Topology Vol. 3, pp. 1–18, Pergamon Press, 1965. Printed in Great Britain

SHEAVES WITH VALUES IN A CATEGORY†

JOHN W. GRAY

(*Received* 3 *September* 1963)

§1. INTRODUCTION

LET T be a topology (i.e., a collection of open sets) on a set X. Consider **T** as a category in which the objects are the open sets and such that there is a unique 'restriction' morphism $U \to V$ if and only if $V \subset U$. For any category **A**, the functor category **F**(**T**, **A**) (i.e., objects are covariant functors from **T** to **A** and morphisms are natural transformations) is called the category of *presheaves* on X (really, on **T**) with values in **A**. A presheaf F is called a *sheaf* if for every open set $U \in$ **T** and for every strong open covering $\{U_\alpha\}$ of U (strong means $\{U_\alpha\}$ is closed under finite, non-empty intersections) we have $F(U) = L\lim\ F(U_\alpha)$ ($L\lim$ means 'generalized' inverse limit. See §4.) The category **S**(**T**, **A**) of *sheaves* on X with values in **A** is the full subcategory (i.e., morphisms the same as in **F**(**T**, **A**)) determined by the sheaves.

In this paper we shall demonstrate several properties of **S**(**T**, **A**):

(i) **S**(**T**, **A**) is left closed in **F**(**T**, **A**). (Theorem 1(i), §8). This says that if a left limit (generalized inverse limit) of sheaves exists as a presheaf then that presheaf is a sheaf.

(ii) If **A** has limits (§4) and enough small objects (§5) then **S**(**T**, **A**) is right reflective (cf. Freyd [3]) in **F**(**T**, **A**); i.e., there is a functor R from presheaves to sheaves which has the inclusion functor of sheaves into presheaves as a right adjoint. Equivalently, there is a natural transformation $r_F : F \to R(F)$ such that if $\phi : F \to G$, where G is a sheaf, then there is a unique $\phi' : R(F) \to G$ with $\phi' \circ r_F = \phi$. As a consequence, right limits (generalized direct limits) of sheaves are the reflection R of the corresponding limits calculated as presheaves. If **A** is an AB5 abelian category (Grothendieck [7]) then R is an exact functor (Theorem (4), §10) from which it follows that **S**(**T**, **A**) is also an AB5 abelian category. Furthermore, in this case, sheaf morphisms are characterized by the induced morphisms on the stalks in the sense that a sequence of sheaves is exact if and only if the induced sequence of stalk morphisms is exact at every point.

(iii) If $\mathbf{T}_X$ (resp. $\mathbf{T}_Y$) is a topology on X (resp., Y) and if $f : X \to Y$ is continuous then f^{-1} may be considered as a functor from $\mathbf{T}_Y$ to $\mathbf{T}_X$. Composition with this fixed functor determines a functor f_* from presheaves on X to presheaves on Y. This functor has a left adjoint f^* which is exact under the same hypotheses on **A** as in (ii). Furthermore f_* and f^* are 'functoral' in f. (Theorem (2), §9) (cf. Grothendieck [8].)

†This work has been partially supported by the National Science Foundation under Grant No. G19022 through Columbia University.

Applications of these results are given in §11 to relative injective sheaves, to induced sheaves on locally closed subspaces and to cosheaves.

In dealing with categories involving topological spaces it is not at all unusual for a significant sub-category to be left closed and right reflective. For example, let $\mathbf{T}_2$ (resp., $\mathbf{T}_{2C}$) be the category of Hausdorff (resp., compact Hausdorff) topological spaces. Then $\mathbf{T}_{2C}$ is left closed (by Tychonoff) and right reflective (by Stone-Cech 'compactification') in $\mathbf{T}_2$. Other examples are Hausdorff or completely regular spaces in the category of all topological spaces, complete Hausdorff uniform spaces in the category of all Hausdorff uniform spaces, and topological groups in the category of completely regular spaces. (This last example is not really a subcategory, but the forgetful functor from topological groups to topological spaces has a left adjoint). The basic proposition (Appendix C3) which gives the existence of the right reflection for sheaves applies more or less without change to the cases cited here as well.

In order not to impede the exposition unnecessarily we shall merely define adjoint functors, reflective sub-categories and limits in the next three paragraphs and relegate the statements and proofs (in those cases in which we cannot cite a published reference) of the results we shall use to an appendix. Most of these properties, except for Appendices C3 and C4, are presumably 'well-known'. We then prove the promised results for the discrete topology on a set and finally show that this implies the general result. All discussion of exactness is reserved for §10 and until that point we do *not* even assume that $\mathbf{A}$ is additive, only that it has limits and small objects. (For an entirely different approach, see Heller and Rowe [10].) The results here are essentially those that were announced in [6].

I would like to thank Peter J. Freyd for many stimulating conversations during the preparation of an earlier draft of this paper.

§2. ADJOINT FUNCTORS

All functors in this paper are covariant.

A pair of functors $S : \mathbf{A} \to \mathbf{B}$ and $T : \mathbf{B} \to \mathbf{A}$ are called *adjoint* via Φ (S the *left* adjoint of T and T the *right* adjoint of S) if Φ is a natural equivalence between the functors $\mathrm{Hom}_{\mathbf{B}}(S(-), -)$ and $\mathrm{Hom}_{\mathbf{A}}(-, T(-))$, both from $\mathbf{A} \times \mathbf{B}$ to the category of sets (or groups in the case of additive functors.) Thus, for each $(A, B) \in \mathbf{A} \times \mathbf{B}$,

$$\Phi_{A,B} : \mathrm{Hom}_{\mathbf{B}}(S(A), B) \to \mathrm{Hom}_{\mathbf{A}}(A, T(B))$$

is an equivalence.

Defining $\psi_B = \Phi^{-1}_{T(B),B}(i_{T(B)})$ and $\theta_A = \Phi_{A,S(A)}(i_{S(A)})$ yields natural transformations $\psi : S \circ T \to I_B$ and $\theta : I_A \to T \circ S$ such that given $f : S(A) \to B$ and $g : A \to T(B)$ there are unique morphisms $f' : A \to T(B)$ and $g' : S(A) \to B$ with $f = \psi_B \circ S(f')$ and $g = T(g') \circ \theta_A$. ψ and θ are called the *induced* natural transformations. (In the preceeding formulas i_A is the identity morphism of the object A and $I_{\mathbf{A}}$ is the identity functor of the category $\mathbf{A}$.)

§3. REFLECTIVE SUBCATEGORIES

A special case arises if one of the functors above is the inclusion functor $I_{\mathbf{AA}'} : \mathbf{A}' \to \mathbf{A}$ of a subcategory. A functor $L : \mathbf{A} \to \mathbf{A}'$ (resp., $R : \mathbf{A} \to \mathbf{A}'$) which has $I_{\mathbf{AA}'}$ as a left (resp.

right) adjoint is called a *left* (resp., *right*) reflection of **A** in **A**′. If such a functor exists, **A**′ is called *left* (resp., *right*) reflective in **A**. (This terminology is chosen since the important 'left' property for a functor is for it to *have* a left adjoint, not for it to be a left adjoint. The term 'right reflective' corresponds to 'reflective' in Freyd [3].) In each case only one of the induced natural transformations is of interest; namely,

$$l : I_{\mathbf{AA}'} \circ L \to I_{\mathbf{A}} \quad \text{and} \quad r : I_{\mathbf{A}} \to I_{\mathbf{AA}'} \circ R.$$

These satisfy the universal property that given $A \in \mathbf{A}$, $A' \in \mathbf{A}'$ and $f : A' \to A$ (resp., $g : A \to A'$) then there is a unique $f' : A' \to L(A)$ (resp., $g' : R(A) \to A'$) such that $f = l_A \circ f'$ (resp., $g = g' \circ r_A$). It is easily checked that these universal properties characterize L and R in the sense that, for example, if for each $A \in \mathbf{A}$ an object $L(A) \in \mathbf{A}'$ together with a morphism $l : L(A) \to A$ is given satisfying the above universal property then there is a unique extension of L to a functor from **A** to **A**′ which is a left reflection.

§4. LIMITS

As an application of the notion of a reflective subcategory, let **D** be a small category (i.e., the collection of objects of **D** is a set). Then the functor category **F(D, A)** is well defined. We identify **A** with the subcategory of **F(D, A)** consisting of constant functors and constant natural transformations, i.e., $A \in \mathbf{A}$ is identified with the functor mapping each object of **D** into A and each morphism of **D** into i_A while a morphism g in **A** is identified with the natural transformation all of whose values are g.

If there is a right reflection $R\lim_{\mathbf{D}} : \mathbf{F}(\mathbf{D}, \mathbf{A}) \to \mathbf{A}$ then for any $D : \mathbf{D} \to \mathbf{A}$, $R\lim_{\mathbf{D}} D$ is called the *right limit* of D (or generalized direct or inductive limit). The induced natural transformation is denoted by

$$rl_{\mathbf{D}} : I_{\mathbf{F}(\mathbf{D},\mathbf{A})} \to I_{\mathbf{F}(\mathbf{D},\mathbf{A}),\mathbf{A}} \circ R\lim_{\mathbf{D}}.$$

Dually, if there is a left reflection $L\lim_{\mathbf{D}} : \mathbf{F}(\mathbf{D}, \mathbf{A}) \to \mathbf{A}$, its values are called *left limits* (or generalized inverse or projective limits.) The induced natural transformation is denoted by

$$ll_{\mathbf{D}} : I_{\mathbf{F}(\mathbf{D},\mathbf{A}),\mathbf{A}} \circ L\lim_{\mathbf{D}} \to I_{\mathbf{F}(\mathbf{D},\mathbf{A})}$$

(cf. Kan [12]).

We leave the translation of the universal mapping property in this case to the reader. From this property it follows immediately that

$$\mathrm{Hom}_{\mathbf{A}}(X, L\lim D) = L\lim \mathrm{Hom}_{\mathbf{A}}(X, D(-))$$

and

$$\mathrm{Hom}_{\mathbf{A}}(R\lim D, X) = L\lim \mathrm{Hom}_{\mathbf{A}}(D(-), X).$$

By the uniqueness of the representing object for a representable functor (Grothendieck [9]) these equations characterize limits.

We shall reserve the term direct (resp., inverse) limit for the case when **D** is an increasing directed category; i.e., the objects of **D** form a directed set (partially ordered with upper bounds) and there is a unique morphism $i \to j$ (resp., $j \to i$) if and only if $i \leq j$. If $D(i) = A_i$, we write $R\lim D = \text{dir lim } A_i$ (resp., $L\lim D = \text{inv lim } A_i$) in this case.

Direct products (or left products!), difference kernels and pullbacks (or fibre products) are other examples of left limits and their duals are examples of right limits. (In general $L\lim_{\mathbf{D}}$ is dual to $R\lim_{\mathbf{D}^\circ}$ where $\mathbf{D}^\circ$ is the opposite category to $\mathbf{D}$.)

The main facts about limits which we shall make use of are: (i) limits in a functor category are computed objectwise, (ii) a functor T with a left adjoint preserves left limits (i.e., $L\lim T \circ D = T\,(L\lim D)$), (iii) all left (resp., right) limits commute with each other. (See Appendix B for proofs and references.)

We shall say that a category *has* left (resp., right) limits if it has left (resp., right) limits of type $\mathbf{D}$ for all small categories $\mathbf{D}$. It is easily shown, for instance, that a category has left limits if and only if it has arbitrary direct products and arbitrary pullbacks. An abelian category has left limits if and only if it has direct products, thus, if and only if it satisfies Grothendieck's axiom AB3* [7].

DEFINITION. *A subcategory* $\mathbf{A}'$ *of* $\mathbf{A}$ *is* left closed *in* $\mathbf{A}$ *if given any* $D : \mathbf{D} \to \mathbf{A}$ *such that* $L\lim(I_{\mathbf{AA}'} \circ D)$ *exists then* $L\lim(I_{\mathbf{AA}'} \circ D) = L\lim\ D \in \mathbf{A}'$ *and* $ll(I_{\mathbf{AA}'} \circ D) = llD$. (I.E., *if a diagram* D *in* $\mathbf{A}'$ *has a limit in* $\mathbf{A}$, *then the limit is, in fact, in* $\mathbf{A}'$.) Right closed *is defined dually.*

If $\mathbf{A}$ *and* $\mathbf{A}'$ *have left limits, then* $\mathbf{A}'$ *is left closed if and only if* $I_{\mathbf{AA}'}$ *preserves left limits* Clearly, a left closed subcategory of a category with left limits also has left limits. Finally, if $\mathbf{A}'$ is a full subcategory then the condition on $ll_{\mathbf{D}}$ is superfluous.

§5. GENERATORS AND SMALL OBJECTS

We shall distinguish between two kinds of generators. A *set* $\{G_\alpha\}$ of objects from a category $\mathbf{A}$ is called a *generating family* in $\mathbf{A}$ if, given any subobject B of an object $A \in \mathbf{A}$, $B \neq A$ (i.e., given any monomorphism $B \to A$ which is not an equivalence), there is a G_α with a morphism $h : G_\alpha \to A$ which does not factor through B. (Cf. Grothendieck [7], §1.9.) For the purposes of this paper a set $\{G'_\alpha\}$ is called an *m-generating family* (*m* for morphism) if given $f, g : A \to C$, $f \neq g$, there is a G'_α with a morphism $h : G'_\alpha \to A$ such that $fh \neq gh$.

If follows immediately from the definitions that if every monomorphism in A is the difference kernel of some pair of morphisms then an m-generating family is a generating family. On the other hand, if every pair of morphisms between the same objects has a difference kernel (in particular if $\mathbf{A}$ has left limits) a generating family is an m-generating family. Consequently, if $\mathbf{A}$ is abelian then the two notions of generators are equivalent.

In [7], §1.9, Grothendieck shows that if $\mathbf{A}$ has a generating family then any object has at most a set of subobjects. Furthermore, Proposition (1.9.2) can easily be interpreted to show that if $\mathbf{A}$ has a generating family then so does $\mathbf{F}(\mathbf{B}, \mathbf{A})$ for any appropriate small category $\mathbf{B}$. Hence if $\mathbf{A}$ has a generating family then any functor $F : \mathbf{B} \to \mathbf{A}$ has at most a set of subfunctors. One can give an alternative proof of this by noting that if, in addition, $\mathbf{A}$ has left limits then the generating family in $\mathbf{F}(\mathbf{B}, \mathbf{A})$ is an m-generating family. From this it can be easily shown that a natural transformation ϕ in $\mathbf{F}(\mathbf{B}, \mathbf{A})$ is a monomorphism if and only if for every $B \in \mathbf{B}$, ϕ_B is a monomorphism in $\mathbf{A}$, which implies directly that any functor has at most a set of subfunctors.

We now introduce a new kind of object, to be called a small object. Let $\mathbf{D}$ be an increasingly directed category and let $D : \mathbf{D} \to \mathbf{A}$, $D(i) = A_i$. The natural transformation $D \to \text{dir lim } A_i$ induces a natural transformation

$$\text{Hom}_{\mathbf{A}}(X, A_i) \to \text{Hom}_{\mathbf{A}}(X, \text{dir lim } A_i)$$

and hence a map

$$\phi_{X,D} : \text{dir lim Hom}_{\mathbf{A}}(X, A_i) \to \text{Hom}_{\mathbf{A}}(X, \text{dir lim } A_i).$$

DEFINITION. $X \in \mathbf{A}$ is called *small* if for every such directed $\mathbf{D}$ and every $D : \mathbf{D} \to \mathbf{A}$, $\phi_{X,D}$ is an injection. Cosmall objects are defined dually. If there is a generating family of small objects then we shall say that $\mathbf{A}$ has *enough small objects*.

In the category of sets or topological spaces, a single point is a small generator. In the category of modules over a ring R, R is a small generator. In the category of torsion abelian groups, $\{Z_p\}$ is a generating family of small objects. Freyd has shown that in the category of group-valued functors on a small category, the representable functors form a generating family of small objects (unpublished.).

§6. DIRECT AND INVERSE IMAGES OF PRESHEAVES

Let $\mathbf{T}_X$ (resp., $\mathbf{T}_Y$) be a topology on X (resp., Y) and let $f : X \to Y$ be $\mathbf{T}_X - \mathbf{T}_Y$-continuous. Then f^{-1} determines a functor

$$\tilde{f}_* = (f^{-1})^A : \mathbf{F}(\mathbf{T}_X, \mathbf{A}) \to \mathbf{F}(\mathbf{T}_Y, \mathbf{A})$$

(See §1, (iii) and appendix A4); i.e., if F is a presheaf on X then $\tilde{f}_* F$ is the presheaf on Y such that for any $V \in \mathbf{T}_Y$, $\tilde{f}_* F(V) = F(f^{-1}(V))$. $\tilde{f}_* F$ is called the *direct image* of F by f.

We shall show here that the functor $\tilde{f}_*$ has a left adjoint. This would follow immediately from Appendix A4 if the functor f^{-1} had a right adjoint. But the existence of such a right adjoint is equivalent to the arbitrary intersection of open sets being open which, of course, is not true in general.

MAIN LEMMA. *If* $\mathbf{A}$ *has direct limits then* $\tilde{f}_*$ *has a left adjoint* $\tilde{f}^*$.
If G *is a presheaf on* Y, *then* $\tilde{f}^* G$ *is called the* presheaf inverse image *of* G *by* f.
Proof. Let G be a presheaf on Y and define

$$[\tilde{f}^* G](U) = \text{dir lim } G(V) \qquad (\text{for } U \subset f^{-1}(V))$$

with the obvious induced morphism when $U' \subset U$. Similarly, if $\phi : G \to G'$ is a presheaf morphism, then define $\tilde{f}^* \phi_U = \text{dir lim } \phi_V$ (for $U \subset f^{-1}(V)$). It is immediate that $\tilde{f}^*$ is a functor from presheaves on Y to presheaves on X. To prove that $\tilde{f}^*$ is the left adjoint of $\tilde{f}_*$, it is sufficient, by Appendix A3, to define natural transformations $\theta : I_{\mathbf{F}(\mathbf{T}_Y, \mathbf{A})} \to \tilde{f}_* \circ \tilde{f}^*$ and $\psi : \tilde{f}^* \circ \tilde{f}_* \to I_{\mathbf{F}(\mathbf{T}_X, \mathbf{A})}$ such that

$$(\psi * \tilde{f}^*) \circ (\tilde{f}^* * \theta) = i * \tilde{f}^* : \tilde{f}^* \to \tilde{f}^* \circ \tilde{f}_* \circ \tilde{f}^* \to \tilde{f}^*$$

$$(\tilde{f}_* * \psi) \circ (\theta * \tilde{f}_*) = i * \tilde{f}_* : \tilde{f}_* \to \tilde{f}_* \circ \tilde{f}^* \circ \tilde{f}_* \to \tilde{f}_*$$

(here i denotes the identity natural transformation and the notation is that of Godement [5, Appendix 1]).

By definition

$$f_*f^*G(V) = \text{dir lim } G(V') \quad (\text{for } f^{-1}(V) \subset f^{-1}(V')).$$

Since V is itself such a V', there is a morphism

$$(\theta_G)_V : G(V) \to f_*f^*G(V).$$

which determines a presheaf morphism θ_G. Similarly

$$f^*f_*F(U) = \text{dir lim } F(f^{-1}(V)) \quad (\text{for } U \subset f^{-1}(V))$$

The morphisms $F(f^{-1}(V)) \to F(U)$ therefore induce a morphism $(\psi_F)_U : f^*f_*F(U) \to F(U)$, which determines a presheaf morphism ψ_F. It is easily seen that θ and ψ are natural transformations. An immediate direct calculation shows that $\psi * f^*$, $f_* * \psi$, $\theta * f_*$ and $f^* * \theta$ are identity natural transformations.

PROPOSITION. *If F is a sheaf on X then f_*F is a sheaf on Y.*

Proof. Let $\{V_\alpha\}$ be a strong open covering of $V \in \mathbf{T}_Y$. Since f^{-1} preserves all lattice operations, $\{f^{-1}(V_\alpha)\}$ is a strong open covering of $f^{-1}(V)$. Hence

$$f_*F(V) = F(f^{-1}(V)) = L\text{lim } F(f^{-1}(V_\alpha)) = L\text{lim } f_*F(V_\alpha)$$

so f_*F is a sheaf. (This also follows from the proof of Theorem (1, i) and the fact that f^{-1} preserves limits.)

We denote the induced functor from $\mathbf{S}(\mathbf{T}_X,\mathbf{A})$ to $\mathbf{S}(\mathbf{T}_Y,\mathbf{A})$ by f_*. If I_X (resp., I_Y) denotes the inclusion functor of sheaves on X (resp., Y) into presheaves on X (resp., Y) then clearly $f_* \circ I_X = I_Y \circ f_*$.

§7. SHEAVES ON A DISCRETE SPACE

Let $\mathbf{T}_d$ denote the discrete topology on X. Since for any $U \subset X$, $\{\{x\}\}_{x \in U}$ is an open covering of U, a presheaf $F \in \mathbf{F}(\mathbf{T}_d,\mathbf{A})$ is a sheaf if and only if $F(U) = \prod_{x \in U} F\{x\}$. Furthermore, if $\phi : F \to F'$ is a sheaf morphism then $\phi_U = \prod_{x \in U} \phi\{x\}$, since for all $x \in U$ the adjoining diagram must be commutative.

$$\begin{array}{ccc} F(U) & \to & F'(U) \\ \downarrow & & \downarrow \\ F\{x\} & \to & F'\{x\} \end{array}$$

Thus there is a natural equivalence:

$$\text{Hom}_{\mathbf{S}(\mathbf{T}_d,\mathbf{A})}(F, F') \approx \prod_{x \in X} \text{Hom}_{\mathbf{A}}(F\{x\}, F'\{x\}).$$

PROPOSITION. *If $\mathbf{A}$ has direct products then there is a right reflection*

$$R_{dX} : \mathbf{F}(\mathbf{T}_d,\mathbf{A}) \to \mathbf{S}(\mathbf{T}_d,\mathbf{A}).$$

Proof. If F is a presheaf, define $R_{dX}F(U) : \prod_{x \in U} F\{x\}$ and $r_F : F \to R_{dX}F$ by $r_{F,U} : F(U) \to R_{dX}F(U)$ where $r_{F,U}$ is the morphism whose co-ordinates are the morphisms $F(U) \to F\{x\}$ for $\{x\} \subset U$. It is immediate that $R_{dX}F$ is a right reflection since if G is a sheaf then

$$\begin{array}{ccccc} F(U) & \longrightarrow & R_{dX}F(U) = \prod_{x\in U} F\{x\} & \longrightarrow & F\{x\} \\ & \searrow^{\phi_U} & \Big\downarrow \prod_{x\in U}\phi_{\{x\}} & & \Big\downarrow \phi_{\{x\}} \\ & & G(U) = \prod_{x\in U} G\{x\} & \longrightarrow & G\{x\} \end{array}$$

is commutative and $\prod_{x\in U} \phi_{\{x\}}$ is the unique morphism that makes it so.

§8. SHEAVES

Let $\mathbf{T}$ be a topology on X and let $\mathbf{T}_d$ be the discrete topology on X as in §7. Since the identity map i_X is $\mathbf{T}_d - \mathbf{T}$-continuous, there is a pair of adjoint functors $(\tilde{\imath}_X)_*$ and $(\tilde{\imath}_X)^*$ by §6. Let $P_X = (i_X)_* \circ [R_{dX} \circ (\tilde{\imath}_X)^*] : \mathbf{F}(\mathbf{T}, \mathbf{A}) \to \mathbf{S}(\mathbf{T}, \mathbf{A})$. To calculate P_X on a presheaf F, let $F_x = \text{dir lim } F(U)(x \in U)$ (F_x is called the *stalk* of F at x). Then $P_X F$ is the sheaf such that for any $U \in \mathbf{T}$, $P_X F(U) = \prod_{x\in U} F_x$.

$$\begin{array}{ccc} \mathbf{F}(\mathbf{T}_d,\mathbf{A}) & \overset{(\tilde{\imath}_X)_*}{\underset{(\tilde{\imath}_X)^*}{\rightleftarrows}} & \mathbf{F}(\mathbf{T},\mathbf{A}) \\ I_d \Big\uparrow \Big\downarrow R_{dX} & & I_X \Big\uparrow \Big\downarrow P_X \\ \mathbf{S}(\mathbf{T}_d,\mathbf{A}) & \underset{(i_X)_*}{\longrightarrow} & \mathbf{S}(\mathbf{T},\mathbf{A}) \end{array}$$

Since $(\tilde{\imath}_X)^*$ is the left adjoint to $(\tilde{\imath}_X)_*$ and R_{dX} is the left adjoint to the inclusion functor I_d, it follows by appendix A2 that $R_{dX} \circ (\tilde{\imath}_X)^*$ is the left adjoint to $(\tilde{\imath}_X)_* \circ I_d = I_X \circ (i_X)_*$. Hence there is an induced natural transformation $\rho : I_{\mathbf{F}(\mathbf{T},\mathbf{A})} \to I_X \circ P_X$. The definitions show that for a presheaf F, $\rho_F : F \to P_X F$ is the presheaf morphism such that on $U \in \mathbf{T}$, $\rho_{F,U} : F(U) \to \prod_{x\in U} F_x$ is the morphism whose co-ordinates are the induced morphisms

$$r_{x,U} : F(U) \to F_x = \text{dir lim } F(U)(x \in U).$$

THEOREM (1). (i) $\mathbf{S}(\mathbf{T}, \mathbf{A})$ *is a left closed* (§4) *subcategory of* $\mathbf{F}(\mathbf{T}, \mathbf{A})$.

(ii) *If* $\mathbf{A}$ *has limits, a zero object and enough small objects* (§5) *then there is a right reflection* $R_X : \mathbf{F}(\mathbf{T}, \mathbf{A}) \to \mathbf{S}(\mathbf{T}, \mathbf{A})$. *If F is a sheaf then* $R_X(F) \approx F$.

Proof. (i) Let $\mathbf{U} = \{U_\alpha\}$ be a strong open covering of $U \in \mathbf{T}$ and let $D : \mathbf{U} \to \mathbf{T}$ be the inclusion functor. Then $L\text{lim } D = \cup U_\alpha = U$. Hence to say that F is a sheaf is to say that $F(L\text{lim } D) = L\text{lim } F \circ D$; i.e., a sheaf is a functor which preserves certain left limits; namely, left limits of inclusions of subcategories of $\mathbf{T}$ corresponding to strong coverings. It therefore follows from Appendix B3 that $\mathbf{S}(\mathbf{T}, \mathbf{A})$ is left closed in $\mathbf{F}(\mathbf{T}, \mathbf{A})$.

(ii) If F is a presheaf, define $R_X F$ to be the intersection of all subsheaves of the sheaf $P_X F$ through which the morphism $\rho_F : F \to P_X F$ factors. This intersection is well defined by §5. It will follow from Appendix C3 that R_X is a right reflection provided that whenever F is a sheaf then ρ_F is a (presheaf) monomorphism. To show this it is sufficient to show that for each $U \in \mathbf{T}$, $\rho_{F,U} : F(U) \to P_X F(U)$ is a monomorphism in $\mathbf{A}$. For this it suffices to show that if $\{G_\alpha\}$ is a family of generators in $\mathbf{A}$ and if $f, f' : G_\alpha \to F(U)$ satisfy $\rho_{F,U} \circ f = \rho_{F,U} \circ f'$ (i.e., $r_{xU} \circ f = r_{xU} \circ f'$ for all $x \in U$) then $f = f'$. By hypothesis we may assume that the G_α's are small.

Let $x \in X$ and $U \in \mathbf{T}$ be fixed and for $x \in V \subset U$ define $f_V = r_{VU} \circ f$ and $f'_V = r_{VU} \circ f'$ where r_{VU} is the 'restriction' morphism $F(U) \to F(V)$. Then $\{f_V\}$ and $\{f'_V\}$ represent elements $\tilde{f}$ and $\tilde{f}'$ in dir lim $\mathrm{Hom}_{\mathbf{A}}(G_\alpha, F(V))$. By hypothesis

$$\phi_{G_\alpha} : \underset{x \in V \subset U}{\text{dir lim}} \ \mathrm{Hom}_{\mathbf{A}}(G_\alpha, F(V)) \to \mathrm{Hom}_{\mathbf{A}}(G_\alpha, F_x)$$

is a monomorphism. Since $r_{xV} f_V = r_{xV} f'_V$ it follows that $\phi_{G_\alpha}(\tilde{f}) = \phi_{G_\alpha}(\tilde{f}')$ and hence $\tilde{f} = \tilde{f}'$. Thus there is a $V(x)$ with $x \in V(x) \subset U$ such that $f_{V(x)} = f'_{V(x)}$. Consider the strong covering of U determined by the $V(x)$, $x \in U$. The morphisms $f_{V(x)}$ satisfy

$$r_{V(xy)V(x)} \circ f_{V(x)} = r_{V(xy)V(y)} \circ f_{V(y)}$$

where $V(xy) = V(x) \cap V(y)$. Since F is a sheaf, there is therefore a unique $f'' : G_\alpha \to F(U)$ such that $r_{V(x)U} \circ f'' = f_{V(x)}$ for all $x \in U$. But f and f' both satisfy this condition and hence $f = f'$.

It is immediate from the definition that if F is a sheaf then $R_X(F) \approx F$. (cf. Note to Appendix C2).

COROLLARY (1). $R_X \circ R_X \approx R_X$.

COROLLARY (2). $\mathbf{S}(\mathbf{T}, \mathbf{A})$ *has limits, left limits being computed as presheaves while right limits are* R_X *applied to the corresponding presheaf limits.*

Proof. If $\mathbf{A}$ has limits, so does $\mathbf{F}(\mathbf{T}, \mathbf{A})$. Since $\mathbf{S}(\mathbf{T}, \mathbf{A})$ is left closed, if $D : \mathbf{D} \to \mathbf{S}(\mathbf{T}, \mathbf{A})$ then $L\lim D = L\lim I_X \circ D$ while, by Appendix C2, $R\lim D = R_X(R\lim I_X \circ D)$.

§9. DIRECT AND INVERSE IMAGES OF SHEAVES

Throughout this section, we assume that $\mathbf{A}$ has limits and enough small objects. Returning to the situation described in §6, our assumptions on $\mathbf{A}$ imply that the functors

$$\begin{array}{ccc} \mathbf{F}(\mathbf{T}_Y, \mathbf{A}) & \underset{\tilde{f}_*}{\overset{\tilde{f}^*}{\rightleftarrows}} & \mathbf{F}(\mathbf{T}_X, \mathbf{A}) \\ R_Y \big\downarrow \big\uparrow I_Y & & R_X \big\downarrow \big\uparrow I_X \\ \mathbf{S}(\mathbf{T}_Y, \mathbf{A}) & \underset{f_*}{\overset{f^*}{\rightleftarrows}} & \mathbf{S}(\mathbf{T}_X, \mathbf{A}) \end{array}$$

R_X, R_Y and $\tilde{f}^*$ exist, R_X (resp., R_Y) being the left adjoint to I_X (resp., I_Y) and $\tilde{f}^*$ the left adjoint to $\tilde{f}_*$. (See the adjoining diagram.) Hence $R_X \circ \tilde{f}^*$ is the left adjoint to $\tilde{f}_* \circ I_X = I_Y \circ f_*$ (by Appendix A2). Define

$$f^* = R_X \circ \tilde{f}^* \circ I_Y : \mathbf{S}(\mathbf{T}_Y, \mathbf{A}) \to \mathbf{S}(\mathbf{T}_X, \mathbf{A}).$$

If G is a sheaf on Y then f^*G is called the (sheaf) *inverse image* of G by f.

THEOREM (2). (i) *f^* is the left adjoint to f_* and there is a natural equivalence $f^* \circ R_Y \approx R_X \circ \tilde{f}^*$.*

(ii) *If $g : Y \to Z$ is also continuous then $(g \circ f)_* = g_* \circ f_*$ and $(g \circ f)^* = f^* \circ g^*$.*

Proof. (i) Let F be a sheaf on X and G a sheaf on Y. Then, by the definition of f^*,

$$\mathrm{Hom}_{\mathbf{S}(\mathbf{T}_X, \mathbf{A})}(f^*G, F) \approx \mathrm{Hom}_{\mathbf{F}(\mathbf{T}_Y, \mathbf{A})}(I_Y G, I_Y \circ f_* F)$$
$$= \mathrm{Hom}_{\mathbf{S}(\mathbf{T}_Y, \mathbf{A})}(G, f_* F)$$

the last equality since I_Y is the inclusion of a full subcategory. Furthermore, $f^* \circ R_Y$ is a left adjoint to $I_Y \circ f_* = \tilde{f}_* \circ I_X$. Since $R_X \circ \tilde{f}^*$ is also a left adjoint to $\tilde{f}_* \circ I_X$ it follows from the uniqueness of adjoints (Appendix A2) that $f^* \circ R_Y$ and $R_X \circ \tilde{f}^*$ are naturally equivalent.

(ii) Since $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$, we get immediately that $(\widetilde{g \circ f})_* = \tilde{g}_* \circ \tilde{f}_*$ and hence $(\widetilde{g \circ f})^* = \tilde{f}^* \circ \tilde{g}^*$ (by Appendix A2). But then

$$I_Z \circ (g \circ f)_* = (\widetilde{g \circ f})_* \circ I_X = \tilde{g}_* \circ \tilde{f}_* \circ I_X = I_Z \circ g_* \circ f_*$$

so $(g \circ f)_* = g_* \circ f_*$ (since I_Z is the inclusion of a full subcategory) and hence $(g \circ f)^* = f^* \circ g^*$.

COROLLARY (1). *$P_X \circ R_X = P_X$; thus, if F is a presheaf on X then $F_x = (R_X F)_x$.*

Proof. The identity map i_X is $\mathbf{T}_d - \mathbf{T}$-continuous. Hence $(i_X)^* \circ R_X = R_{dX} \circ (i_X)^*$. Thus

$$P_X \circ R_X = (i_X)_* \circ R_{dX} \circ (\tilde{i}_X)^* \circ R_X = (i_X)_* \circ (i_X)^* \circ R_X \circ R_X$$
$$= (i_X)_* \circ (i_X)^* \circ R_X = P_X$$

since $R_X \circ R_X = R_X$ (Corollary 1 to Theorem 1).

COROLLARY (2). *If G is a sheaf on Y, then $(f^*G)_x = G_{f(x)}$.*

Proof. Let $\{p\}$ be a topological space consisting of a single point. Then, clearly, $\mathbf{F}(\{p\}, \mathbf{A}) = \mathbf{S}(\{p\}, \mathbf{A}) \approx \mathbf{A}$; and if $g_x : \{p\} \to X$ is the map such that $g_x(p) = x$, then with respect to this identification $(g_x)^*F = F_x$ for a sheaf F on X. Hence

$$(f^*G)_x = (g_x)^* f^* G = (f \circ g_x)^* G = (g_{f(x)})^* G = G_{f(x)}.$$

COROLLARY (3). *If F is a sheaf on X then $f_*F(Y) = F(X)$.*

Proof. The proof is dual to that of Corollary (2), using the constant map $h : Y \to \{p\}$.

§10. EXACTNESS

In an additive category the kernel (resp., cokernel) of a morphism is defined to be the difference kernel (resp., difference cokernel) of the morphism and zero. An additive functor between additive categories is called *left* (resp., *right*) *exact* if it preserves kernels (resp., cokernels). By Appendix B2 a functor with a left (resp., right) adjoint is left (resp., right) exact.

If $\mathbf{A}$ is an additive (resp., abelian—we assume Grothendieck's axioms [7]) category then so is $\mathbf{F}(\mathbf{D}, \mathbf{A})$ (by Appendix B1) and left (resp., right) limits are left (resp., right) exact. Grothendieck's axiom AB5 for an abelian category is just the requirement that for every increasingly directed category $\mathbf{D}$, dir $\lim_{\mathbf{D}}$ exists and is an *exact* functor (i.e., both left and right exact).

In this section we assume that $\mathbf{A}$ is an AB5-abelian category with direct products (hence limits) and with enough small objects. We remark in passing that in such a category it can be shown that an object G is small if and only if, given an increasingly directed family of subobjects $A_i \subset B$ and a morphism $g : G \to$ l.u.b. A_i, there is a factorization of g through some A_i.

THEOREM (4). *The functors* $\tilde{f}^*$, R_{dX}, R_X *and* f^* *are exact.*

Proof. The category of sheaves is a full subcategory of the abelian category of presheaves and hence is an additive category, but we have not yet shown that it is an abelian category so we may not use the usual characterizations of left and right exactness of functors in terms of preserving certain short exact sequences which are only valid when both the domain and the range category are abelian. All of the functors listed in the theorem have right adjoints (namely, f_*, I_{dX}, I_X, and f_*) so they all preserve cokernels. Hence we need only show that they preserve kernels.

(i) Since $\tilde{f}^*G(U) =$ dir lim $G(V)$ (for $U \subset f^{-1}(V)$) and since dir lim is exact, $\tilde{f}^*$ is exact.

(ii) Let $E : 0 \to F' \to F \to F''$ be a left exact sequence in the (abelian) category of presheaves on $\mathbf{T}_d$. By the characterization of limits in §4, it is sufficient to show that the sequence

$$\mathrm{Hom}_{\mathbf{S}(\mathbf{T}_d, \mathbf{A})}(G, R_{dX}(E)) \approx \prod_{x \in X} \mathrm{Hom}_{\mathbf{A}}(G\{x\}, E\{x\})$$

is left exact. But

$$E\{x\} : 0 \to F'\{x\} \to F\{x\} \to F''\{x\}$$

is left exact in $\mathbf{A}$ since exactness for presheaves means exactness for each open set in $\mathbf{T}_d$ (by Appendix B1), and $\prod_{x \in X}$ is a left exact functor. Hence R_{dX} is an exact functor.

(iii) By Appendix C4, to show that R_X is exact, we must verify three things:

(a) $P_X \circ R_X \approx P_X$. This is just Corollary (1) to Theorem (2), §9.

(b) A sheaf morphism $\phi : F \to F'$ is an equivalence if and only if $P_X\phi$ is an equivalence.

Suppose $P_X\phi$ is an equivalence. We prove first that ϕ is a sheaf epimorphism. Let Cok ϕ (resp., Cok$'\phi$) be the presheaf (resp., sheaf) cokernel of ϕ. Then it is sufficient to show that Cok$'\,\phi = R$(Cok $\phi) = 0$ (since the category of sheaves is additive). But $F \to F' \to$ Cok $\phi \to 0$ is exact so $F_x \to F'_x \to (\text{Cok } \phi)_x \to 0$ is exact in $\mathbf{A}$. Since $P_X\phi$ is an equivalence, ϕ_x is an equivalence. Hence $(\text{Cok } \phi)_x = \text{Cok}(\phi_x) = 0$, so $(\text{Cok}'\phi)_x = 0$ (since $P_X \circ R_X = P_X$) and therefore $P_X(\text{Cok}'\phi) = 0$. Since $\rho : \text{Cok}'\phi \to P_X\ \text{Cok}'\phi$ is a monomorphism, $\text{Cok}'\phi = 0$.

Now let $\psi : P_XF' \to P_XF$ be the inverse of $P_X\phi$ and let $\psi' = \psi \circ \rho_{F'} : F' \to P_XF$. Then $\psi' \circ \phi = \rho_F$ (since, after composition with the equivalence $P_X\phi$, this relation holds.) Hence in the abelian category of presheaves (cok $\rho_F) \circ \psi' \circ \phi = 0$ so (cok $\rho_F) \circ \psi' = 0$ (since ϕ is an epimorphism.) But, since ρ_F is a monomorphism in an abelian category, $\rho_F = \ker(\text{cok } \rho_F)$, so $\psi' = \rho_F \circ \psi''$ where $\psi'' : F' \to F$. It is easily checked that $\psi'' = \phi^{-1}$.

(c) $P_X = (i_X)_* \circ R_{dX} \circ (\bar{i}_X)^*$ is left exact, since $(\bar{i}_X)^*$ and R_{dX} are exact by parts (i) and (ii) above and $(i_X)_*$ is left exact since it has a left adjoint $(i_X)^*$ (by Theorem (2)).

(iv) $f^* = R_X \circ \bar{f}^* \circ I_Y$ is left exact since R_X and f^* are exact by parts (i) and (iii) above and I_Y is left exact since it has a left adjoint R_Y. f^* is automatically right exact since it has a right adjoint f_*, although this is not at all evident from its definition since I_Y is not right exact.

COROLLARY (1). **S**(**T**, **A**) *is an* AB5-*abelian category with limits and enough small objects* (i.e., **S**(**T**, **A**) *inherits everything we have assumed about* **A**.)

Proof. If **A** is AB5-abelian then so is the functor category **F**(**T**, **A**). Since R_X is exact, it follows immediately from Appendix C5 that **S**(**T**, **A**) is AB5-abelian. By Corollary (2) to Theorem (1), **S**(**T**, **A**) has limits. Finally, let $\{G_\alpha\}$ be a generating family of small objects in **A**. For each α and for each $U \in$ **T**, let $F_{\alpha,U}$ be the presheaf such that $F_{\alpha,U}(V) = G_\alpha$ if $V \subset U$ and zero otherwise, with the obvious morphism (either 0 or i_{G_α}) when $V' \subset V$. It is easily checked that the $F_{\alpha,U}$'s form a generating family of small *sheaves*.

Remark. Since the direct sum of a family of generators is a generator, **S**(**T**, **A**) is an AB5-abelian category with a generator. Hence, by Grothendieck [7], Théorème 1.10.1, **S**(**T**, **A**) has enough injectives.

COROLLARY (2). *A sequence* $0 \to F' \to F \to F'' \to 0$ *of sheaves is exact in* **S**(**T**, **A**) *if and only if*

$$0 \to F'_x \to F_x \to F''_x \to 0$$

is exact in **A** for every $x \in X$.

Proof. Let g be the map of a single point into X whose image is $x \in X$. Then it is immediate from the definitions that $g^*F = F_x$. Since g^* is exact it follows that if a sequence of sheaves is exact, then the sequence of stalk morphisms is also exact. Conversely suppose $F' \xrightarrow{i} F \xrightarrow{\phi} F''$ is a pair of morphisms such that

$$0 \to F'_x \to F_x \to F''_x \to 0$$

is exact for all $x \in X$. Then

$$0 \to P_X F' \to P_X F \to P_X F''$$

is exact as sheaves since $\prod$ is left exact and the category of sheaves is left closed in the category of presheaves. As in the proof in Appendix C5, consider the commutative diagram

$$\begin{array}{ccccccc} & & F' & \xrightarrow{i} & F & \xrightarrow{\phi} & F'' \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & P_X F' & \to & P_X F & \to & P_X F''. \end{array}$$

The vertical arrows are the monomorphisms ρ. Hence i is a monomorphism and $\phi \circ i = 0$, so there is an induced morphism $\psi : F' \to \operatorname{Ker} \phi$. Since $P_X F' = \operatorname{Ker}(P_X \phi) = P_X(\operatorname{Ker} \phi)$, $P_X \psi$ is an equivalence. Hence, by the proof of (iii, part (b)) above, ψ is an equivalence, so $F' = \operatorname{Ker} \phi$. Finally, the proof that ϕ is an epimorphism is almost identical with part (iii (b)) of the proof of Theorem (4) since ϕ_X is an epimorphism.

§11. APPLICATIONS

1. Relative injectives

Let **A** be an AB5 abelian category with limits and enough small objects. It follows from Butler and Horrocks [2], §13 that the sheaves $P_X F$ are Φ-injective where Φ is the class of all short exact sequences E of sheaves such that, for all sheaves P on the trivial topology, $\mathrm{Hom}(E,(i_X)_*P)$ is also exact. Equivalently, Φ is the class of short exact sequences which split on each stalk, since, in general, if S is the left adjoint to T then the objects of the form $T(B)$ are relatively injective for sequences E such that $S(E)$ is split exact.

2. Locally closed subspaces

Let **A** be as above. Let $A \subset X$ with $i : A \to X$ as the inclusion map. If F is a sheaf on X then i^*F is called the *sheaf induced* on A.

PROPOSITION. *If G is a sheaf on A then i_*G is a sheaf on X which induces G on A.*

Proof. One shows easily that $\psi_G : i^*i_*G \to G$ is an isomorphism on each stalk and hence an isomorphism.

PROPOSITION. *Let $A \subset X$ be locally closed.*

(i) *If F is a sheaf on X, then there is a sheaf F_A on X which induces the same sheaf on A as F and zero on $X \sim A$. If A is closed then there is an exact sequence*

$$0 \to F_{X \sim A} \to F \to F_A \to 0$$

(ii) *If G is a sheaf on A then there is a sheaf G_X on X which induces G on A and zero on $X \sim A$.*

Proof. By the preceeding proposition, (i) implies (ii). It is sufficient to prove (i) for A first closed and then open. For A closed, define $F_A = i_*i^*F$. Since $i^*i_*i^*F = i^*F$ (i_* is locally epimorphic, so this follows from Appendix A3) F_A induces the same sheaf on A as F and if $x \in A$, then $(F_A)_x = 0$ (since A is closed) so F_A induces zero on $X \sim A$. Furthermore the natural transformation $\theta_F : F \to i_*i^*F = F_A$ is an epimorphism since it is an epimorphism on each stalk. Finally, if $X \sim A$ is open, define $F_{X \sim A} = \mathrm{Ker}(F \to F_A)$. Then since i^* is an exact functor, $F_{X \sim A}$ has the desired properties.

3. Cosheaves

A *copresheaf* (resp., *cosheaf*) on X with values in **A** is a presheaf (resp., sheaf) on X with values in $\mathbf{A}^0$ (the dual category to **A**) (cf. Kultze [13] and Luft [14].)

Since the theory of sheaves is completely categorical, there is nothing to prove about cosheaves. If **A** satisfies the dual assumptions to those made above, then the category of cosheaves is a right closed, left reflective subcategory of the category of copresheaves. If, for example, **A** is AB5* then the left reflection is exact so the category of cosheaves is abelian with enough projectives. Furthermore, it follows immediately from §4 that if F

is a cosheaf then $G(U) = Hom_{\mathbf{A}}(F(U), A)$ for A a fixed object of **A**, defines a sheaf. An example of a cosheaf on a locally compact space is given by assigning to each open set the set of sections with compact supports of a fixed sheaf of groups. (See Borel and Moore [1].)

APPENDICES

A. Adjoint functors

(1). Let $S : \mathbf{A} \to \mathbf{B}$ and $T : \mathbf{B} \to \mathbf{A}$ be adjoint functors via Φ with ψ and θ the induced natural transformations (§2). Φ is determined by the other data since, if $f: S(A) \to B$, then $\Phi(f) = T(f) \circ \theta_A$ while, if $g : A \to T(B)$, then $\Phi^{-1}(g) = \psi_B \circ S(g)$. Similarly, S and T are determined on morphisms since if $f: A' \to A$ then $S(f) = \Phi\psi^1(f \circ \theta_A)$ and if $g : B \to B'$ then $T(g) = \Phi(\psi_B \circ g)$. (See Huber [11, §4] and Butler and Horrocks [2, §13.])

(2). If S^i is the left adjoint to T^i via Φ^i with induced natural transformations ψ^i and θ^i $(i = 0, 1)$ then given a natural transformation $\alpha : T^0 \to T^1$ (resp., $\beta : S^1 \to S^0$) there is a unique $\beta : S^1 \to S^0$ (resp., $\alpha : T^0 \to T^1$) making the obvious diagrams commutative; namely, $\beta_A = (\Phi^1)^{-1}(\alpha_{S^0(A)} \circ \theta^0_A)$ (resp., $\alpha_B = \Phi^1(\psi^0_B \circ \beta_{T^0(B)})$.) As a corollary, the adjoint to a functor, if it exists, is unique up to a unique equivalence. Furthermore the adjoint of a composition of two functors is the composition of their adjoints in the opposite order (Freyd [3]).

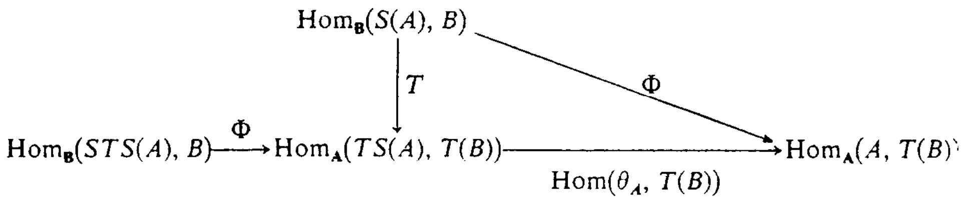
(3). If ψ and θ are given, define Φ and Φ^{-1} by the equations in (1). Then S and T are adjoint via Φ with ψ and θ as the induced natural transformations if and only if

$$(\psi * S) \circ (S * \theta) = i * S : S \to STS \to S$$

$$(T * \psi) \circ (\theta * T) = i * T : T \to TST \to T$$

Furthermore, if S and T are adjoint then S and STS (resp., T and TST) are naturally equivalent if and only if T (resp., S) is locally epimorphic. (I.E., for all $B, B' \in \mathbf{B}$, T: $\text{Hom}_{\mathbf{B}}(B, B') \to \text{Hom}_{\mathbf{A}}(T(B), T(B'))$ is surjective.

Proof. For the first part see Huber [11], §4 or Shih Weishu [15]. To prove the second part, it is sufficient, by (2)., to show that STS is a left adjoint to T. Consider



Since the diagonal Φ is a bijection, T is an injection and by hypothesis it is a surjection, hence a bijection. Hence $\text{Hom}(\theta_A, T(B))$ is a bijection. The desired natural equivalence is $\text{Hom}(\theta_A, T(B)) \circ \Phi$.

(4). Let $S : \mathbf{A} \to \mathbf{B}$ and let $\mathbf{C}$ be a fixed category. Let $S_{\mathbf{C}} : \mathbf{F}(\mathbf{C}, \mathbf{A}) \to \mathbf{F}(\mathbf{C}, \mathbf{B})$ be composition on the left with S and $S^{\mathbf{C}} : \mathbf{F}(\mathbf{B}, \mathbf{C}) \to \mathbf{F}(\mathbf{A}, \mathbf{C})$ composition on the right. (The notation $S_{\mathbf{C}}$ agrees with Kan [12], §7, but $S^{\mathbf{C}}$ does not agree with [12], §8.)

PROPOSITION. *If S is the left adjoint to T then S_C is the left adjoint to T_C and T^C is the left adjoint to S^C.*

Proof. We outline a proof of the first half whose dual is a proof of the second half. Let ψ and θ be the induced natural transformations. If $\alpha : S \circ F \to G$, define

$$\Phi(\alpha) = (T * \alpha) \circ (\theta * F) : F \to T \circ G$$

and if $\beta : F \to T \circ G$, define

$$\Phi'(\beta) = (\psi * G) \circ (S * \beta) : SF \to G.$$

The proof that $\Phi' = \Phi^{-1}$ is an immediate consequence of the five rules of functorial calculus of Godement [5], Appendix 1. For an alternative proof of the first part, see Kan [12, Theorem (12.1)].

B. Limits

(1). PROPOSITION. *If* **A** *has left or right limits of type* **D**, *then so does* **F(B, A)** *for any small category* **B**, *the limit being calculated objectwise. Hence if* **A** *is abelian* (*resp.*, AB5) *so is* *F*(**B**, **A**), *a sequence being exact if and only if it is exact on each object.*

Proof. $\mathbf{F}(\mathbf{B}, \mathbf{A}) \subset \mathbf{F}(\mathbf{D}, \mathbf{F}(\mathbf{B}, \mathbf{A})) \approx \mathbf{F}(\mathbf{D} \times \mathbf{B}, \mathbf{A}) \approx \mathbf{F}(\mathbf{B}, \mathbf{F}(\mathbf{D}, \mathbf{A}))$ the categorical equivalences being given by the usual formulas. Thus, if $L\text{lim} : \mathbf{F}(\mathbf{D}, \mathbf{A}) \to \mathbf{A}$ is adjoint to the inclusion $\mathbf{A} \subset \mathbf{F}(\mathbf{D}, \mathbf{A})$ then, by *A*4,

$$(L\text{lim})_{\mathbf{B}} : \mathbf{F}(\mathbf{B}, \mathbf{F}(\mathbf{D}, \mathbf{A})) \to \mathbf{F}(\mathbf{B}, \mathbf{A})$$

is adjoint to the above inclusion.

COROLLARY. *Let $S : \mathbf{A} \to \mathbf{B}$. Then S^C preserves limits.*

Proof. Let $D : \mathbf{D} \to \mathbf{F}(\mathbf{B}, \mathbf{C})$ be such that $L\text{lim}\ D$ exists. Since limits are computed objectwise, for any $A \in \mathbf{A}$,

$$[L\text{lim}\ S^C \circ D](A) = L\text{lim}[S^C \circ D(A)] = [L\text{lim}\ D](S(A))$$
$$= [S^C(L\text{lim}\ D)](A).$$

Similarly, S^C preserves right limits. We know of no analogous facts concerning S_C.

(2). PROPOSITION. *If $S : \mathbf{A} \to \mathbf{B}$ has a right* (*resp., left*) *adjoint then S preserves right limits and epimorphisms* (*resp., left limits and monomorphisms*).

Proof. We repeat the proof of Freyd [3] since we make crucial use of this proposition. Let T be the right adjoint to S and let $D : \mathbf{D} \to \mathbf{A}$ be such that $R\text{lim}\ D$ exists. Then for any $B \in \mathbf{B}$,

$$\text{Hom}_{\mathbf{B}}(S(R\text{lim}\ D), B) = \text{Hom}_{\mathbf{A}}(R\text{lim}\ D, T(B))$$
$$= L\text{lim}[\text{Hom}_{\mathbf{A}}(D(-), T(B))] = L\text{lim}[\text{Hom}_{\mathbf{B}}(S \circ D(-), B)]$$
$$= \text{Hom}_{\mathbf{B}}(R\text{lim}\ S \circ D, B).$$

The representable functors defined by $S(R\text{lim}\ D)$ and $R\text{lim}\ S \circ D$ being thus equivalent, it follows that the objects $S(R\text{lim}\ D)$ and $R\text{lim}(S \circ D)$ are equivalent (Grothendieck [9]).

To show that S preserves epimorphisms, recall that $f: A \to B$ is an epimorphism if and only if $\mathrm{Hom}(f, Y)$ is a monomorphism for all Y. But

$$\mathrm{Hom}_{\mathbf{B}}(S(f), Y) = \Phi^{-1} \circ \mathrm{Hom}_{\mathbf{A}}(f, T(Y)) \circ \Phi$$

so $S(f)$ is an epimorphism if f is.

COROLLARY. *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are abelian and* $S : \mathbf{A} \to \mathbf{B}$ *has a right (resp., left) adjoint then* S *is right (resp., left) exact.*

Proof. Right exactness is equivalent to preserving cokernels which are special cases of right limits.

COROLLARY. *All types of left (resp., right) limits commute with each other and a left (resp., right) limit of monomorphisms (resp., epimorphisms) is a monomorphism (resp., epimorphism). In the abelian case, left (resp., right) limits are left (resp., right) exact.*

Proof. $L\lim : \mathbf{F}(\mathbf{D}, \mathbf{A}) \to \mathbf{A}$ has the inclusion functor $\mathbf{A} \subset \mathbf{F}(\mathbf{D}, \mathbf{A})$ as a left adjoint.

(3). Let $\{D_\alpha : \mathbf{D}_\alpha \to \mathbf{A}\}$ be a collection of functors such that $L\lim D_\alpha$ exists in $\mathbf{A}$ and let $\mathbf{F}_L(\mathbf{A}, \mathbf{B})$ be the full subcategory of $\mathbf{F}(\mathbf{A}, \mathbf{B})$ determined by those functors such that $F(L\lim D_\alpha) = L\lim(F \circ D_\alpha)$ for all D_α in the collection.

PROPOSITION. $\mathbf{F}_L(\mathbf{A}, \mathbf{B})$ *is left closed* (§4) *in* $\mathbf{F}(\mathbf{A}, \mathbf{B})$.

Proof. Let $D : \mathbf{D} \to \mathbf{F}_L(\mathbf{A}, \mathbf{B})$ be such that $L\lim D$ exists in $\mathbf{F}(\mathbf{A}, \mathbf{B})$. Changing notation slightly, if $D(i) = F_i$, $i \in \mathbf{D}$ and $D_\alpha(j) = A_j$, $j \in \mathbf{D}_\alpha$, then

$$\begin{aligned} L\lim_j[(L\lim_i F_i)(A_j)] &= L\lim_j[L\lim_i(F_i(A_j))] \\ &= L\lim_i[L\lim_j(F_i(A_j))] \\ &= L\lim_i[F_i(L\lim_j A_j)] \\ &= [L\lim_i F_i](L\lim_j A_j) \end{aligned}$$

the first and last equalities since limits of functors are computed objectwise, the second since all types of left limits commute and the third since $F_i \in \mathbf{F}_L(\mathbf{A}, \mathbf{B})$. Hence $L\lim D \in \mathbf{F}_L(\mathbf{A}, \mathbf{B})$.

C. Reflective subcategories

(1). Two immediate facts are the following:

PROPOSITION. *Let* $R : \mathbf{A} \to \mathbf{A}'$ *be a right reflection. If* $\{G_\alpha\}$ *is a family of generators in* $\mathbf{A}$, *then* $\{R(G_\alpha)\}$ *is a family of generators in* $\mathbf{A}'$.

PROPOSITION. *Let* $\mathbf{A}'$ *be a full subcategory of* $\mathbf{A}$ *and* $R : \mathbf{A} \to \mathbf{A}'$ *a locally epimorphic (see* A1*) right reflection. If* $A \in \mathbf{A}$ *is injective then* $R(A) \in \mathbf{A}'$ *is injective.* (This does not apply to sheaves.)

(2). PROPOSITION. *Let* $\mathbf{A}'$ *be a full subcategory of a category* $\mathbf{A}$ *with right limits and let* $R : \mathbf{A} \to \mathbf{A}'$ *be a right reflection such that* $R \circ I_{\mathbf{A}\mathbf{A}'} \approx I_{\mathbf{A}'}$. *Then* $\mathbf{A}'$ *has right limits.*

Proof. Since R has a right adjoint, R preserves right limits. Hence

$$R[R\lim I_{\mathbf{A}\mathbf{A}'} \circ D] = R\lim[R \circ I_{\mathbf{A}\mathbf{A}'} \circ D] = R\lim D.$$

Note. The hypothesis $R \circ I_{\mathbf{AA}'} \approx I_{\mathbf{A}'}$ is a consequence of the requirement that $\mathbf{A}'$ be a full subcategory of $\mathbf{A}$. However, in the case of sheaves, this hypothesis is obviously satisfied so that we omit the proof of this fact.

(3). PROPOSITION. *Assume*

(i) $\mathbf{A}$ *has left limits and a set of generators;*

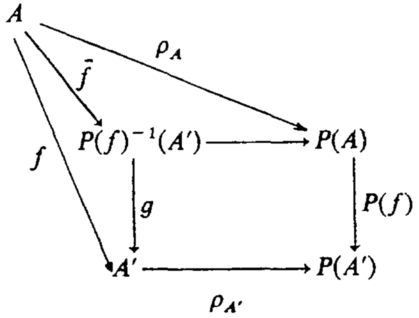
(ii) $\mathbf{A}'$ *is a full, left closed subcategory of* $\mathbf{A}$;

(iii) *There exists a functor* $P : \mathbf{A} \to \mathbf{A}'$ *and a natural transformation* $\rho : I_{\mathbf{A}} \to I_{\mathbf{AA}'} \circ P$ *such that for* $A' \in \mathbf{A}'$, $\rho_{A'}$ *is a monomorphism.*

Then there is a right reflection $R : \mathbf{A} \to \mathbf{A}'$ *with* $R(A') = A'$ *for* $A' \in \mathbf{A}'$.

Proof. Define $R(A)$ to be the intersection of all of the subobjects of $P(A)$ which belong to $\mathbf{A}'$ and through which $\rho_A : A \to P(A)$ factors. Since $\mathbf{A}$ has a set of generators, $P(A)$ has at most a set of subobjects (See §5) so $R(A)$ is well-defined. (Clearly, $R(A') = A'$ if $A' \in \mathbf{A}'$.) Since an intersection means a left limit of monomorphisms, our hypotheses imply that $R(A) \in \mathbf{A}'$, that $R(A)$ is a subobject of $P(A)$ and, by the definition of left limits, that ρ_A still factors through $R(A)$. Thus $\rho_A = \mu_A \circ r_A$, where $r_A : A \to R(A)$ and $\mu_A : R(A) \to P(A)$ is a monomorphism (in $\mathbf{A}$).

To show that $r_A : A \to R(A)$ satisfies the required universal mapping property (see §3), consider $f : A \to A'$, $A' \in \mathbf{A}'$. In the adjoining diagram $P(f)^{-1}(A')$ is the pullback of the morphisms $P(f)$ and $\rho_{A'}$. Since $\rho_{A'}$ is a monomorphism, $P(f)^{-1}(A')$ is a subobject of $P(A)$. It belongs to $\mathbf{A}'$ because A', $P(A')$ and $P(A)$ are all objects of $\mathbf{A}'$ and $\mathbf{A}'$ is left closed (a pullback is a left limit). Since ρ is a natural transformation, the outer rectangle commutes and hence there is a morphism $\bar{f} : A \to P(f)^{-1}(A')$; i.e. $P(f)^{-1}(A')$ is an $\mathbf{A}'$-subobject of $P(A)$. through which ρ_A factors. Thus $R(A) \subset P(f^{-1})(A)$. We define $f' : R(A) \to A'$ by $f' = g | R(A)$ (g as indicated in the diagram). Then, clearly, $f' \circ r_A = f$.



To show that f' is unique, suppose f'' also has this property. Let $k : K \to R(A)$ be the difference kernel of f' and f''. Then, again since $\mathbf{A}'$ is left closed, $K \in \mathbf{A}'$ and since $f' \circ r_A = f'' \circ r_A$, r_A factors through K. Since k is a monomorphism, it follows from the definition of $R(A)$ that k is an equivalence. Hence $f' = f''$.

(4). PROPOSITION. *Let* $\mathbf{A}'$ *be a subcategory of an* abelian *category* $\mathbf{A}$ *satisfying the hypotheses in* (3) *above. If, in addition,*

(i) $P \circ R \approx P$ ($\approx$ *means 'naturally equivalent'*);

(ii) *A morphism* ϕ *in* $\mathbf{A}'$ *is an equivalence if and only if* $P\phi$ *is an equivalence;*

(iii) P *is left exact.*

Then R *is exact.*

Proof. See §10 for the definitions of exactness. R preserves cokernels since it has a right adjoint. To show that R preserves kernels, let $A' = \mathrm{Ker}(A \to A'')$ (in $\mathbf{A}$). Then $P(A') = \mathrm{Ker}(P(A) \to P(A''))$ and, since $I_{\mathbf{AA}'}$ is left exact, we have a diagram

$$\begin{array}{ccccccc} & & R(A') & \to & R(A) & \to & R(A'') \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & P(A') & \to & P(A) & \to & P(A'') \end{array}$$

in $\mathbf{A}$ with the bottom row left exact. The vertical arrows are monomorphisms so $R(A') \to R(A)$ is a monomorphism and the composition in the top row is zero since R is an additive functor. Hence there is a morphism

$$\phi : R(A') \to K = \mathrm{Ker}[R(A) \to R(A'')].$$

Since $\mathbf{A}'$ is left closed, $K \in \mathbf{A}'$. Furthermore, $P\phi$ is an equivalence since

$$P(K) = \mathrm{Ker}[PR(A) \to PR(A'')] = \mathrm{Ker}[P(A) \to P(A'')] = P(A').$$

Hence, ϕ is an equivalence, so $R(A') = \mathrm{K}$.

(5). PROPOSITION. *Let* $\mathbf{A}'$ *be a full, left closed subcategory of an abelian category* $\mathbf{A}$. *If* $R : \mathbf{A} \to \mathbf{A}'$ *is an exact right reflection then* $\mathbf{A}'$ *is abelian. If, in addition,* $\mathbf{A}$ *is* AB5 *then so is* $\mathbf{A}'$.

Proof. (i) (F. W. Lawvere, unpublished). Since $\mathbf{A}'$ is full, $\mathbf{A}'$ is additive, and since it is left closed it has kernels and finite products. By C2, $\mathbf{A}'$ has cokernels. If f is a morphism in $\mathbf{A}'$, let $\mathrm{Ker}' f$ (resp., $\mathrm{Ker} f$) denote the kernel of f in $\mathbf{A}'$ (resp., $\mathbf{A}$). Similarly for $\mathrm{Cok}' f$, etc. To verify Grothendieck's axioms [7] for an abelian category, it remains to show that $\mathrm{Coim}' f = \mathrm{Im}' f$. But $\mathrm{Ker}' f = \mathrm{Ker}\, f$ and $\mathrm{Cok}' f = R[\mathrm{Cok}\, f]$. Hence

$$\mathrm{Coim}' f = \mathrm{Cok}'(\ker' f) = R[\mathrm{Cok}(\ker f)] = R[\mathrm{Coim}\, f]$$

and, since R is exact,

$$\mathrm{Im}' f = \mathrm{Ker}'(\mathrm{cok}' f) = \mathrm{Ker}'[R(\mathrm{cok}\, f)] = R[\mathrm{Ker}(\mathrm{cok}\, f)] = R[\mathrm{Im}\, f]$$

Since $\mathrm{Coim}\, f = \mathrm{Im}\, f$, we have that $\mathrm{Coim}' f = \mathrm{Im}' f$.

(ii) Now suppose $\mathbf{A}$ is AB5. Let $0 \to D' \to D \to D'' \to 0$ be an exact sequence of directed systems in $\mathbf{A}'$. The direct limit is automatically right exact and we must show it is left exact. But, since $I_{\mathbf{AA}'}$ has a left adjoint,

$$0 \to I_{\mathbf{AA}'} \circ D' \to I_{\mathbf{AA}'} \circ D \to I_{\mathbf{AA}'} \circ D''$$

is exact in $\mathbf{A}$. Hence, since $\mathbf{A}$ is AB5,

$$0 \to \mathrm{dir\,lim}\, I \circ D' \to \mathrm{dir\,lim}\, I \circ D \to \mathrm{dir\,lim}\, I \circ D''$$

is exact in $\mathbf{A}$. Since R is an exact functor, the sequence remains exact after applying R. But $R(\mathrm{dir\,lim}\, I \circ D') = \mathrm{dir\,lim}\, D'$ (by B2) etc., giving the desired result.

REFERENCES

1. A. Borel and J. C. Moore: Homology Theory for locally compact spaces, *Michigan Math. J.* **7** (1960), 137–159.
2. M. C. R. Butler and G. Horrocks: Classes of Extensions and resolutions. *Phil. Trans.* A, **254** (1961), 155–222.
3. P. J. Freyd: Functor theory, thesis, Princeton University (1960).
4. P. Gabriel: Des catégories abeliennes, thesis, Faculté des Sciences, Université de Paris (1961).
5. R. Godement: *Topologie algébrique et théorie des faisceaux*, Hermann, Paris, 1958.
6. J. W. Gray: Category-valued sheaves, *Bull. Amer. Math. Soc.* **68** (1962), 451–453.
7. A. Grothendieck: Sur quelques points d'algèbre homologique, *Tôhoku Math. J.* ser. 2, **9** (1957), pp. 119–221.
8. A. Grothendieck: Eléments de géométrie algébrique—I, *Publ. Inst. Hautes Études Sci., Paris* No. 4 (1960).
9. A. Grothendieck: Techniques de Construction en Géométrie Analytique—IV Formalisme Général des Foncteurs Représentable. Exposé 11, *Seminaire H. Cartan, Paris*, 1960–61.
10. A. Heller and K. A. Rowe: On the category of sheaves, *Amer. J. Math.* **84** (1962), 205–216.
11. P. J. Huber: Homotopy Theory in general categories, *Math. Ann.* **144** (1961), 361–385.
12. D. M. Kan: Adjoint functors, *Trans. Amer. Math. Soc.* **87** (1958), 295–329.
13. R. Kultze: Lokalholomorphe Funktionen und das Geschlecht kompakter Riemannscher Flachen, *Math. Ann.* **143** (1961), 163–186.
14. E. Luft: Eine Verallgemeinerung der Čechschen Homologietheorie, *Bonn. Math. Schr.* No. 8 (1959).
15. Shih Weishu: Ensembles simpliciaux et opérations cohomologiques, Exposé 7, *Seminaire H. Cartan, Paris*, 1958–59.

Note added in press. An earlier version of this paper appeared as *NSF Report G1 9022 M.* Without my being aware of it, much of this report was reproduced in Bourgin: *Modern Algebraic Topology*, Chapter 17, under the notion of "general sheaves". Unfortunately, in this version the exactness proofs are incomplete, and the adjointness proof for the functors on presheaves induced by a continuous map contains an incorrect statement.

J.W.G.

University of Illinois,
Urbana, Illinois, USA.