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A relationship between three analytical approaches to nonlinear problems

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ABSTRACT

In this work, the relationship between three analytical techniques is demonstrated. The direct relationship between the variational approach (VA) and the Hamiltonian approach (HA) is illustrated for a first approximation, and subsequently the relationship between the variational approach and the harmonic balance method (HBM) is concluded. Moreover, the relationship between HA and VA is investigated for higher order solutions.

0. Introduction

A number of mathematical techniques have been applied for solving nonlinear problems. Energy Balance [1–3], homotopy analysis [4], variational iteration [5], homotopy perturbation [6,7], the max–min approach [8,9], frequency amplitude formulation [10] and perturbation methods [11] are the most prominent approaches that those have been employed for solving a large number of nonlinear equations. Researchers have used these methods for solving nonlinear problems in wave propagation [12], biomathematics [13] and structural dynamics [14]. It should be noted that both variational and Hamiltonian approaches were proposed by He over the past years. The variational approach is a powerful method for solving nonlinear equations, and a lot of researchers have employed it as a tool for solving nonlinear problems [15–18]. Similarly, the Hamiltonian approach has been used for solving conservative oscillator problems, and many authors have applied it to obtain the amplitude–frequency relationship of a number of conservative systems [19–29]. More recently, Beléndez et al. [30] have shown a relationship between the harmonic balance method (HBM) and the Hamiltonian approach (HA) in a first approximation with the cosine solution. Beléndez et al. [30] demonstrated HBM and HA to have the same solutions in a first approximation, and then they examined this worthwhile and valuable conclusion for generalized conservative oscillators and obtained a highly valuable result which is the resemblance of the Hamiltonian approach and harmonic balance method for a large number of oscillators. The aforementioned conclusion motivated us to investigate a relationship between other approaches in first-order and higher approximations. In the present study, the relationship between the variational approach (VA) and the Hamiltonian approach (HA) is exhibited and then, according to Ref. [30], the direct relationship between VA and HBM is concluded. Additionally, the relationship between HA and VA is analytically demonstrated for higher order approximations.

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1. The relationship between VA, HA and HBM in the first approximation

Consider conservative oscillators of a general kind with a single degree of freedom:

$$\ddot{x} + f(x) = 0 \qquad x(0) = A, \qquad \dot{x}(0) = 0. \tag{1}$$

Their variational principle can be easily established using the semi-inverse method:

$$J(x) = \int_0^{\frac{T}{4}} \left\{ -\frac{1}{2}\dot{x}^2 + F(x) \right\} dt \tag{2}$$

where T is period of the nonlinear oscillator, and $\frac{\partial F}{\partial x} = f$.

Assume that its solution can be expressed as

$$x(t) = A\cos\omega t \tag{3}$$

where A and ω are the amplitude and the frequency of the oscillator. Substituting (3) into (2) leads to

$$J(A, \omega) = \int_0^{\frac{T}{4}} \left\{ -\frac{1}{2}A^2\omega^2 \sin^2 \omega\, t + F(A\cos\omega\, t) \right\} dt. \tag{4}$$

In the variational approach, on applying $\frac{\partial J}{\partial A} = 0$, the frequency response is obtained. Eq. (4) can be easily changed to the Hamiltonian in the Hamiltonian approach:

$$J(A, \omega) = \int_0^{\frac{T}{4}} \left\{ -\frac{1}{2}A^2\omega^2 \sin^2 \omega\, t + F(A\cos\omega\, t) \right\} dt = \frac{\partial}{\partial\left(\frac{1}{\omega}\right)} \left\{ \int_0^{\frac{T}{4}} \frac{1}{2}A^2\omega \sin^2 \omega t + \frac{1}{\omega}F(A\cos\omega t) \right\} \tag{5}$$

and by applying the Ritz method to Eq. (5) we obtain

$$\begin{aligned} \frac{\partial J(A, \omega)}{\partial A} &= \frac{\partial}{\partial A} \int_0^{\frac{T}{4}} \left\{ -\frac{1}{2}A^2\omega^2 \sin^2 \omega\, t + F(A\cos\omega\, t) \right\} dt \\ &= \frac{\partial}{\partial A}\left[\frac{\partial}{\partial\left(\frac{1}{\omega}\right)} \left\{ \int_0^{\frac{T}{4}} \frac{1}{2}A^2\omega \sin^2 \omega t + \frac{1}{\omega}F(A\cos\omega t) \right\} \right] = H(A, \omega) = 0. \end{aligned} \tag{6}$$

The right part of Eq. (6) is the Hamiltonian in the Hamiltonian approach [19]. According to Eq. (6) the variational and Hamiltonian approaches yield the same result. Beléndez et al. [30] have exhibited that HA and HBM have the same results and due to this, we can conclude that VA and HBM have equal solutions.

2. Investigation of the relationship between VA and HA in higher order approximation

In this section, the relationship between VA and HA is examined analytically. Firstly, the correlation between these approaches is exposed in the second approximation, and then similarly, this idea is extended to the third approximation, and afterwards the relationship of this method is shown for the general cosine solution.

2.1. Studying the second-order approximation

Consider Eq. (1) which is a kind of general oscillator. At this stage, $x = A_1 \cos\omega t + A_2 \cos 3\omega t$ is postulated as the second-order approximation for the elicitation of the relationship between these approaches. Like in the first section, Eq. (2) is considered as a variational function and by applying the second-order approximation, Eq. (7) is obtained:

$$\begin{aligned} &J(A_1, A_2, \omega) = \int_0^{\frac{T}{4}} \left\{ -\frac{1}{2}\,\omega^2 (A_1 \sin\omega\, t + 3A_2 \sin 3\omega t)^2 + F(A_1\cos\omega\, t + A_2\cos 3\omega\, t) \right\} dt \\ &A_1 + A_2 = A. \end{aligned} \tag{7}$$

Eq. (7) can easily be altered to the Hamiltonian function in the Hamiltonian approach:

$$\begin{aligned} J(A_1, A_2, \omega) = {}& \int_0^{\frac{T}{4}} \left\{ -\frac{1}{2}\,\omega^2 (A_1 \sin\omega\, t + 3A_2 \sin 3\omega t)^2 + F(A_1\cos\omega\, t + A_2\cos 3\omega\, t) \right\} dt \\ & \times \frac{\partial}{\partial\left(\frac{1}{\omega}\right)} \int_0^{\frac{T}{4}} \left\{ \frac{1}{2}\,\omega (A_1 \sin\omega\, t + 3A_2 \sin 3\omega t)^2 + \frac{1}{\omega}F(A_1\cos\omega\, t + A_2\cos 3\omega\, t) \right\} dt \end{aligned} \tag{8}$$

and by applying the Ritz method to Eq. (8), we obtain

$$\begin{aligned}\frac{\partial J(A_1,A_2,\omega)}{\partial A_1} &= \frac{\partial}{\partial A_1}\int_0^{\frac{T}{4}}\left\{-\frac{1}{2}\,\omega^2(A_1\sin\omega t+3A_2\sin 3\omega t)^2+F(A_1\cos\omega t+A_2\cos 3\omega t)\right\}dt\\ &= \frac{\partial}{\partial A_1}\frac{\partial}{\partial\left(\frac{1}{\omega}\right)}\int_0^{\frac{T}{4}}\left\{\frac{1}{2}\,\omega(A_1\sin\omega t+3A_2\sin 3\omega t)^2+\frac{1}{\omega}F(A_1\cos\omega t+A_2\cos 3\omega t)\right\}dt\\ &= H(A_1\,,A_2\,,\omega)=0 \end{aligned} \tag{9}$$

and

$$\begin{aligned}\frac{\partial J(A_1,A_2,\omega)}{\partial A_2} &= \frac{\partial}{\partial A_2}\int_0^{\frac{T}{4}}\left\{-\frac{1}{2}\,\omega^2(A_1\sin\omega t+3A_2\sin 3\omega t)^2+F(A_1\cos\omega t+A_2\cos 3\omega t)\right\}dt\\ &= \frac{\partial}{\partial A_2}\frac{\partial}{\partial\left(\frac{1}{\omega}\right)}\int_0^{\frac{T}{4}}\left\{\frac{1}{2}\,\omega(A_1\sin\omega t+3A_2\sin 3\omega t)^2+\frac{1}{\omega}F(A_1\cos\omega t+A_2\cos 3\omega t)\right\}dt\\ &= H(A_1\,,A_2\,,\omega)=0. \end{aligned} \tag{10}$$

According to (9) and (10), HA and VA have a same result in the second approximation.

2.2. Studying the third-order approximation

In this part, $x=A_1\cos\omega t+A_2\cos 3\omega t+A_3\cos 5\omega t$ is considered as the third approximation. By inserting it into Eq. (2), we can obtain Eq. (11) as a variational function:

$$\begin{aligned}J(A_1,A_2,A_3,\omega) \;=\; &\int_0^{\frac{T}{4}}\left\{-\frac{1}{2}\,\omega^2(A_1\sin\omega t+3A_2\sin 3\omega t+5A_3\sin 5\omega t)^2\right.\\ &\left.+\,F(A_1\cos\omega t+A_2\cos 3\omega t+A_3\cos 5\omega t)\right\}dt \end{aligned} \tag{11}$$

$$A=A_1+A_2+A_3.$$

Eq. (11) readily yields

$$\begin{aligned}\frac{\partial J(A_1,A_2,A_3,\omega)}{\partial A_1} &= \frac{\partial}{\partial A_1}\int_0^{\frac{T}{4}}\left\{-\frac{1}{2}\,\omega^2(A_1\sin\omega t+3A_2\sin 3\omega t+5A_3\sin 5\omega t)^2\right.\\ &\quad\left.+F(A_1\cos\omega t+A_2\cos 3\omega t+A_3\cos 5\omega t)\right\}dt\\ &= \frac{\partial}{\partial A_1}\frac{\partial}{\partial\left(\frac{1}{\omega}\right)}\int_0^{\frac{T}{4}}\left\{\frac{1}{2}\,\omega(A_1\sin\omega t+3A_2\sin 3\omega t+5A_3\sin 5\omega t)^2\right.\\ &\quad\left.+\frac{1}{\omega}F(A_1\cos\omega t+A_2\cos 3\omega t+A_3\cos 5\omega t)\right\}dt\\ &= H(A_1\,,A_2\,,A_3,\omega)=0 \end{aligned} \tag{12}$$

$$\begin{aligned}\frac{\partial J(A_1,A_2,A_3,\omega)}{\partial A_2} &= \frac{\partial}{\partial A_2}\int_0^{\frac{T}{4}}\left\{-\frac{1}{2}\,\omega^2(A_1\sin\omega t+3A_2\sin 3\omega t+5A_3\sin 5\omega t)^2\right.\\ &\quad\left.+F(A_1\cos\omega t+A_2\cos 3\omega t+A_3\cos 5\omega t)\right\}dt\\ &= \frac{\partial}{\partial A_2}\frac{\partial}{\partial\left(\frac{1}{\omega}\right)}\int_0^{\frac{T}{4}}\left\{\frac{1}{2}\,\omega(A_1\sin\omega t+3A_2\sin 3\omega t+5A_3\sin 5\omega t)^2\right.\\ &\quad\left.+\frac{1}{\omega}F(A_1\cos\omega t+A_2\cos 3\omega t+A_3\cos 5\omega t)\right\}dt\\ &= H(A_1\,,A_2\,,A_3,\omega)=0 \end{aligned} \tag{13}$$

and

$$\begin{aligned}\frac{\partial J(A_1,A_2,A_3,\omega)}{\partial A_3} &= \frac{\partial}{\partial A_3}\int_0^{\frac{T}{4}}\left\{-\frac{1}{2}\,\omega^2(A_1\sin\omega t+3A_2\sin 3\omega t+5A_3\sin 5\omega t)^2\right.\\ &\quad\left.+F(A_1\cos\omega t+A_2\cos 3\omega t+A_3\cos 5\omega t)\right\}dt\\ &= \frac{\partial}{\partial A_3}\frac{\partial}{\partial\left(\frac{1}{\omega}\right)}\int_0^{\frac{T}{4}}\left\{\frac{1}{2}\,\omega(A_1\sin\omega t+3A_2\sin 3\omega t+5A_3\sin 5\omega t)^2\right.\\ &\quad\left.+\frac{1}{\omega}F(A_1\cos\omega t+A_2\cos 3\omega t+A_3\cos 5\omega t)\right\}dt\\ &= H(A_1\,,A_2\,,A_3,\omega)=0.\end{aligned} \tag{14}$$

Eqs. (12)–(14) reveal the fact that HA and VA have the same result in the third approximation. It can be readily concluded that these two techniques yield analogous results for the general cosine solution ($\sum_{n=0}^{n} A_n\cos(2n+1)\omega t$). For the demonstration of this fact, $\sum_{n=0}^{n} A_n\cos(2n+1)\omega t$ is selected as a solution for the generalized nonlinear equation, and subsequently, it is inserted in Eq. (5); doing this, we obtain

$$\begin{aligned}J(A_1,\ldots,A_n,\omega) &= \int_0^{\frac{T}{4}}\left\{-\frac{1}{2}\omega^2\left(\sum_{n=0}^{n}A_n(2n+1)\sin^2(2n+1)\omega t\right)^2+F\left(\sum_{n=0}^{n}A_n\cos\omega t\right)\right\}dt\\ &= \frac{\partial}{\partial\left(\frac{1}{\omega}\right)}\left\{\int_0^{\frac{T}{4}}\frac{1}{2}\omega\left(\sum_{n=0}^{n}A_n(2n+1)\sin^2(2n+1)\omega t\right)^2+\frac{1}{\omega}F\left(\sum_{n=0}^{n}A_n\cos(2n+1)\omega t\right)\right\}\end{aligned} \tag{15}$$

and by applying the Ritz method to Eq. (7), we obtain

$$\begin{aligned}\frac{J(A_1,\ldots,A_n,\omega)}{\partial A_1} &= \frac{\partial}{\partial A_1}\int_0^{\frac{T}{4}}\left\{-\frac{1}{2}\omega^2\left(\sum_{n=0}^{n}A_n(2n+1)\sin^2(2n+1)\omega t\right)^2+F\left(\sum_{n=0}^{n}A_n\cos\omega t\right)\right\}dt\\ &= \frac{\partial}{\partial A_1}\,\frac{\partial}{\partial\left(\frac{1}{\omega}\right)}\left\{\int_0^{\frac{T}{4}}\frac{1}{2}\omega\left(\sum_{n=0}^{n}A_n(2n+1)\sin^2(2n+1)\omega t\right)^2\right.\\ &\quad\left.+\frac{1}{\omega}F\left(\sum_{n=0}^{n}A_n\cos(2n+1)\omega t\right)\right\}=H(A_1,\ldots,A_n,\omega)=0\end{aligned} \tag{16}$$

$$A=A_1+A_2+\cdots+A_n.$$

It is obvious that after applying the Ritz method to Eq. (15), we can obtain

$$\frac{J(A_1,\ldots,A_n,\omega)}{\partial A_1}=H(A_1,\ldots,A_n,\omega)=0 \tag{17}$$

$$\frac{J(A_1,\ldots,A_n,\omega)}{\partial A_2}=H(A_1,\ldots,A_n,\omega)=0. \tag{18}$$

Consequently, we have

$$\frac{J(A_1,\ldots,A_n,\omega)}{\partial A_n}=H(A_1,\ldots,A_n,\omega)=0. \tag{19}$$

Eqs. (16)–(18) prove the relationship of these approaches for the general cosine solution.

3. Conclusion

In this study, it was proved that HA and VA have equal responses in the first approximation, and afterwards it was concluded that HBM and VA have the same result. Similarly, the relationship of HA and VA was analytically demonstrated for the second and third approximations. As a result, it was concluded that these approaches yield equal solutions for the general cosine approximation. Indeed, HA and VA have the same basis and, consequently, these methods yield the same amplitude–frequency relationship. Obviously, the idea of this work can be useful for researchers in this field investigating the relationships of these kinds of methods analytically.

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