Fixed Point Theorems for Uniformly Lipschitzian Semigroups in Uniformly Convex Spaces

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1. INTRODUCTION

Let $C$ be a nonempty closed convex subset of a Banach space $X$. A mapping $T: C \rightarrow C$ is said to be uniformly $k$-Lipschitzian if

$$\|T^n x - T^n y\| \leq k \|x - y\|$$

for all $x, y$ in $C$ and each $n = 1, 2, \ldots$. Goebel and Kirk [2] first studied these mappings and they proved that such $T$ has a fixed point if $C$ is a closed bounded convex subset of a uniformly convex Banach space $X$ and $k < k^*$ ($k^*$ is the unique solution of the equation $(1 - \delta_X(1/k^*))k^* = 1$ with $\delta_X(\cdot)$ being the modulus of convexity of $X$). Lifschitz [5] showed that in a Hilbert space $H$, a uniformly $k$-Lipschitzian mapping $T$ with $k < 2^{1/2}$ has a fixed point. Lim [6, 7] and Lim et al. [8] extended Goebel–Kirk’s theorem in $L^p$ spaces. Recently, Downing and Ray [1] and Ishihara and Takahashi [3] verified that Lifschitz’s theorem is valid for uniformly Lipschitzian semigroups in Hilbert spaces.

The purpose of this paper is to prove fixed point theorems for a uniformly $k$-Lipschitzian semigroup in uniformly convex Banach spaces which is left reversible or for which the space of right uniformly continuous functions on the semigroup has a left invariant mean. The results generalize those of Downing and Ray [1], Lau [4], Lim [6, 7], and Ishihara and Takahashi [3].

2. PRELIMINARIES AND LEMMAS

Let $X$ be a Banach space. Then the modulus of convexity of $X$ is defined as $\delta_X(\epsilon) = \inf \{1 - \frac{1}{2} \|x + y\| : x, y \in B_X \text{ and } \|x - y\| \geq \epsilon\}$, where
$B_X = \{ x \in X : \| x \| \leq 1 \}$ is the closed unit ball of $X$. We recall that $X$ is said to have the modulus of convexity of power type $p \geq 2$ (and $X$ is said to be $p$-uniformly convex) if there exists a constant $c > 0$ such that

$$\delta_X(e) \geq ce^p \quad \text{for} \quad 0 < e \leq 2.$$ 

In this section we establish some inequalities which play key roles in the proofs of the main results (Theorems 1 and 2 below) of this paper, which also have applications in strongly unique best approximation (cf. [8, 11, 12]).

**Lemma 1.** Let $X$ be a uniformly convex Banach space. Then for any $p > 1$, there exists a continuous nondecreasing function $\varphi_p : [0, \infty) \to \mathbb{R}^+$ with $\varphi_p(0) = 0$ and $\varphi_p(t) > 0$ for $t > 0$ such that

$$\| tx + (1 - t)y \|^p \leq t \| x \|^p + (1 - t) \| y \|^p - W_p(t) \varphi_p(\| x - y \|) \quad (2.1)$$

for all $x, y$ in $B_X$ and $t$ in $(0, 1)$, and

$$\| tx + (1 - t)y \|^p \leq t \| x \|^p + (1 - t) \| y \|^p - W_p(t)(\| x \| \vee \| y \|)^p \varphi_p(\frac{\| x - y \|}{\| x \| \vee \| y \|}) \quad (2.2)$$

for all $x, y$ in $X$ not both zero and $t$ in $(0, 1)$, where $W_p(t) = t(1-t)^p + t^p(1-t)$ and $a \vee b = \max(a, b)$ for real numbers $a$ and $b$.

**Proof.** By results of Zalinescu [14, Theorem 4.1 and Remark 4.3], the functional $\| \cdot \|^p$ is uniformly convex on $B_X$ and therefore we have for $0 < e \leq 2$

$$\varphi_p(e) := \inf \left\{ \frac{t \| x \|^p + (1 - t) \| y \|^p - \| tx + (1 - t)y \|^p}{W_p(t)} \right\} > 0,$$

where the infimum is taken over all $x, y$ in $B_X$ with $\| x - y \| \geq e$ and $t$ in $(0, 1)$. It is easy to see that $\varphi_p(\cdot)$ is continuous and nondecreasing. We also see that the inequality (2.1) is valid for all $x, y$ in $B_X$ and $t$ in $(0, 1)$ by definition of $\varphi_p$. Now for $x, y$ in $X$ not both zero, replacing $x, y$ in (2.1) by $x/(\| x \| \vee \| y \|)$ and $y/(\| x \| \vee \| y \|)$, respectively, we obtain the inequality (2.2) and the proof is complete.

Since $X$ is $p$-uniformly convex if and only if there exists a constant $d > 0$ such that $\varphi_p(e) \geq de^p$ for $0 < e \leq 2$, we have the following

**Corollary 1** (Prus and Smarzewski [11, Lemma 2.1]). Let $X$ be a
p-uniformly convex Banach space. Then there exists a constant $d > 0$ such that
\[
\|tx + (1 - t)y\|^p \leq t\|x\|^p + (1 - t)\|y\|^p - dW_p(t)\|x - y\|^p
\]
for all $x, y$ in $X$ and $t$ in $(0, 1)$.

Now let $G$ be a semitopological semigroup; i.e., $G$ is a semigroup with a Hausdorff topology such that for each $a \in G$ the mappings $s \to a \cdot s$ and $s \to s \cdot a$ from $G$ to $G$ are continuous. Denote by $C(G)$ the Banach space of bounded continuous real valued functions on $G$. Then for $f \in C(G)$ and $a \in G$, we define $(l_a f)(s) = f(as)$ and $(r_a f)(s) = f(sa)$ for all $s \in G$. If $X$ is a closed subspace of $C(G)$ containing constants and $l_a(X) \subseteq X$ for all $a \in G$, then $m \in X^*$ is called a left invariant mean if $\|m\| = m(1) = 1$ and $m(l_a f) = m(f)$ for all $a \in G$ and $f \in X$. Let $RUC(G)$ be the space of bounded right uniformly continuous functions on $G$, i.e., all $f \in C(G)$ such that the function $a \to r_a f$ is continuous when $C(G)$ has the norm topology. Then $RUC(G)$ is a closed translation invariant subalgebra of $C(G)$ containing constants (see [9] for more details). Let $\{x_s : s \in G\}$ be a bounded family of elements of $X$. Then, as in the proof of Lau [4, Lemma 3.4], we have for each $p > 1$ and $x$ in $X$ that the function $h(s) := \|x_s - x\|^p$ is in $RUC(G)$. For a mean $m$ on $RUC(G)$, we denote by $m_s \|x_s - x\|^p$ the value of $m$ at the function $h$.

**Lemma 2.** Let $C$ be a closed convex subset of a p-uniformly convex Banach space $X$, $m$ a left invariant mean on $RUC(G)$, and $\{x_s : s \in G\}$ a bounded family of elements of $X$. Then there exists a unique element $z \in C$ such that
\[
m_s \|x_s - z\|^p \leq m_s \|x_s - x\|^p - d\|x - z\|^p
\]
for all $x \in C$, where the constant $d$ is as in Corollary 1.

**Proof.** Since $X$ is uniformly convex, there exists a unique element $z \in C$ such that
\[
f(z) = \min_{x \in C} f(x),
\]
where $f(x) = m_s \|x_s - x\|^p$. We have from Corollary 1
\[
\|z + t(x - z) - x_s\|^p \leq t\|x - x_s\|^p + (1 - t)\|z - x_s\|^p - dW_p(t)\|x - z\|^p
\]
for all $x \in C$, $t \in (0, 1)$, and $s \in G$. It then follows that
\[
0 \leq \frac{f(z + t(x - z)) - f(z)}{t} \leq f(x) - f(z) - \frac{W_p(t)}{t} d\|x - z\|^p
\]
for $x \in C$ and $t \in (0, 1)$. By taking the limit as $t \to 0$, we arrive at the desired inequality (2.4) and the proof is complete.

3. THE RESULTS

Let $C$ be a nonempty closed convex subset of a Banach space $X$, let $G$ be a semitopological semigroup, and let $\mathcal{F} = \{T_s : s \in G\}$ be a family of self-mappings of $C$ into itself. Then $\mathcal{F}$ is said to be a uniformly $k$-Lipschitzian semigroup on $C$ if the following conditions are satisfied:

(i) $T_t(x) = T_s T_t(x)$ for $t, s \in G$ and $x \in C$;

(ii) the mapping $(s, x) \mapsto T_s(x)$ from $G \times C$ into $C$ is continuous when $G \times C$ has the product topology;

(iii) $\|T_s x - T_s y\| \leq k \|x - y\|$ for $x, y \in C$ and $s \in G$.

Now we prove a fixed point theorem for uniformly $k$-Lipschitzian semigroups in a $p$-uniformly convex Banach space. In the Hilbert space setting this theorem was proved by Ishihara and Takahashi [3].

**Theorem 1.** Let $C$ be a nonempty closed convex subset of a $p$-uniformly convex Banach space $X$ and let $\mathcal{F} = \{T_s : s \in G\}$ be a uniformly $k$-Lipschitzian semigroup on $C$ with $k < (1 + d)^{1/p}$ and $d$ being as in Corollary 1. Suppose that $\{T_s x_0 : s \in G\}$ is bounded for some $x_0 \in C$ and that $RUC(G)$ has a left invariant mean. Then there exists a $z \in C$ such that $T_s z = z$ for all $s \in G$.

**Proof.** Let $m$ be a left invariant mean on $RUC(G)$. Then, by Lemma 2, we can inductively define a sequence $\{x_n\}_{n \geq 1}$ in $C$ such that

$$m_t \|T_t x_{n-1} - x_n\|^p \leq m_t \|T_t x_{n-1} - x\|^p - d \|x - x_n\|^p$$

(3.1)

for $x \in C$ and $n = 1, 2, \ldots$

Putting $x = T_s x_n$ into (3.1), we have

$$d \|T_s x_n - x_n\|^p \leq m_t \|T_t x_{n-1} - T_s x_n\|^p - m_t \|T_t x_{n-1} - x_n\|^p$$

$$= m_t \|T_t x_{n-1} - x_n\|^p - m_t \|T_t x_{n-1} - x_n\|^p$$

$$\leq (k^p - 1) m_t \|T_t x_{n-1} - x_n\|^p$$

and hence

$$m_s \|T_s x_n - x_n\|^p \leq \frac{k^p - 1}{d} m_t \|T_t x_{n-1} - x_n\|^p.$$  

(3.2)
Inserting \( x = T_s x_{n-1} \) into (3.1) and in a similar way to above, we obtain

\[
m_t \| T_s x_{n-1} - x_n \|^p \leq \frac{k^p}{1 + d} m_t \| T_s x_{n-1} - x_{n-1} \|^p.
\]

Combining (3.2) and (3.3) yields

\[
m_t \| T_s x_n - x_n \|^p \leq \frac{k^p(k^p - 1)}{d(1 + d)} m_t \| T_s x_n - x_{n-1} \|^p \cdot \ldots \cdot \leq A^p m_t \| T_s x_0 - x_0 \|^p,
\]

where \( A = \frac{k^p(k^p - 1)}{d(1 + d)} \). Since \( k < (1 + d)^{1/p} \), it follows that

\[
\| x_n - x_{n-1} \|^p \leq m_t (\| x_n - T_s x_{n-1} \| + \| T_s x_{n-1} - x_{n-1} \|)^p
\]

\[
\leq 2^{p-1} (m_t \| x_n - T_s x_{n-1} \|^p + m_t \| T_s x_{n-1} - x_{n-1} \|^p)
\]

\[
\leq 2^{p-1} \left(1 + k^p/(1 + d)\right) m_t \| T_s x_{n-1} - x_{n-1} \|^p
\]

\[
\leq 2^{p-1} \left(1 + k^p/(1 + d)\right) A^{n-1} m_t \| T_s x_0 - x_0 \|^p,
\]

which shows that \( \{x_n\} \) is Cauchy. Let \( z = \lim_{n \to \infty} x_n \). Then for each \( s \in G \) we have

\[
\| z - T_s z \|^p \leq \left(\| z - x_n \| + \| x_n - T_s x_n \| + \| T_s x_n - T_s z \|\right)^p
\]

\[
\leq (1 + k) \| z - x_n \| + \| x_n - T_s x_n \|^p
\]

\[
\leq 2^{p-1} \left((1 + k)^p \| z - x_n \|^p + \| x_n - T_s x_n \|^p\right) \to 0
\]
as \( n \to \infty \). Therefore \( T_s z = z \) for all \( s \in G \). The proof is complete.

Next by using the method similar to that in the proof of Theorem 1, we extend Downing and Ray's theorem in [1] to \( p \)-uniformly convex Banach spaces. A semitopological semigroup \( G \) is said to be left reversible if any two closed right ideals have nonvoid intersection. In this case, \( (G, \leq) \) is a directed system when the binary relation "\( \leq \)" on \( G \) is defined by \( a \leq b \) if and only if \( \{a\} \cup aG = \{b\} \cup bG \).

Let \( \{x_a : a \in S\} \) be a bounded net in a uniformly convex Banach space \( X \) and \( C \) a nonempty closed convex subset of \( X \). For a fixed \( p > 1 \), let us set

\[
r(x) = \inf_{b \in S} \sup_{a \geq b} \| x_a - x \|^p
\]

and

\[
r = \inf \{r(x) : x \in C\}.
\]
Then we have a unique point $z \in C$ (called the asymptotic center of the net \{${x_a}$\} in $C$) such that $r(z) = r$. Exactly as in the proof of Lemma 2, we have the following

**Lemma 3.** Let $X$ be a $p$-uniformly convex Banach space for some $p > 1$. Then, we have

$$r(z) \leq r(x) - d \|x - z\|^p$$

for all $x \in C$, where the constant $d$ is as in Corollary 1.

**Theorem 2.** Let $X$ be a $p$-uniformly convex Banach space for some $p > 1$, $C$ a nonempty closed convex subset of $X$, and $\mathcal{F} = \{T_s : s \in G\}$ a uniformly $k$-Lipschitzian semigroup on $C$ with $k < (1 + d)\frac{1}{p}$ and $d$ the constant in Corollary 1. Suppose that $G$ is left reversible and $\{T_s x_0 : s \in G\}$ is bounded for some $x_0$ in $C$. Then there exists an element $z \in C$ such that $T_s(z) = z$ for every $s \in G$.

*Proof.* Define a sequence $\{x_n\} \subset C$ in the following way: $x_{n+1}$ is the asymptotic center of the net $\{T_s x_n\}$ in $C$. Then, by Lemma 3, we have for $x \in C$ and $n = 1, 2, ...$

$$d \|x_{n+1} - x\|^p \leq \inf_{s \geq s} \sup_{t \geq s} \|T_t x_n - x\|^p - \inf_{s \geq s} \sup_{t \geq s} \|T_t x_n - x_{n+1}\|^p.$$  \hspace{1cm} (3.4)

Noting the inequality

$$\inf_{s \geq s} \sup_{t \geq s} \|T_t y - x\|^p \leq \inf_{s \geq s} \sup_{t \geq s} \|T_{at} y - x\|^p$$

is valid for all $x, y \in C$ and every $a \in G$, we get from (3.4)

$$d \|x_{n+1} - T_a x_{n+1}\|^p \leq \inf_{s \geq s} \sup_{t \geq s} \|T_t x_n - T_a x_{n+1}\|^p - \inf_{s \geq s} \sup_{t \geq s} \|T_t x_n - x_{n+1}\|^p \leq \inf_{s \geq s} \sup_{t \geq s} \|T_{at} x_n - T_a x_{n+1}\|^p \leq (k^p - 1) \inf_{s \geq s} \sup_{t \geq s} \|T_t x_n - x_{n+1}\|^p \leq (k^p - 1) \inf_{s \geq s} \sup_{t \geq s} \|T_t x_n - x_n\|^p.$$

Then as in the proof of Theorem 1, it follows that the sequence $\{x_n\}$ converges in norm to some element $z \in C$ for which $T_s(z) = z$ for each $s \in G$. The proof is complete.
Remark 1. Since a Hilbert space $H$ is 2-uniformly convex and the identity
\[ \|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2 \]
holds for all $x, y \in H$ and $t \in (0, 1)$, we have the following

**Corollary 2** (Downing-Ray [1] and Ishihara and Takahashi [3]). Let $C$ be a closed convex subset of a Hilbert space $H$ and let $\mathcal{F} = \{ T_s : s \in G \}$ be a uniformly $k$-Lipschitzian semigroup on $C$ with $k < 2^{1/2}$. Suppose that there is $x_0 \in C$ such that $\{ T_s x_0 \}$ is bounded and that either $G$ is left reversible or the space $RUC(G)$ has a left invariant mean. Then there is some $z \in C$ such that $T_s(z) = z$ for every $s \in G$.

Remark 2. Since for $L^p$ spaces, $1 < p < \infty$, we have (cf. [6, 8, 12])
\[ \|tx + (1 - t)y\|^q \leq t\|x\|^q + (1 - t)\|y\|^q - c_p W_q(t)\|x - y\|^q \]
for all $x, y \in L^p$ and $t \in (0, 1)$, where $q = \max(2, p)$, $W_q(t) = t^q(1 - t) + t(1 - t)^{q/2}$, and
\[ c_p = \begin{cases} \frac{(1 + b^{p-1})}{(1 + b)^{p-1}} & \text{if } 2 < p < \infty, \\ p - 1 & \text{if } 1 < p \leq 2, \end{cases} \]
with $b$ being the unique solution of the equation
\[ (p - 2) t^{p-1} + (p - 1) t^{p-2} - 1 = 0, \ t \in (0, 1). \]
As a consequence of Theorems 1 and 2, we have

**Corollary 3**. Let $C$ be a nonempty closed subset of $L^p$, $1 < p < \infty$, $G$ a semitopological semigroup which is left reversible or for which $RUC(G)$ has a left invariant mean, and $\mathcal{F} = \{ T_s : s \in G \}$ a uniformly $k$-Lipschitzian semigroup on $C$ with $k < p^{1/2}$ if $1 < p \leq 2$ or $k < (1 + (1 + b^{p-1})/(1 + b))^{1/p}$ if $2 < p < \infty$ ($b$ is as above). Suppose that there is some $x_0 \in C$ such that $\{ T_s x_0 : s \in G \}$ is bounded. Then there exists an element $z \in C$ such that $T_s(z) = z$ for each $s \in G$.

Remark 3. As remarked in [3, 4], there exist topological semigroups for which $RUC(G)$ (or even $C(G)$) has a left invariant mean and yet $G$ is not left reversible. Hence the conditions "$RUC(G)$ has a left invariant mean" and "$G$ is left reversible" are, in general, independent. Also, when $G$ is discrete, "$RUC(G)$ has a left invariant mean" implies "$G$ is left reversible," so in this case Theorem 2 implies Theorem 1.
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