Asymmetric multi-channel sampling in shift invariant spaces

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We develop an asymmetric multi-channel sampling on a shift invariant space $V(\phi)$ with a Riesz generator $\phi(t)$ in $L^2(\mathbb{R})$, where each channeled signal is assigned a uniform but distinct sampling rate. We use Fourier duality between $V(\phi)$ and $L^2[0, 2\pi]$ to find conditions under which there is a stable asymmetric multi-channel sampling formula on $V(\phi)$.

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1. Introduction

Reconstructing a signal from samples which are taken from its several channeled versions is called multi-channel sampling. The multi-channel sampling method goes back to the works of Shannon [18] and Fogel [7], where reconstruction of a band-limited signal from samples of the signal and its derivatives was suggested. Generalized sampling expansion using arbitrary multi-channel sampling on the Paley–Wiener space was introduced first by Papoulis [16]. Since Papoulis’ fundamental work, there have been many generalizations and applications of multi-channel sampling. See [1,5,6,14,15,17] and references therein.

Papoulis’ result has also been extended to a general shift invariant space by using the filter banks technique (see [4,19,20]). More recently García and Pérez-Villalón [9] derived stable generalized sampling in a shift invariant space. Most previous work related to multi-channel sampling has assumed that the sampling rates of all channels are the same.

In this paper we consider an asymmetric multi-channel sampling in a shift invariant space $V(\phi)$ with a suitable Riesz generator $\phi(t)$, where each channeled signal is sampled with a uniform but distinct rate. In Section 2, we introduce concepts and definitions needed throughout the paper. In Section 3, using Fourier duality between $V(\phi)$ and $L^2[0, 2\pi]$ [8–10], we derive a stable shifted asymmetric multi-channel sampling formula in $V(\phi)$. The corresponding symmetric multi-channel sampling in $V(\phi)$ was handled in [9], where $\phi(t)$ is a continuous Riesz generator satisfying $\sup_{t \in \mathbb{R}} \sum_{n \in \mathbb{Z}} |\phi(t - n)|^2 < \infty$. In this case all signals in $V(\phi)$ are continuous on $\mathbb{R}$ [21]. In this work, we require only that the Riesz generator $\phi(t)$ is pointwise well defined everywhere on $\mathbb{R}$ and $\sum_{n \in \mathbb{Z}} |\phi(t - n)|^2 < \infty$, $t \in \mathbb{R}$. Hence we essentially allow any Riesz generator in $L^2(\mathbb{R})$. On the other hand, we allow more general filters than the ones in [9] by asking only that the impulse responses of filters belong to $L^2(\mathbb{R})$ (or the frequency responses of filters belong to $L^2(\mathbb{R}) \cup L^\infty(\mathbb{R})$ when $\sum_{n \in \mathbb{Z}} |\hat{\phi}(\xi + 2n\pi)| \in L^2[0, 2\pi]$), whereas they belong to $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ in [9]. Finally, in Section 4, we give an illustrative example.

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2. Preliminaries

We take the Fourier transform to be normalized as
\[ \mathcal{F}[\phi](\xi) = \hat{\phi}(\xi) := \int_{-\infty}^{\infty} \phi(t)e^{-i\xi t} \, dt, \quad \phi(t) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \]
so that \( \frac{1}{\sqrt{2\pi}}\mathcal{F}[\cdot] \) extends to a unitary operator from \( L^2(\mathbb{R}) \) onto \( L^2(\mathbb{R}) \). For any \( \phi(t) \in L^2(\mathbb{R}) \), let
\[ C_\phi(t) := \sum_{n \in \mathbb{Z}} |\phi(t+n)|^2 \quad \text{and} \quad G_\phi(\xi) := \sum_{n \in \mathbb{Z}} |\hat{\phi}(\xi + 2n\pi)|^2. \]
Then \( C_\phi(t) = C_\phi(t+1) \in L^1[0,1] \), \( G_\phi(\xi) = G_\phi(\xi + 2\pi) \in L^1[0,2\pi] \) and
\[ \|\phi(t)\|_{L^2(\mathbb{R})}^2 = \|C_\phi(t)\|_{L^1[0,1]} = \frac{1}{2\pi} \|G_\phi(\xi)\|_{L^1[0,2\pi]}. \]
In particular, \( C_\phi(t) < \infty \) for a.e. \( t \in \mathbb{R} \). We also let
\[ Z_\phi(t,\xi) := \sum_{n \in \mathbb{Z}} \phi(t+n)e^{-i\xi t} \]
be the Zak transform [12] of \( \phi(t) \) in \( L^2(\mathbb{R}) \). Then \( Z_\phi(t,\xi) \) is well defined a.e. on \( \mathbb{R}^2 \) and is quasi-periodic in the sense that
\[ Z_\phi(t+1,\xi) = e^{i\xi}Z_\phi(t,\xi) \quad \text{and} \quad Z_\phi(t,\xi + 2\pi) = Z_\phi(t,\xi). \]
A Hilbert space \( \mathcal{H} \) consisting of complex valued functions on a set \( E \) is called a reproducing kernel Hilbert space (RKHS in short) if there is a function \( q(s,t) \) on \( E \times E \), called the reproducing kernel of \( \mathcal{H} \), satisfying

- \( q(.,t) \in \mathcal{H} \) for each \( t \in E \),
- \( (f(s),q(s,t)) = f(t), \ f \in \mathcal{H} \).

In an RKHS \( \mathcal{H} \), any norm converging sequence also converges uniformly on any subset of \( E \), on which \( \|q(.,t)\|_{\mathcal{H}}^2 = q(t,t) \) is bounded. A sequence \( \{\phi_n; \ n \in \mathbb{Z}\} \) of vectors in a separable Hilbert space \( \mathcal{H} \) is

- a Bessel sequence with a bound \( B \ (> 0) \) if
  \[ \sum_{n \in \mathbb{Z}} |\langle \phi, \phi_n \rangle|^2 \leq B\|\phi\|^2, \quad \phi \in \mathcal{H}; \]
- a frame of \( \mathcal{H} \) with bounds \( B \geq A \ (> 0) \) if
  \[ A\|\phi\|^2 \leq \sum_{n \in \mathbb{Z}} |\langle \phi, \phi_n \rangle|^2 \leq B\|\phi\|^2, \quad \phi \in \mathcal{H}; \]
- a Riesz basis of \( \mathcal{H} \) with bounds \( B \geq A \ (> 0) \) if it is complete in \( \mathcal{H} \) and
  \[ A\|c\|^2 \leq \sum_{n \in \mathbb{Z}} c(n)\phi_n \|c\|^2 \leq B\|c\|^2, \quad c = \{c(n)\}_{n \in \mathbb{Z}} \in l^2, \]
where \( \|c\|^2 := \sum_{n \in \mathbb{Z}} |c(n)|^2 \).

In the rest of the paper, we let \( V(\phi) \) be the shift invariant space, where \( \phi(t) \) is a Riesz generator, that is, \( \{\phi(t-n); \ n \in \mathbb{Z}\} \) is a Riesz basis of \( V(\phi) \). Then
\[ V(\phi) = \left\{ (c \ast \phi)(t) := \sum_{n \in \mathbb{Z}} c(n)\phi(t-n); \ c = \{c(n)\}_{n \in \mathbb{Z}} \in l^2 \right\}. \]
It is well known (cf. Theorem 7.2.3 in [2]) that \( \phi(t) \) is a Riesz generator if and only if there are constants \( B \geq A > 0 \) such that
\[ A \leq G_\phi(\xi) \leq B \quad \text{a.e. on} \ [0,2\pi]. \]
In this case, \( \{\phi(t-n); \ n \in \mathbb{Z}\} \) is a Riesz basis of \( V(\phi) \) with bounds \( B \geq A \). We assume further that
\( \phi(t) \) is everywhere well defined on \( \mathbb{R} \);
\( C_\phi(t) < \infty, t \in \mathbb{R}, \) i.e., \( \{\phi(t+n) : n \in \mathbb{Z}\} \in l^2 \) for each \( t \) in \( \mathbb{R} \).

We then allow essentially all Riesz generators since for any \( \phi(t) \in L^2(\mathbb{R}), C_\phi(t) < \infty \) a.e. so that \( \phi(t) \) has an equivalent representative satisfying the above two conditions. Then for each \( c \in l^2 \), \( (c \ast \phi)(t) \) converges both in \( L^2(\mathbb{R}) \) and absolutely for each \( t \) in \( \mathbb{R} \). Hence \( V(\phi) \) becomes an RKHS with the reproducing kernel (cf. Proposition 2.3 in [13])

\[
q(s, t) := \sum_{n \in \mathbb{Z}} \phi(s-n)\overline{\phi(t-n)},
\]
where \( \{\hat{\phi}(t-n) : n \in \mathbb{Z}\} \) is the dual Riesz basis of \( \{\phi(t-n) : n \in \mathbb{Z}\} \) with bounds \( \frac{1}{\pi} \geq \frac{1}{\|q\|} \). As in [9,10], we introduce an isomorphism \( \mathcal{J} \) from \( l^2[0,2\pi] \) onto \( V(\phi) \) defined as:

\[
(\mathcal{J}F)(t) := \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \langle F(\xi), e^{-int}\rangle_{L^2[0,2\pi]} \phi(t-n) = \left( F(\xi), \frac{1}{2\pi} Z_\phi(t, \xi) \right)_{L^2[0,2\pi]},
\]
We then have:

- \( (\mathcal{J}F)(\xi) = F(\xi) \hat{\phi}(\xi) \)
- \( \mathcal{J}(F(\xi)e^{-int}) = (\mathcal{J}F)(t-n), n \in \mathbb{Z} \)

3. Asymmetric multi-channel sampling

The aim of this paper is as follows. Let \( L_j[f] : 1 \leq j \leq N \) be \( N \) LTI (linear time-invariant) systems with impulse responses \( L_j(t) : 1 \leq j \leq N \). Develop a stable shifted multi-channel sampling formula for any signal \( f(t) \in V(\phi) \) using discrete sample values from \( L_j[f](t) \) \( 1 \leq j \leq N \), where each channeled signal \( L_j[f](t) \) for \( 1 \leq j \leq N \) is assigned with a distinct sampling rate:

\[
f(t) = \sum_{j=1}^{N} \sum_{n \in \mathbb{Z}} L_j[f](\sigma_j + r_j n)s_{j,n}(t), \quad f(t) \in V(\phi),
\]
where \( \{s_{j,n}(t) : 1 \leq j \leq N, n \in \mathbb{Z}\} \) is a frame or a Riesz basis of \( V(\phi) \), \( \{\sigma_j : 1 \leq j \leq N\} \) are positive integers, and \( \{\sigma_j : 1 \leq j \leq N\} \) are real constants.

Not that the shifting of sampling instants is unavoidable in some uniform sampling [12] and arises naturally when we allow rational sampling periods in (1). See Remark 3.6 below.

Here, we assume that each \( L_j[\cdot] \) is one of the following three types: the impulse response \( l(t) \) of an LTI system \( L[\cdot] \) is such that

- \( l(t) = \delta(t+a), a \in \mathbb{R} \) or
- \( l(t) \in L^2(\mathbb{R}) \) or
- \( \hat{l}(\xi) \in L^\infty(\mathbb{R}) \cup L^2(\mathbb{R}) \) when

\[
H_\phi(\xi) := \sum_{n \in \mathbb{Z}} \left| \hat{\phi}(\xi + 2n\pi) \right|^2 \in L^2[0,2\pi]
\]

For type (i), \( L[f](t) = f(t+a), f \in L^2(\mathbb{R}) \) so that \( L[\cdot] : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) is an isomorphism. In particular, for any \( f(t) = (c \ast \phi)(t) \in V(\phi), L[f](t) = (c \ast \psi)(t) \) converges absolutely on \( \mathbb{R} \) since \( C_\phi(t) = \sum_{n \in \mathbb{Z}} |\hat{\psi}(t+n)|^2 < \infty, t \in \mathbb{R} \), where \( \psi(t) := L[\phi](t) = \phi(t+a) \). For types (ii) and (iii), we have:

**Lemma 3.1.** Let \( L[\cdot] \) be an LTI system with the impulse response \( l(t) \) of the type (ii) or (iii) as above and \( \psi(t) := L[\phi](t) = (\phi \ast l)(t) \).

Then

- \( \psi(t) \in C_\infty(\mathbb{R}) := \{u(t) \in C(\mathbb{R}) : \lim_{|f| \to \infty} u(t) = 0\} \)
- \( \sup_{t \in \mathbb{R}} C_\phi(t) < \infty \)
- \( \text{for any } f(t) = (c \ast \phi)(t) \in V(\phi), L[f](t) = (c \ast \psi)(t) \) converges absolutely and uniformly on \( \mathbb{R} \). Hence \( L[f](t) \in C(\mathbb{R}) \).

**Proof.** First assume \( l(t) \in L^2(\mathbb{R}) \). Then \( \psi(t) \in C_\infty(\mathbb{R}) \) by the Riemann–Lebesgue lemma since \( \psi(t) = \hat{\phi}(\xi)\hat{l}(\xi) \in L^1(\mathbb{R}) \). Since \( \sum_{n \in \mathbb{Z}} |\hat{\psi}(\xi + 2n\pi)|^2 \leq \sup_{0 \leq \xi \leq 2\pi} G_\phi(\xi)G_l(\xi) d\xi \leq 2\pi \|G_\phi(\xi)\|_{L^\infty(\mathbb{R})} \|l\|_{L^2(\mathbb{R})}^2 \).

Thus for any \( t \in \mathbb{R} \), we have by the Poisson summation formula (cf. Lemma 5.1 in [13])
\[
\sum_{n \in \mathbb{Z}} \hat{\psi}(\xi + 2n\pi) e^{i(t+2n\pi)} = \sum_{n \in \mathbb{Z}} \psi(t + n)e^{-in\xi} \quad \text{in } L^2[0, 2\pi].
\]

Therefore for any \( t \) in \( \mathbb{R} \)
\[
C_\psi(t) = \sum_{n \in \mathbb{Z}} |\psi(t + n)|^2 = \frac{1}{2\pi} \left\| \sum_{n \in \mathbb{Z}} \psi(t + n)e^{-in\xi} \right\|_{L^2[0, 2\pi]}^2
\]
\[
= \frac{1}{2\pi} \left\| \sum_{n \in \mathbb{Z}} \hat{\psi}(\xi + 2n\pi)e^{i(t+2n\pi)} \right\|_{L^2[0, 2\pi]}^2 \leq \left\| C_\psi(\xi) \right\|_{L^\infty(\mathbb{R})} \left\| H(\xi) \right\|_{L^2(\mathbb{R})}^2.
\]

By Young's inequality on the convolution product, \( \|f \ast g\|_{L^1(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})} \) so that \( L[1] : L^2(\mathbb{R}) \rightarrow L^\infty(\mathbb{R}) \) is a bounded linear operator. Hence for any \( f(t) = (c \ast \phi)(t) = \sum_{n \in \mathbb{Z}} c(n)\phi(t - n) \in V(\phi), \)
\[
L[f](t) = \sum_{n \in \mathbb{Z}} c(n)L[\phi(t - n)] = \sum_{n \in \mathbb{Z}} c(n)\psi(t - n),
\]
which converges absolutely and uniformly on \( \mathbb{R} \) by (b). Now assume that \( H_\phi(\xi) \in L^2[0, 2\pi] \). The case \( \hat{\psi}(\xi) \in L^2(\mathbb{R}) \) is reduced to type (ii). So let \( \hat{\psi}(\xi) \in L^2(\mathbb{R}) \). Then \( \phi(\xi) \in L^1(\mathbb{R}) \) and \( H(\xi) \in L^1(\mathbb{R}) \) and so \( \psi(\xi) \in C_{\infty}(\mathbb{R}) \cap L^2(\mathbb{R}). \)

\[
C_\psi(t) = \frac{1}{2\pi} \left\| \sum_{n \in \mathbb{Z}} \hat{\psi}(\xi + 2n\pi)e^{i(t+2n\pi)} \right\|_{L^2[0, 2\pi]}^2 \leq \frac{1}{2\pi} \left\| \hat{\psi}(\xi) \right\|_{L^2(\mathbb{R})}^2 \left\| H(\xi) \right\|_{L^2(\mathbb{R})}^2
\]

so that \( \sup_{\mathbb{R}} C_\psi(t) < \infty \). For any \( f \in L^2(\mathbb{R}), \)
\[
\|L[f]\|_{L^2(\mathbb{R})} = \|f \ast H\|_{L^2(\mathbb{R})} = \frac{1}{\sqrt{2\pi}} \left\| \hat{\phi}(\xi) \right\|_{L^2(\mathbb{R})} \left\| \hat{\psi}(\xi) \right\|_{L^2(\mathbb{R})} \leq \left\| \hat{\psi}(\xi) \right\|_{L^2(\mathbb{R})} \|f\|_{L^2(\mathbb{R})}.
\]

Hence \( L[1] : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \) is a bounded linear operator. Now assume that \( \psi(t) \in L^2(\mathbb{R}) \) and \( \phi(t) \) may not converge in \( L^2(\mathbb{R}) \) unless \( \psi(t - n) : n \in \mathbb{Z} \) is a Bessect sequence. Lemma 3.1(b) improves Lemma 1 in [9], for which the proof uses \( l(t) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}), \sup_{\mathbb{R}} C_l(t) < \infty, \) and the integral version of Minkowski inequality. Note that the condition \( H_\phi(\xi) \in L^2[0, 2\pi] \) implies \( \phi(t) \in L^2(\mathbb{R}) \cap C_{\infty}(\mathbb{R}) \) and \( \sup_{\mathbb{R}} C_\phi(t) < \infty \) (see Proposition 2.4 in [13]). Note also that \( H_\phi(\xi) \in L^2[0, 2\pi] \). If \( \phi(\xi) = O((1 + |\xi|)^{-r}), r > 1, \) then holds e.g. for \( \phi_n(t) := (\phi_0 * \phi_{n-1})(t) \) the cardinal B-spline of degree \( n \geq 1, \) where \( \phi_0 := \chi_{[0,1]}(t). \)

We have as a consequence of Lemma 3.1: Let \( L[1] \) be an LTI system with impulse response \( l(t) \) of type (i) or (ii) or (iii) as above and \( \psi(t) := L[\psi](t) \). Then for any \( f(t) = (\mathcal{J}F)(t) \in V(\phi), \)
\[
F(\xi) = \sum_{j=1}^{N} \sum_{n \in \mathbb{Z}} \{F(\xi), \overline{g_j(\xi)e^{-irjn\xi}}\}_{L^2[0, 2\pi]} S_{j,n}(\xi), \quad F(\xi) \in L^2[0, 2\pi],
\]
where \( \{S_{j,n}(\xi) : 1 \leq j \leq N, n \in \mathbb{Z}\} \) is a frame or a Riesz basis of \( L^2[0, 2\pi]. \) This observation leads us to consider the problem: when is \( \{g_j(\xi)e^{-irjn\xi} : 1 \leq j \leq N, n \in \mathbb{Z}\} \) a frame or a Riesz basis of \( L^2[0, 2\pi]? \)
Note that
\[
\left\{ \hat{g}_j(\xi)e^{-ir_jn\xi} : 1 \leq j \leq N, \ n \in \mathbb{Z} \right\} = \left\{ g_{j,m}(\xi)e^{-ir_jn\xi} : 1 \leq j \leq N, \ 1 \leq m_j \leq \frac{r}{r_j}, \ n \in \mathbb{Z} \right\}
\]
where \( r := \text{lcm}(r_j : 1 \leq j \leq N) \) and \( g_{j,m}(\xi) := g_j(\xi)e^{ir_j(m_j-1)\xi} \) for \( 1 \leq j \leq N \).

Let \( D \) be the unitary operator from \( L^2[0, 2\pi] \) onto \( L^2(I)^r \), where \( I = [0, \frac{2\pi}{r}] \), defined by
\[
DF := \left[ F\left( x + (k-1)\frac{2\pi}{r_j} \right) \right]_{k=1}^T , \quad F(\xi) \in L^2[0, 2\pi].
\]
We also let
\[
G(\xi) := \left[ Dg_{1,1}(\xi), \ldots, Dg_{1,\frac{r}{r_j}}(\xi), \ldots, Dg_{N,1}(\xi), \ldots, Dg_{N,\frac{r}{r_j}}(\xi) \right]^T
\]
be a \((\sum_{j=1}^N \frac{r}{r_j}) \times r\) matrix on \( I \) and \( \lambda_m(\xi), \lambda_M(\xi) \) be the smallest and the largest eigenvalues of the positive semi-definite \( r \times r \) matrix \( G(\xi)^*G(\xi) \), respectively.

**Lemma 3.2.** Let \( \alpha_G := \|\lambda_m(\xi)\|_0 \) and \( \beta_G := \|\lambda_M(\xi)\|_\infty \) be the essential infimum of \( \lambda_m(\xi) \) and the essential supremum of \( \lambda_M(\xi) \) respectively. Then \( \{\hat{g}_j(\xi)e^{-ir_jn\xi} : 1 \leq j \leq N, \ n \in \mathbb{Z}\} \)

(a) a Bessel sequence in \( L^2[0, 2\pi] \) if and only if \( \beta_G < \infty \) or equivalently \( \{Z_\psi(\sigma_j, \xi) : 1 \leq j \leq N\} \in L^{\infty}[0, 2\pi] \);

(b) a frame of \( L^2[0, 2\pi] \) if and only if \( 0 < \alpha_G \leq \beta_G < \infty \);

(c) a Riesz basis of \( L^2[0, 2\pi] \) if and only if \( 0 < \alpha_G < \beta_G < \infty \) and \( \sum_{j=1}^N \frac{1}{r_j} = 1 \).

**Proof.** Since \( \{\hat{g}_j(\xi)e^{-ir_jn\xi} : 1 \leq j \leq N, \ n \in \mathbb{Z}\} \) is a Bessel sequence or a frame or a Riesz basis of \( L^2[0, 2\pi] \) if and only if \( \{\hat{g}_j(\xi)e^{-ir_jn\xi} : 1 \leq j \leq N, \ 1 \leq m_j \leq \frac{r}{r_j}, \ n \in \mathbb{Z}\} \) is a Bessel sequence or a frame or a Riesz basis of \( L^2[0, 2\pi] \) respectively, all of the conclusions follow from Lemma 3 in [9]. \( \square \)

Note that in [9], the authors use the Fourier transform \( \hat{f}(\xi) = \int_{-\infty}^{\infty} f(t)e^{-2\pi i t \xi} dt \) so that they use \( L^2[0, 1] \) instead of \( L^2[0, 2\pi] \).

Assume that \( 0 < \alpha_G \leq \beta_G < \infty \) so that \( \{\hat{g}_j(\xi)e^{-ir_jn\xi} : 1 \leq j \leq N, \ n \in \mathbb{Z}\} \) or equivalently \( \{\hat{g}_{j,m}(\xi)e^{-ir_jn\xi} : 1 \leq j \leq N, \ 1 \leq m_j \leq \frac{r}{r_j}, \ n \in \mathbb{Z}\} \) is a frame of \( L^2[0, 2\pi] \). Then we can show easily (see the arguments in the proof of Theorem 2 in [9]) that \( \{\hat{g}_{j,m}(\xi)e^{-ir_jn\xi} : 1 \leq j \leq N, \ 1 \leq m_j \leq \frac{r}{r_j}, \ n \in \mathbb{Z}\} \) has a dual frame of the form \( \{S_j(\xi)e^{-ir_jn\xi} : 1 \leq j \leq N, \ 1 \leq m_j \leq \frac{r}{r_j}, \ n \in \mathbb{Z}\} \) with \( S_{j,m}(\xi) \in L^{\infty}[0, 2\pi] \) for \( 1 \leq j \leq N \) and \( 1 \leq m_j \leq \frac{r}{r_j} \) satisfying
\[
[D_{S1,1}(\xi), \ldots, D_{S1,\frac{r}{r_j}}(\xi), \ldots, D_{SN,1}(\xi), \ldots, D_{SN,\frac{r}{r_j}}(\xi)] = \frac{r}{2\pi} \left[ G(\xi)^\dagger + B(\xi)(I - G(\xi)G(\xi)^\dagger) \right],
\]
where \( G(\xi)^\dagger := [G(\xi)^*G(\xi)]^{-1}G(\xi)^* \) is the pseudo-inverse of \( G(\xi) \), \( B(\xi) \) is any \( r \times \sum_{j=1}^N \frac{r}{r_j} \) matrix with entries in \( L^{\infty}(I) \), and \( I \) is the \((\sum_{j=1}^N \frac{r}{r_j}) \times (\sum_{j=1}^N \frac{r}{r_j})\) identity matrix. In particular, when we choose \( B(\xi) = 0 \) in (5), we have the canonical dual frame of the form \( \{\hat{g}_{j,m}(\xi)e^{-ir_jn\xi} : 1 \leq j \leq N, \ 1 \leq m_j \leq \frac{r}{r_j}, \ n \in \mathbb{Z}\} \).

We are now ready to give the main results of this paper. We first discuss the sampling expansion (1), which is a frame expansion in \( V(\phi) \).

**Theorem 3.3.** Let \( \alpha_G \) and \( \beta_G \) be the same as in Lemma 3.2. Assume \( \beta_G < \infty \). Then the following are all equivalent.

(a) There is a frame \( \{s_{j,m}(t - rm) : 1 \leq j \leq N, \ 1 \leq m_j \leq \frac{r}{r_j}, \ n \in \mathbb{Z}\} \) of \( V(\phi) \) for which
\[
f(t) = \sum_{j=1}^N \sum_{m_j=1}^{r/r_j} \sum_{n \in \mathbb{Z}} L_j[f](\sigma_j + r_j(m_j - 1) + rm)s_{j,m}(t - rm), \quad f(t) \in V(\phi).
\]
(b) There is a frame \( \{s_{j,n}(t) : 1 \leq j \leq N, \ n \in \mathbb{Z}\} \) of \( V(\phi) \) for which
\[
f(t) = \sum_{j=1}^N \sum_{n \in \mathbb{Z}} L_j[f](\sigma_j + rjn)s_{j,n}(t), \quad f(t) \in V(\phi).
\]
(c) $0 < \alpha_C$.

**Proof.** Assume $\beta_G < \infty$. Then by Lemma 3.2 $\{\overline{g_j(\xi)}e^{-ir_n\xi}\}, \ 1 \leq j \leq N, \ n \in \mathbb{Z}$ is a Bessel sequence in $L^2[0,2\pi]$. First (a) implies (b) trivially. Assume (b). Applying the isomorphism $\mathcal{J}^{-1}$ to (7) gives (3)

$$F(\xi) = \sum_{j=1}^{N} \sum_{n \in \mathbb{Z}} \langle F(\xi), \overline{g_j(\xi)e^{-ir_n\xi}} \rangle_{L^2[0,2\pi]} S_{j,n}(\xi), \quad F(\xi) \in L^2[0,2\pi],$$

where $\{S_{j,n}(\xi), 1 \leq j \leq N, \ n \in \mathbb{Z}\}$ is a frame of $L^2[0,2\pi]$. Then the Bessel sequence $\{\overline{g_j(\xi)}e^{-ir_n\xi}\}, \ 1 \leq j \leq N, \ n \in \mathbb{Z}$ is in fact a dual frame of $\{S_{j,n}(\xi), 1 \leq j \leq N, \ n \in \mathbb{Z}\}$ (cf. Lemma 5.6.2 in [2]). Hence (c) must hold by Lemma 3.2. Finally assume (c). Then $0 < \alpha_C \leq \beta_G < \infty$ so that $\{\overline{g_j(\xi)}e^{-ir_n\xi}\}, \ 1 \leq j \leq N, \ 1 \leq m_j \leq \frac{r_j}{n}, \ n \in \mathbb{Z}$ is a frame of $L^2[0,2\pi]$. Then we have a frame expansion on $L^2[0,2\pi]$:

$$F(\xi) = \sum_{j=1}^{N} \sum_{m_j=1}^{r_j} \sum_{n \in \mathbb{Z}} \langle F(\xi), \overline{g_j(\xi)e^{-ir_n\xi}} \rangle_{L^2[0,2\pi]} S_{j,m_j}(\xi)e^{-ir_n\xi}, \quad F(\xi) \in L^2[0,2\pi].$$

(8)

where $S_{j,m_j}(\xi)$'s are given by (5). Then the sampling expansion (6) comes from (8) by applying the isomorphism $\mathcal{J}$ since

$$\langle F(\xi), \overline{g_j(\xi)e^{-ir_n\xi}} \rangle_{L^2[0,2\pi]} = \left(F(\xi), \frac{1}{2\pi} Z_{\phi_j}(\sigma_j + r_j(m_j - 1) + rn, \xi) \right)_{L^2[0,2\pi]}$$

$$= L_j[f](\sigma_j + r_j(m_j - 1) + rn)$$

for $(\mathcal{J}F)(t) = f(t)$. $\square$

Note that when $0 < \alpha_C \leq \beta_G < \infty$, the sampling series (6) converges not only in $L^2(\mathbb{R})$ but also uniformly on any subset of $\mathbb{R}$, on which $C_0(t)$ is bounded. Moreover since $\alpha_C > 0$, the rank of $G(\xi)$ is $r$ a.e. so that $1 \leq \sum_{j=1}^{N} \frac{1}{r_j}$, which means that the total sampling rate $\sum_{j=1}^{N} \frac{1}{r_j}$ of the sampling expansion (6) must be at least 1, the Nyquist sampling rate for signals in $V(\phi)$.

In the extreme case we have:

**Theorem 3.4.** Let $\alpha_C$ and $\beta_G$ be the same as in Lemma 3.2. Then there is a Riesz basis $\{s_{j,m_j,n}(t), 1 \leq j \leq N, 1 \leq m_j \leq \frac{r_j}{n}, \ n \in \mathbb{Z}\}$ of $V(\phi)$ for which

$$f(t) = \sum_{j=1}^{N} \sum_{m_j=1}^{r_j} \sum_{n \in \mathbb{Z}} L_j[f](\sigma_j + r_j(m_j - 1) + rn)s_{j,m_j,n}(t), \quad f(t) \in V(\phi)$$

(9)

if and only if $0 < \alpha_C \leq \beta_G < \infty$ and $\sum_{j=1}^{N} \frac{1}{r_j} = 1$.

In this case, we also have:

- $s_{j,m_j,n}(t) = s_{j,m_j}(t - rn)$ for $1 \leq j \leq N, 1 \leq m_j \leq \frac{r_j}{n}$, and $n \in \mathbb{Z}$;
- $L_j[s_{k,m_n}](\sigma_j + r_j(m_j - 1) + rn) = \delta_{j,k}\delta_{m,n}$ for $1 \leq j, k \leq N$ and $n \in \mathbb{Z}$.

**Proof.** Assume $0 < \alpha_C \leq \beta_G < \infty$ and $\sum_{j=1}^{N} \frac{1}{r_j} = 1$. Then by Lemma 3.2, $\{\overline{g_j(\xi)}e^{-ir_n\xi}\}, \ 1 \leq j \leq N, \ 1 \leq m_j \leq \frac{r_j}{n}, \ n \in \mathbb{Z}$ is a Riesz basis of $L^2[0,2\pi]$. Then we have

$$F(\xi) = \sum_{j=1}^{N} \sum_{m_j=1}^{r_j} \sum_{n \in \mathbb{Z}} \langle F(\xi), \overline{g_j(\xi)e^{-ir_n\xi}} \rangle_{L^2[0,2\pi]} S_{j,m_j}(\xi)e^{-ir_n\xi}, \quad F(\xi) \in L^2[0,2\pi].$$

(10)

where $\{S_{j,m_j}(\xi)e^{-ir_n\xi} : 1 \leq j \leq N, \ 1 \leq m_j \leq \frac{r_j}{n}, \ n \in \mathbb{Z}\}$ is the dual of $\{\overline{g_j(\xi)}e^{-ir_n\xi}, 1 \leq j \leq N, \ 1 \leq m_j \leq \frac{r_j}{n}, \ n \in \mathbb{Z}\}$. Applying the isomorphism $\mathcal{J}$ to (10) gives (9), where $s_{j,m_j,n}(t) = \mathcal{J}(S_{j,m_j}(\xi)e^{-ir_n\xi}) = s_{j,m_j}(t - rn)$ and $\mathcal{J}(S_{j,m_j}(\xi)) = s_{j,m_j}(t)$. Conversely assume that the Riesz basis expansion (9) holds on $V(\phi)$. Applying the isomorphism $\mathcal{J}^{-1}$ to (9) gives

$$F(\xi) = \sum_{j=1}^{N} \sum_{m_j=1}^{r_j} \sum_{n \in \mathbb{Z}} \langle F(\xi), \overline{g_j(\xi)e^{-ir_n\xi}} \rangle_{L^2[0,2\pi]} \mathcal{J}^{-1}(s_{j,m_j,n})(\xi), \quad F(\xi) \in L^2[0,2\pi].$$
which is a Riesz basis expansion on $L^2[0, 2\pi]$. Then $\{\overline{g_{j,m}}(\xi)e^{-in\xi}: 1 \leq j \leq N, 1 \leq m_j \leq \frac{r_j}{\ell}, n \in \mathbb{Z}\}$ must be a Riesz basis of $L^2[0, 2\pi]$ so that $0 < \alpha_G \leq \beta_G < \infty$ and $\sum_{j=1}^{N} \frac{1}{\ell} = 1$ by Lemma 3.2. As the dual Riesz basis of $\overline{g_{j,m}(\xi)e^{-in\xi}}: 1 \leq j \leq N, 1 \leq m_j \leq \frac{r_j}{\ell}, n \in \mathbb{Z}$, $\{J^{-1}_{\xi}(s_{j,m,n}(t))$: $1 \leq j \leq N, 1 \leq m_j \leq \frac{r_j}{\ell}, n \in \mathbb{Z}\}$ must be of the form $\{S_{j,m}(\xi)e^{in\xi}: 1 \leq j \leq N, 1 \leq m_j \leq \frac{r_j}{\ell}\}$ satisfy Eq. (5) with $B(\xi) = 0$. Hence

$$s_{j,m,n}(t) = J(S_{j,m}(\xi)e^{in\xi}) = s_{j,m}(t - mn), \quad 1 \leq j \leq N \text{ and } n \in \mathbb{Z}.$$ 

Finally, we have

$$s_{k,m}(t) = \sum_{j=1}^{N} \sum_{m_j=1}^{\frac{r_j}{\ell}} L_j[s_{k,m}]_j(\sigma_j + r_j(m_j - 1) + mn)s_{j,m}(t - mn)$$

so that $L_j[s_{k,m}]_j(\sigma_j + r_j(m_j - 1) + mn) = \delta_{j,k}\delta_{n,0}$. \hfill \Box

When $N = 1$, write $L_1[\cdot]$, $l(t)$, $\sigma_1$, $r_1$, and $\psi_1(t)$ as $L[\cdot]$, $l(t)$, $\sigma$, $r$, and $\psi(t)$.

**Corollary 3.5.** (Cf. Theorem 3.1 in [11].) Let $N = 1$. Then there is a Riesz basis $\{s_n(t): n \in \mathbb{Z}\}$ of $V(\phi)$ such that

$$f(t) = \sum_{n \in \mathbb{Z}} L[f](\sigma + rn)s_n(t), \quad f(t) \in V(\phi) \tag{11}$$

if and only if $r = 1$ and

$$0 < \|Z_\psi(\sigma, \xi)\|_0 < \|Z_\psi(\sigma, \xi)\|_\infty < \infty. \tag{12}$$

In this case, we also have:

- $s_n(t) = s(t - n), \quad n \in \mathbb{Z}$;
- $\hat{s}(\xi) = \frac{\hat{\phi}(\xi)}{Z_\psi(\sigma, \xi)}$;
- $L[s](\sigma + n) = \delta_{n,0}, \quad n \in \mathbb{Z}$.

**Proof.** Note that for $r = 1$, $G(\xi) = \frac{1}{2\pi} Z_\psi(\sigma, \xi)$ and $\lambda_m(\xi) = \lambda_M(\xi) = (\frac{1}{2\pi})^2|Z_\psi(\sigma, \xi)|^2$ so that $0 < \alpha_G \leq \beta_G < \infty$ if and only if (12) holds. Therefore, everything except (13) follows from Theorem 3.4. Finally applying (11) to $\phi(t)$ gives

$$\phi(t) = \sum_{n \in \mathbb{Z}} \psi(\sigma + n)s(t - n)$$

from which we have (13) by taking the Fourier transform. \hfill \Box

When $l(t) = \delta(t)$ so that $L[\cdot]$ is the identity operator, Corollary 3.5 reduces to a regular shifted sampling on $V(\phi)$ (see Theorem 3.3 in [13]).

**Remark 3.6.** In (1), we may allow rational sampling periods. If $r_j = \frac{p_j}{q_j}$, where $p_j$ and $q_j$ are coprime positive integers, then

$$\left\{L_j[f](\sigma_j + r_jn): n \in \mathbb{Z}\right\} = \left\{L_j[f](\sigma_j + r_j(k - 1) + p_jn): 1 \leq k \leq q_j, n \in \mathbb{Z}\right\}.$$ 

Hence the case of rational sampling periods $\{r_j\}_{j=1}^{N}$ can be reduced to the case of integer sampling periods $\{p_j\}_{j=1}^{N}$ by extending the number of LTI systems involved.

For example when $N = 1$, we have:

**Corollary 3.7.** Let $N = 1$ and $q (\geq 2)$ be an integer. Assume $Z_\psi(\sigma, \xi) \in L^\infty[0, 2\pi], 1 \leq j \leq q$, where $\sigma_j := \sigma + \frac{1}{q}(j - 1)$. Then the following are all equivalent.

(a) There is a frame $\{s_n(t): n \in \mathbb{Z}\}$ of $V(\phi)$ for which

$$f(t) = \sum_{n \in \mathbb{Z}} L[f](\sigma + \frac{1}{q}n)s_n(t), \quad f(t) \in V(\phi).$$
(b) There is a frame \( \{s_j(t-n) : 1 \leq j \leq q, \, n \in \mathbb{Z} \} \) of \( V(\phi) \) for which
\[
f(t) = \sum_{j=1}^{q} \sum_{n \in \mathbb{Z}} L[f](\sigma_j + n)s_j(t-n), \quad f(t) \in V(\phi).
\]

(c) \( \| \sum_{j=1}^{q} |Z(\sigma_j, \xi)| \|_0 > 0 \).

**Proof.** Since \( \{ L[f](\sigma + \frac{1}{q} n) : n \in \mathbb{Z} \} = \{ L[f](\sigma + n) : 1 \leq j \leq q, \, n \in \mathbb{Z} \} \), we have a shifted symmetric multi-channel sampling for \( q \) LTI systems \( \{ L_j[-] : 1 \leq j \leq q \} \) with \( L_j[-] = L[-1], \, 1 \leq j \leq q \). Then \( g_j(\xi) = \frac{1}{2\pi} Z(\sigma_j, \xi), \, 1 \leq j \leq q \) and \( G(\xi)^*G(\xi) = \frac{1}{2\pi} \sum_{j=1}^{q} |Z(\sigma_j, \xi)|^2 \). Hence \( \alpha_G > 0 \) if and only if \( \| \sum_{j=1}^{q} |Z(\sigma_j, \xi)| \|_0 > 0 \). Therefore, Corollary 3.7 is a consequence of Theorem 3.3. \( \square \)

4. Example

Let \( \phi := \chi_{[0,1)}(t) \) be the Haar scaling function and
\[\phi_1(t) = (\phi \ast \phi_0)(t) = t \chi_{[0,1)}(t) + (2 - t) \chi_{(1,2)}(t)\]
a B-spline of degree 1. Then \( \phi_1(t) \) is a continuous Riesz generator [3] and \( \sup_{\mathbb{R}} \mathcal{C}_{\phi_1}(t) = \sup_{\mathbb{R}} \sum_{n \in \mathbb{Z}} |\phi_1(t + n)|^2 < \infty \). First we take \( N = 2, \, \sigma_1 = \sigma_2 = 0, \, r_1 = 1, \, r_2 = 2 \), and two LTI systems \( L_1[-] \) and \( L_2[-] \) with impulse responses \( l_1(t) = \chi_{[-\frac{1}{2},0)}(t) \) and \( l_2(t) = \chi_{[-1,-\frac{1}{2})}(t) \). Then it’s easy to see that
\[
g_1(\xi) = \frac{1}{2\pi} Z_{\phi_1}(0, \xi) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \psi_1(n)e^{-im\xi} = \frac{1}{16\pi}(1 + 3e^{-i\xi}),
\]
\[
g_2(\xi) = \frac{1}{2\pi} Z_{\phi_2}(0, \xi) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \psi_2(n)e^{-im\xi} = \frac{1}{16\pi}(3 + e^{-i\xi}),
\]
where \( \psi_j(t) = L_j[\phi](t) \). Hence \( g_{1,1}(\xi) = g_1(\xi), \, g_{1,2}(\xi) = g_1(\xi)e^{i\xi}, \, g_{2,1}(\xi) = g_2(\xi) \) so that (cf. (4))
\[
G(\xi) = [Dg_{1,1}, \, Dg_{1,2}, \, Dg_{2,1}]^T = \frac{1}{16\pi}\begin{bmatrix}1 + 3e^{-i\xi} & 1 - 3e^{-i\xi} \\3 + e^{i\xi} & 3 - e^{i\xi} \\3 + e^{-i\xi} & 3 - e^{-i\xi}\end{bmatrix}
\]
and
\[
G(\xi)^*G(\xi) = \frac{1}{(16\pi)^2}\begin{bmatrix}30 + 18 \cos \xi & 8 + 6i \sin \xi & 30 + 18 \cos \xi \\8 - 6i \sin \xi & 8 - 6i \sin \xi & 30 - 18 \cos \xi\end{bmatrix}.
\]
The eigenvalues of \( G(\xi)^*G(\xi) \) are \( \frac{1}{(16\pi)^2}[30 + 18 \cos \xi \pm \sqrt{100 - 36 \cos^2 \xi}] \) so that \( \frac{2}{(16\pi)^2} \leq \alpha_G = \|\lambda_m(\xi)\|_0 < \beta_G = \|\lambda_M(\xi)\|_\infty \leq \frac{58}{(16\pi)^2} \). Hence by Theorem 3.3, there is a frame \( \{s_j(t-2n) : j = 1, 2, 3 \text{ and } n \in \mathbb{Z}\} \) of the space of linear splines \( V(\phi_1) \) for which the following asymmetric multi-channel sampling expansion holds:
\[
f(t) = \sum_{n \in \mathbb{Z}} \{L_1[f](2n)s_1(t-2n) + L_1[f](2n+1)s_2(t-2n) + L_2[f](2n)s_3(t-2n)\}, \quad f(t) \in V(\phi_1),
\]
which converges in \( L^2(\mathbb{R}) \) and absolutely and uniformly on \( \mathbb{R} \).

We now take \( N = 1 \) and \( l(t) = \delta(t) \) so that \( L[-] \) is the identity operator. Let \( q \geq 1 \) be an integer and \( 0 \leq \alpha < \frac{1}{q} \). Note first that for any fixed \( t \in \mathbb{R} \), \( Z_\phi(t, \xi) = \sum_{n \in \mathbb{Z}} \phi(t + n)e^{-in\xi} \in C[0, 2\pi] \) since \( \phi(t) \) has compact support. Hence \( \|Z_\phi(t, \xi)\|_{L^0[0,2\pi]} < \infty \) for each \( t \in \mathbb{R} \). Since \( Z_\phi(\sigma, \xi) = \sigma(1 - \sigma)e^{-in\xi} \) for \( 0 \leq \sigma < 1 \), \( \|Z_\phi(\sigma, \xi)\|_0 = 2|\sigma - \frac{1}{2}| \) and \( \|Z_\phi(\sigma, \xi)\|_\infty = 1 \). Therefore, by Corollary 3.5, for any \( \sigma \) with \( 0 \leq \sigma < 1 \), there is a Riesz basis \( \{s(t-n) : n \in \mathbb{Z}\} \) of \( V(\phi) \) such that
\[
f(t) = \sum_{n \in \mathbb{Z}} f(\sigma + n)s(t-n), \quad f(t) \in V(\phi)
\]
if and only if \( \sigma \neq \frac{1}{2} \). On the other hand, by Corollary 3.7, for any \( q \geq 2 \) and any \( \sigma \) with \( 0 \leq \sigma < \frac{1}{q} \), there is a frame \( \{s_j(t-n) : 1 \leq j \leq q, \, n \in \mathbb{Z}\} \) such that
\[
f(t) = \sum_{j=1}^{q} \sum_{n \in \mathbb{Z}} f\left(\sigma + \frac{1}{q}(j-1) + n\right)s_j(t-n), \quad f(t) \in V(\phi_1).
\]
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References