \delta \). Thus the lemma is proved, taking \( w = w_\delta, u = u_\delta \) for \( \delta > 0 \) suitably small.

**Example 7.4.** Let \( \sigma > 0 \) be arbitrary, \( I := [0, \infty[ \), and let \( w_1 \) be a \( C^1 \)-function on \( I \) such that \( w_1(0) > 0 \) and \( w_1'(t) > 0 \) for every \( t \geq 0 \). As is well known (see, e.g., [9] or [10]), the Dirichlet problem

\[
-u'' + w_1(t) |u|^{\sigma} u = \lambda u,
\]

\[ u(0) = u(1) = 0 \]

has a unique positive solution \( u_1 \) provided \( \lambda \) is sufficiently large. Pick a fixed \( \lambda \) for which \( u_1 \) exists, and consider the energy function

\[
E := \frac{1}{2}(u_1'^2 + \lambda u_1^2) - \frac{1}{p} w_1 |u_1|^{p'}
\]

on \( I \) (where \( p := \sigma + 2 \), as always). We have \( E' = -(1/p) w_1' |u_1|^{p'} \), which implies

\[
\xi_0^2 - \xi_1^2 = 2E(0) - 2E(1) = -\int_0^1 w_1' |u_1|^{p'} \, dt > 0,
\]

where we have put \( \xi_k := u_1'(k) \) (\( k = 0, 1 \)). Hence \( \beta_1 := |\xi_1/\xi_0| < 1 \), and we pick \( q \) such that \( \beta_1 < q < 1 \).

Next, we construct an increasing sequence \((b_k)_{k \geq 1}\) of points of \( I \) and sequences of functions \((w_k)_{k \geq 1}, (u_k)_{k \geq 1}\) such that \( u_k \) and \( w_k \) are defined on \([b_{k-1}, b_k]\), \((-1)^{k-1} u_k\) is the positive solution of the Dirichlet problem

\[
-u'' + w_k(t) |u|^{\sigma} u = \lambda u \quad (7.8)
\]

\[ u(b_{k-1}) = u(b_k) = 0 \quad (7.9)\]

and such that for \( \xi_k := u_k'(b_k), \beta_k := |\xi_k/\xi_0| \) we have

\[
\beta_k < q^k \quad (7.10)
\]

for every \( k \geq 1 \). We start with \( b_0 := 0, b_1 := 1 \), and the \( u_1, w_1 \) already described. Now consider \( k \geq 2 \) and suppose \( b_0, \ldots, b_{k-1}, u_1, \ldots, u_{k-1} \) and
$w_1, ..., w_{k-1}$ are already constructed. Then put $\bar{h}_k := b_k - 1 + 1$ and define $\tilde{u}_k, \tilde{w}_k$ by

$$\tilde{u}_k(t) := \beta_{k-1} u_1(t - b_{k-1}) \quad (b_{k-1} \leq t \leq \bar{h}_k),$$
$$\tilde{w}_k(t) := \beta_{k-1} w_1(t - b_{k-1}) \quad (t \geq b_{k-1}).$$

These functions clearly satisfy (7.1), (7.2) on the interval $[b_{k-1}, \bar{h}_k]$. Thus, taking $r := w_{k-1}(b_{k-1})$ in Lemma 7.3, we can find a point $b_k \in [\bar{h}_k - \frac{1}{2}, \bar{h}_k + \frac{1}{2}]$ and functions $u_k, w_k$ such that $(-1)^k u_k$ is the positive solution of (7.8), (7.9) and such that $b_k, u_k, w_k$ have the following additional properties:

$$w_k(b_{k-1}) = w_{k-1}(b_{k-1})$$  \hspace{1cm} (7.11)

$$\int_{b_{k-1}}^{b_k} |w_k - \tilde{w}_k| \, dt < 2^{-k}$$  \hspace{1cm} (7.12)

$$u_k(b_k) = 0 = u_k(b_{k-1})$$  \hspace{1cm} (7.13)

$$|u_k'(b_{k-1})| = |\tilde{w}_{k-1}|$$  \hspace{1cm} (7.14)

the last relation being due to (7.4) and $|\tilde{u}_k(b_{k-1})| = \beta_{k-1} \tilde{w}_k = |\tilde{w}_{k-1}|$. By construction the signs of $\tilde{w}_{k-1}$ and $u_k'(b_{k-1})$ also agree, and hence (7.14) implies

$$u_k'(b_{k-1}) = u_{k-1}'(b_{k-1}).$$  \hspace{1cm} (7.15)

Moreover, the induction hypothesis implies

$$|\tilde{u}_k(\bar{h}_k)| = \beta_{k-1} |\tilde{w}_k| < q^{k-1} |\tilde{w}_k| = q^{k-1} \beta_1 \xi_0 < q^k \xi_0,$$

and hence by Lemma 7.3(iv) everything may even be arranged so as to have (7.10) again. This completes the construction.

From the choice of the $\bar{h}_k, b_k$ it is clear that $k/2 \leq b_k \leq (3/2)k$ for every $k \geq 1$. Hence the intervals $[b_{k-1}, b_k]$ cover $I$, and therefore it follows from (7.11), (7.13) and (7.15) that the $w_k$ (resp. the $u_k$) are the restrictions to $[b_{k-1}, b_k]$ of a unique function $w$ (resp. $u$) defined on $I$. It is also clear that $w$ is continuous and that $u$ is a solution of (6.1), (6.3) for this $w$. Moreover, the zeroes of $u$ are the points $b_k$ ($k \in \mathbb{N}$). Finally, (7.10) and (7.12) yield

$$\int_{b_{k-1}}^{b_k} w^{-2/\sigma} \, dt \leq \int_{b_{k-1}}^{b_k} \tilde{w}_k^{-2/\sigma} \, dt + \int_{b_{k-1}}^{b_k} |w_k^{-2/\sigma} - \tilde{w}_k^{-2/\sigma}| \, dt$$

$$< \int_{b_k}^{b_k + 3/2} \tilde{w}_k^{-2/\sigma} \, dt + 2^{-k} = \beta_{k-1}^2 \int_0^{3/2} w_1^{-2/\sigma} \, dt + 2^{-k}$$

$$< q^{2(k-1)} \int_0^{3/2} w_1^{-2/\sigma} \, dt + 2^{-k}.$$
and hence

$$\int_0^\infty w^{-2/\sigma} \, dt < \sum_{k=1}^\infty q^{2k-2} \int_0^{2/2} w^{-2/\sigma} \, dt + \sum_{k=1}^\infty 2^{-k} < \infty$$

because of $q < 1$. This means that (4.5) indeed holds for the present choice of $w$.

**Remarks 7.5.** (a) In Lemma 7.3 the piecewise linear interpolation (7.5) can obviously be replaced by a smooth interpolation. Starting with some $w_1 \in C^\infty$, we then obtain an example of the kind just described, but with $w \in C^\infty$.

(b) It is clear from Remark 7.2(a) that the equation constructed in Example 7.4 cannot satisfy the modified condition (U). (Note that the construction yields $w(t) > w_1(0) > 0$ for all $t > 0$ because of $\beta_k < 1$ and (i) in Lemma 7.3.) In other words, for some $b > 0$ and $n \geq 2$ there exist two different solutions which vanish for $t = 0$ and $t = b$, have $n - 1$ distinct zeroes for $0 < t < b$, and have positive slope for $t = 0$. It is clear from the uniqueness of nonvanishing solutions that the zeroes of two such solutions must have different locations. However, an explicit example for this phenomenon does not seem to be available for Eq. (6.1) as yet.

**REFERENCES**


