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On Antipodes in Pointed Hopf Algebras

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If S is the antipode of a Hopf algebra H, the *order* of S is defined to be the smallest positive integer n such that $S^n = I$ (in case such integers exist) or ∞ (if no such integers exist). Although in most familiar examples of Hopf algebras the antipode has order 1 or 2, examples are known of infinite dimensional Hopf algebras in which the antipode has infinite order or arbitrary even order [1, 4, 6] and also of finite dimensional Hopf algebras in which the antipode has arbitrary even order [3, 5]. Some sufficient conditions for the antipode to have order $\leqslant 4$ are known [2, 4], but the following questions remain open: Does the antipode of a finite dimensional Hopf algebra necessarily have finite order? If the antipode S of a Hopf algebra H has finite order is that order bounded by some function of dim H?

In this paper, by constructing a certain basis for an arbitrary pointed coalgebra and studying the action of the antipode on the elements of such a basis for a pointed Hopf algebra, we obtain affirmative answers to the second question in case H is pointed and to the first question in case H is pointed over a field of prime characteristic.

We use freely the definitions, notation, and results of [4].

1. STATEMENT OF RESULTS

Let C be a pointed coalgebra with comultiplication Δ and counit ϵ over an arbitrary field Φ. Let $G(C)$ be the set of group-like elements and let $(0) = C_{-1} \subseteq C_0 = \Phi G(C) \subseteq C_1 \subseteq \cdots$ be the coradical filtration. For each $i \geqslant 1$ choose a subspace $K_i \subseteq C_i$ such that $C_{i-1} \oplus K_i = C_i$. If $a, b \in G(C)$ define

$$K_{i,a,b} = \{k \in K_i \mid \Delta(k) - k \otimes a - b \otimes k \in C_{i-1} \otimes C_{i-1}\}.$$

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PROPOSITION 1. *$K_i = \sum_{a,b \in G(C)} K_{i,a,b}$ for all $i \geqslant 1$.*

If $a, b \in G(C)$ define $C_{1,a,b} = \{k \in C_1 \mid \Delta(k) = k \otimes a + b \otimes k\}$.

PROPOSITION 2. *$C_1 = C_0 \oplus \sum_{a,b \in G(C)} C_{1,a,b}$.*

We apply Proposition 2 to the case of a pointed Hopf algebra H to obtain:

PROPOSITION 3. *If $G(H)$ has exponent e then $(S^{2e} - I)(H_1) = (0)$.*

The following proposition allows us to extend the above result to higher terms in the coradical filtration.

PROPOSITION 4. *Let $i \geqslant 1$ and let φ be a homomorphism of C such that $(\varphi - I)(C_j) \subseteq C_{j-1}$ for all j, $0 \leqslant j \leqslant i$. Then $(\varphi - I)(C_{i+1}) \subseteq C_i$.*

Propositions 3 and 4 lead to our main result.

THEOREM 5. *Let H be a pointed Hopf algebra over an arbitrary field. If $H = H_n$ and if $G(H)$ has exponent e then $(S^{2e} - I)^n = 0$.*

Finally, the following corollary provides partial answers to the questions raised in the introduction.

COROLLARY 6. *Let H be a finite dimensional pointed Hopf algebra with antipode S over a field Φ. Assume $G(H)$ has exponent e and that $H = H_n$. If Φ has characteristic 0 and S has finite order then the order of S divides $2e$. If Φ has characteristic p then $S^{2ep^m} = I$, where $p^m \geqslant n > p^{m-1}$.*

(Note that n and e are both $\leqslant \dim H$, so the corollary does give a bound (in terms of $\dim H$) on the order of S.)

2. ON THE CORADICAL FILTRATION

In this section we will prove Propositions 1, 2, and 4.

Let η denote the projection of $C_i = C_{i-1} \oplus K_i$ onto K_i. Note that for $x \in C_i$ we have

$$(\eta \otimes I)\, \Delta\eta(x) = (\eta \otimes I)\, \Delta(x), \tag{1'}$$

and

$$(I \otimes \eta)\, \Delta\eta(x) = (I \otimes \eta)\, \Delta(x). \tag{1}$$

Then $\rho_R = (\eta \otimes I)\Delta$ (respectively $\rho_L = (I \otimes \eta)\Delta$) gives K_i the structure of a right (respectively left) C_0-comodule. (For using (1) we see that

$$(I \otimes \Delta)\rho_R = (\eta \otimes I \otimes I)(\Delta \otimes I)\Delta = (\rho_R \otimes I)\rho_R,$$

and clearly $(I \otimes \epsilon)\rho_R = I$.) Furthermore, as $(\eta \otimes \eta)\Delta C_i = (0)$ we have

$$(\Delta - \rho_R - \rho_L)(K_i) \subseteq \ker(I \otimes \eta) \cap \ker(\eta \otimes I) = C_{i-1} \otimes C_{i-1}.$$

Hence for a and $b \in G(C)$ we have

$$K_{i,a,b} = \{k \in K_i \mid \rho_R(k) = k \otimes a, \rho_L(k) = b \otimes k\}. \tag{2}$$

Now if $c \in K_i$ then c generates a finite dimensional subcoalgebra $D \subseteq C_i$. We claim that $\rho_R(\eta(D)) \subseteq \eta(D) \otimes D_0$ and hence that $\eta(D)$ is a (finite dimensional) right D_0-comodule with structure map ρ_R (and similarly a left D_0-comodule with structure map ρ_L). As $\rho_R\eta(x) = \rho_R(x)$ for all $x \in C_i$ (by (1)) it suffices to show that $\rho_R(D) \subseteq \eta(D) \otimes D_0$. But

$$\rho_R(D) \subseteq (\eta(D) \otimes D) \cap (K_i \otimes C_0).$$

As $\eta(D) \subseteq K_i$ and $D \cap C_0 = D_0$ the desired result follows. Let $G(D) = \{g_j \mid 1 \leqslant j \leqslant n\}$. Then the dual basis $\{g_j^* \mid 1 \leqslant j \leqslant n\}$ for D_0^* is a set of orthogonal idempotents and $\sum_{j=1}^n g_j^* = \epsilon$, the identity of D_0^*.

Using (1) and (1′) we see that

$$(\rho_L \otimes I)\rho_R = (I \otimes \eta \otimes I)(\Delta \otimes I)\Delta = (I \otimes \rho_R)\rho_L.$$

From this it follows that if, for a and $b \in D_0^*$ and $q \in \eta(D)$, we define

$$a \cdot q = (I \otimes a)\rho_R(q)$$

and

$$q \cdot b = (b \otimes I)\rho_L(q),$$

then

$$(a \cdot q) \cdot b = (b \otimes I \otimes a)(\rho_L \otimes I)\rho_R = (b \otimes I \otimes a)(I \otimes \rho_R)\rho_L = a \cdot (q \cdot b).$$

Hence $\eta(D)$ has the structure of a D_0^*-bimodule. It also follows that

$$\rho_L(q) = \sum_{j=1}^n g_j \otimes (q \cdot g_j^*)$$

and

$$\rho_R(q) = \sum_{j=1}^n (g_j^* \cdot q) \otimes g_j.$$

It is then clear by (2) that $g_j^* \cdot \eta(D) \cdot g_l^* \subseteq K_{i,g_j,g_l}$. Hence

$$c \in \eta(D) = \sum_{j,l=1}^{n} g_j^* \cdot \eta(D) \cdot g_l^* \subseteq \sum_{a,b \in G(C)} K_{i,a,b}$$

(where the equality follows from the well known representation theory for direct sums of fields). This completes the proof of Proposition 1.

We now prove Proposition 2. Let $c \in C_1$. Then c generates a finite dimensional subcoalgebra D. It is clear that $D_0 = C_0 \cap D$ and $C_1 \cap D \subseteq D_1$. In fact, $C_1 \cap D = D_1$. To see this let $x \in C_1 \cap D$. Then

$$\Delta(x) \in (D \otimes D) \cap (C \otimes C_0 + C_0 \otimes C).$$

Now we can find subspaces $A_i \subseteq C$, $1 \leqslant i \leqslant 4$, such that $A_1 = D_0$, $A_1 \oplus A_2 = D$, $A_1 \oplus A_3 = C_0$, and $\oplus_{i=1}^{4} A_i = C$. Then $C \otimes C = \oplus_{i,j=1}^{4} (A_i \otimes A_j)$. Now as $D \otimes D = \oplus_{i,j=1}^{2} (A_i \otimes A_j)$ and $A_2 \otimes A_2$ is not among the summands of $C \otimes C_0 + C_0 \otimes C$, we have

$$\Delta(x) \in A_1 \otimes A_1 + A_1 \otimes A_2 + A_2 \otimes A_1 = D \otimes D_0 + D_0 \otimes D.$$

Thus $x \in D_1$, proving $C_1 \cap D = D_1$. Hence $c \in C_1 \cap D = D_1$. Now if the proposition holds for all finite dimensional coalgebras we have

$$c \in D_1 = D_0 + \sum_{a,b \in G(D)} D_{1,a,b} \subseteq C_0 + \sum_{a,b \in G(C)} C_{1,a,b},$$

verifying the proposition for C. Thus it is sufficient to prove the proposition for finite dimensional coalgebras.

Now assume that C is finite dimensional. Let $G(C) = \{g_i \mid 1 \leqslant i \leqslant n\}$. Since $C^*/C_0^{\perp}$ is separable (being a direct sum of copies of Φ) and $C_0^{\perp}$ is the Jacobson radical of C^*, the Wedderburn principal theorem shows that $C^* = A \oplus C_0^{\perp}$ where A is a subalgebra. It is then immediate that $C = C_0 \oplus A^{\perp}$. Setting $K_1 = C_1 \cap A^{\perp}$ we see that $C_1 = C_0 \oplus K_1$.

Now as A is a subalgebra of C^* it is clear that

$$\Delta K_1 \subseteq (A \otimes A)^{\perp} \cap (C_1 \otimes C_0 + C_0 \otimes C_1).$$

We claim

$$(A \otimes A)^{\perp} \cap (C_1 \otimes C_0 + C_0 \otimes C_1) = K_1 \otimes C_0 + C_0 \otimes K_1. \tag{3}$$

As $K_1 \otimes C_0 + C_0 \otimes K_1 \subseteq (A \otimes A)^{\perp}$ and

$$C_1 \otimes C_0 + C_0 \otimes C_1 = K_1 \otimes C_0 + C_0 \otimes C_0 + C_0 \otimes K_1$$

the claim (3) is equivalent to $(A \otimes A)^{\perp} \cap (C_0 \otimes C_0) = (0)$. But this is immediate since

$$(A \otimes A)^{\perp} = C \otimes A^{\perp} = A^{\perp} \otimes C = C_0 \otimes A^{\perp} + A^{\perp} \otimes C_0 + A^{\perp} \otimes A^{\perp}$$

and $C_0 \cap A^{\perp} = (0)$. Thus $\Delta K_1 \subseteq K_1 \otimes C_0 + C_0 \otimes K_1$. Then for a and $b \in G(C)$ we have $K_{1,a,b} \subseteq C_{1,a,b}$ and so Proposition 1 shows that

$$K_1 = \sum_{a,b \in G(C)} K_{1,a,b} \subseteq \sum_{a,b \in G(C)} C_{1,a,b},$$

completing the proof of Proposition 2.

To prove Proposition 4 it is sufficient to show that $\Delta(\varphi - I)(C_{i+1}) \subseteq C \otimes C_0 + C_{i-1} \otimes C$. To this end note that, as φ is a coalgebra homomorphism, $\Delta(\varphi - I) = (\varphi \otimes \varphi - I \otimes I)\Delta = (\varphi \otimes (\varphi - I) + (\varphi - I) \otimes I)\Delta$ so that

$$\begin{aligned}\Delta(\varphi - I)(C_{i+1}) &= (\varphi \otimes (\varphi - I) + (\varphi - I) \otimes I)\Delta C_{i+1} \\ &\subseteq (\varphi \otimes (\varphi - I) + (\varphi - I) \otimes I) \sum_{j=0}^{i+1} C_j \otimes C_{i+1-j} \\ &\subseteq C_0 \otimes C + \sum_{j=1}^{i-1} C_j \otimes C_{i-j} + C \otimes C_0 \subseteq C \otimes C_0 + C_{i-1} \otimes C,\end{aligned}$$

as required.

3. Properties of the Antipode

It is well known that $S(a) = a^{-1}$ for $a \in G(H)$. If $k \in H_{1,a,b}$ then $0 = \epsilon(k) = (S \otimes I)\,\Delta(k) = S(k)a + b^{-1}k$ so $S(k) = -b^{-1}ka^{-1}$. It then follows that $S^2(k) = ab^{-1}ka^{-1}b$ and $S^{2t}(k) = (ab^{-1})^t k(a^{-1}b)^t$ for all $t \geqslant 1$. Thus, if e is the exponent of $G(H)$ we have $S^{2e}(k) = k$ for all $k \in H_{1,a,b}$. In view of Proposition 2 this implies $(S^{2e} - I)(H_1) = (0)$, proving Proposition 3.

We, thus, have the conditions of Proposition 4 satisfied for $i = 1$. Repeated use of this proposition gives $(S^{2e} - I)(H_i) \subseteq H_{i-1}$ for all $i \geqslant 1$, and so $(S^{2e} - I)^i(H_i) = (0)$, proving the theorem.

Corollary 6 follows from the theorem by noting that if Φ has characteristic 0 then $((x^{2e} - 1)^n, x^q - 1)$ divides $x^{2e} - 1$ for any positive integers n and q (since the roots of $x^q = 1$ are distinct), while if Φ has characteristic p then $0 = (S^{2e} - I)^{p^m} = S^{2ep^m} - I$.

4. Remarks

Since extension of the base field leaves the order of the antipode fixed these results apply to Hopf algebras which become pointed upon extension of the base field.

It is interesting to note that the antipode of any (not necessarily pointed) finite dimensional Hopf algebra over a finite field has finite order. For the antipode is bijective [4, p. 101] and hence is a permutation of the finite set H.

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