Asymmetric multi-channel sampling in shift invariant spaces

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ARTICLE INFO

Article history:
Received 11 August 2009
Available online 11 December 2009
Submitted by P.G. Lemarie-Rieusset

Keywords:
Shift invariant space
Multi-channel sampling
Frame
Riesz basis

ABSTRACT

We develop an asymmetric multi-channel sampling on a shift invariant space \( V(\phi) \) with a Riesz generator \( \phi(t) \) in \( L^2(\mathbb{R}) \), where each channeled signal is assigned a uniform but distinct sampling rate. We use Fourier duality between \( V(\phi) \) and \( L^2[0,2\pi] \) to find conditions under which there is a stable asymmetric multi-channel sampling formula on \( V(\phi) \).

1. Introduction

Reconstructing a signal from samples which are taken from its several channeled versions is called multi-channel sampling. The multi-channel sampling method goes back to the works of Shannon [18] and Fogel [7], where reconstruction of a band-limited signal from samples of the signal and its derivatives was suggested. Generalized sampling expansion using arbitrary multi-channel sampling on the Paley–Wiener space was introduced first by Papoulis [16]. Since Papoulis’ fundamental work, there have been many generalizations and applications of multi-channel sampling. See [1,5,6,14,15,17] and references therein.

Papoulis’ result has also been extended to a general shift invariant space by using the filter banks technique (see [4,19,20]). More recently García and Pérez-Villalón [9] derived stable generalized sampling in a shift invariant space. Most previous work related to multi-channel sampling has assumed that the sampling rates of all channels are the same.

In this paper we consider an asymmetric multi-channel sampling in a shift invariant space \( V(\phi) \) with a suitable Riesz generator \( \phi(t) \), where each channeled signal is sampled with a uniform but distinct rate. In Section 2, we introduce concepts and definitions needed throughout the paper. In Section 3, using Fourier duality between \( V(\phi) \) and \( L^2[0,2\pi] \) [8–10], we derive a stable shifted asymmetric multi-channel sampling formula in \( V(\phi) \). The corresponding symmetric multi-channel sampling in \( V(\phi) \) was handled in [9], where \( \phi(t) \) is a continuous Riesz generator satisfying \( \sup_{\mathbb{R}} \sum_{n \in \mathbb{Z}} |\phi(t-n)|^2 < \infty \). In this case all signals in \( V(\phi) \) are continuous on \( \mathbb{R} \) [21]. In this work, we require only that the Riesz generator \( \phi(t) \) is pointwise well defined everywhere on \( \mathbb{R} \) and \( \sum_{n \in \mathbb{Z}} |\phi(t-n)|^2 < \infty, \, t \in \mathbb{R} \). Hence we essentially allow any Riesz generator in \( L^2(\mathbb{R}) \). On the other hand, we allow more general filters than the ones in [9] by asking only that the impulse responses of filters belong to \( L^2(\mathbb{R}) \) (or the frequency responses of filters belong to \( L^2(\mathbb{R}) \)) when \( \sum_{n \in \mathbb{Z}} |\phi(t+2\pi n)| \in L^2[0,2\pi] \), whereas they belong to \( L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \) in [9]. Finally, in Section 4, we give an illustrative example.

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2. Preliminaries

We take the Fourier transform to be normalized as
\[ \mathcal{F} \{ \phi \} (\xi) = \hat{\phi}(\xi) := \int_{-\infty}^{\infty} \phi(t) e^{-it\xi} \, dt, \quad \phi(t) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \]
so that \( \frac{1}{\sqrt{2\pi}} \mathcal{F} \{ \cdot \} \) extends to a unitary operator from \( L^2(\mathbb{R}) \) onto \( L^2(\mathbb{R}) \). For any \( \phi(t) \in L^2(\mathbb{R}) \), let
\[ C_\phi(t) := \sum_{n \in \mathbb{Z}} |\phi(t + n)|^2 \quad \text{and} \quad G_\phi(\xi) := \sum_{n \in \mathbb{Z}} |\hat{\phi}(\xi + 2n\pi)|^2. \]
Then \( C_\phi(t) = C_\phi(t + 1) \in L^1[0, 1] \), \( G_\phi(\xi) = G_\phi(\xi + 2\pi) \in L^1[0, 2\pi] \) and
\[ \left\| \phi(t) \right\|_{L^2(\mathbb{R})}^2 = \left\| C_\phi(t) \right\|_{L^1[0,1]} = \frac{1}{2\pi} \left\| G_\phi(\xi) \right\|_{L^1[0,2\pi]} \]
In particular, \( C_\phi(t) < \infty \) for a.e. \( t \) in \( \mathbb{R} \). We also let
\[ Z_\phi(t, \xi) := \sum_{n \in \mathbb{Z}} \phi(t + n)e^{-in\xi} \]
be the Zak transform \([12]\) of \( \phi(t) \) in \( L^2(\mathbb{R}) \). Then \( Z_\phi(t, \xi) \) is well defined a.e. on \( \mathbb{R}^2 \) and is quasi-periodic in the sense that
\[ Z_\phi(t + 1, \xi) = e^{it\xi} Z_\phi(t, \xi) \quad \text{and} \quad Z_\phi(t, \xi + 2\pi) = Z_\phi(t, \xi). \]
A Hilbert space \( \mathcal{H} \) consisting of complex valued functions on a set \( E \) is called a reproducing kernel Hilbert space (RKHS in short) if there is a function \( q(s, t) \) on \( E \times E \), called the reproducing kernel of \( \mathcal{H} \), satisfying
- \( q(., t) \in \mathcal{H} \) for each \( t \) in \( E \),
- \( (f(s), q(s, t)) = f(t), f \in \mathcal{H} \).

In an RKHS \( \mathcal{H} \), any norm converging sequence also converges uniformly on any subset of \( E \), on which \( \left\| q(., t) \right\|_{\mathcal{H}}^2 = q(t, t) \) is bounded. A sequence \( \{\phi_n; n \in \mathbb{Z}\} \) of vectors in a separable Hilbert space \( \mathcal{H} \) is
- a Bessel sequence with a bound \( B (> 0) \) if
  \[ \sum_{n \in \mathbb{Z}} |\langle \phi, \phi_n \rangle|^2 \leq B \|\phi\|^2, \quad \phi \in \mathcal{H}; \]
- a frame of \( \mathcal{H} \) with bounds \( B \geq A (> 0) \) if
  \[ A \|\phi\|^2 \leq \sum_{n \in \mathbb{Z}} |\langle \phi, \phi_n \rangle|^2 \leq B \|\phi\|^2, \quad \phi \in \mathcal{H}; \]
- a Riesz basis of \( \mathcal{H} \) with bounds \( B \geq A (> 0) \) if it is complete in \( \mathcal{H} \) and
  \[ A \|\mathbf{c}\|^2 \leq \sum_{n \in \mathbb{Z}} c(n) \phi_n \|^2 \leq B \|\mathbf{c}\|^2, \quad \mathbf{c} = \{c(n)\}_{n \in \mathbb{Z}} \in l^2, \]
where \( \|\mathbf{c}\|^2 := \sum_{n \in \mathbb{Z}} |c(n)|^2 \).

In the rest of the paper, we let \( V(\phi) \) be the shift invariant space, where \( \phi(t) \) is a Riesz generator, that is, \( \{\phi(t - n); n \in \mathbb{Z}\} \) is a Riesz basis of \( V(\phi) \). Then
\[ V(\phi) = \left\{ (\mathbf{c} \ast \phi)(t) := \sum_{n \in \mathbb{Z}} c(n) \phi(t - n); \mathbf{c} = \{c(n)\}_{n \in \mathbb{Z}} \in l^2 \right\}. \]
It is well known (cf. Theorem 7.2.3 in \([2]\)) that \( \phi(t) \) is a Riesz generator if and only if there are constants \( B \geq A > 0 \) such that
\[ A \leq G_\phi(\xi) \leq B \quad \text{a.e. on } [0, 2\pi]. \]
In this case, \( \{\phi(t - n); n \in \mathbb{Z}\} \) is a Riesz basis of \( V(\phi) \) with bounds \( B \geq A \). We assume further that...
We then allow essentially all Riesz generators since for any isomorphism $t$ isomorphism

$$q(s, t) := \sum_{n \in \mathbb{Z}} \hat{\phi}(s - n) \hat{\phi}(t - n),$$

where $\{\hat{\phi}(t - n): n \in \mathbb{Z}\}$ is the dual Riesz basis of $\{\phi(t - n): n \in \mathbb{Z}\}$ with bounds $\frac{1}{\alpha} \leq \frac{1}{\beta}$. As in [9,10], we introduce an isomorphism $\mathcal{J}$ from $L^2[0,2\pi]$ onto $V(\phi)$ defined as:

$$\mathcal{J}F(t) := \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \{F(\xi), e^{-in\xi}\}_{L^2[0,2\pi]} \phi(t - n) = \left( F(\xi), \frac{1}{2\pi} Z_\phi(t, \xi) \right)_{L^2[0,2\pi]}.$$

We then have:

- $\mathcal{J}F(\xi) = F(\xi) \hat{\phi}(\xi)$;
- $\mathcal{J}(F(\xi)e^{-in\xi}) = (\mathcal{J}F)(t - n), n \in \mathbb{Z}$.

### 3. Asymmetric multi-channel sampling

The aim of this paper is as follows. Let $\{L_j[\cdot]: 1 \leq j \leq N\}$ be $N$ LTI (linear time-invariant) systems with impulse responses $L_j(t): 1 \leq j \leq N$. Develop a stable shifted multi-channel sampling formula for any signal $f(t) \in V(\phi)$ using discrete sample values from $\{L_j[f](t): 1 \leq j \leq N\}$, where each channeled signal $L_j[f](t)$ for $1 \leq j \leq N$ is assigned with a distinct sampling rate:

$$f(t) = \sum_{j=1}^{N} \sum_{n \in \mathbb{Z}} L_j[f](\sigma_j + rn) s_{j,n}(t), \quad f(t) \in V(\phi),$$

where $\{s_{j,n}(t): 1 \leq j \leq N, n \in \mathbb{Z}\}$ is a frame or a Riesz basis of $V(\phi)$, $\{\sigma_j: 1 \leq j \leq N\}$ are positive integers, and $\{r_j: 1 \leq j \leq N\}$ are real constants.

Note that the shifting of sampling instants is unavoidable in some uniform sampling [12] and arises naturally when we allow rational sampling periods in (1). See Remark 3.6 below.

Here, we assume that each $L_j[\cdot]$ is one of the following three types: the impulse response $l(t)$ of an LTI system $L[\cdot]$ is such that

1. $l(t) = \delta(t + a), a \in \mathbb{R}$ or
2. $l(t) \in L^2(\mathbb{R})$ or
3. $\hat{l}(\xi) \in L^\infty(\mathbb{R}) \cup L^2(\mathbb{R})$ when $H_\phi(\xi) := \sum_{n \in \mathbb{Z}} |\hat{\phi}(\xi + 2n\pi)| \in L^2[0,2\pi]$.

For type (i), $L[f](t) = f(t + a), f \in L^2(\mathbb{R})$ so that $L[\cdot]: L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is an isomorphism. In particular, for any $f(t) = (c \ast \phi)(t) \in V(\phi)$, $L[f](t) = (c \ast \psi)(t)$ converges absolutely on $\mathbb{R}$ since $C_\phi(t) = \sum_{n \in \mathbb{Z}} |\psi(t + n)|^2 < \infty, t \in \mathbb{R}$, where $\psi(t) := L[\phi](t) = \phi(t + a)$. For types (ii) and (iii), we have:

**Lemma 3.1.** Let $L[\cdot]$ be an LTI system with the impulse response $l(t)$ of the type (ii) or (iii) as above and $\psi(t) := L[\phi](t) = (\phi \ast l)(t)$. Then

(a) $\psi(t) \in C_\infty(\mathbb{R}) := \{u(t) \in C(\mathbb{R}): \lim_{|t| \rightarrow \infty} u(t) = 0\}$;
(b) $\sup_{t \in \mathbb{R}} C_\phi(t) < \infty$;
(c) for any $f(t) = (c \ast \phi)(t) \in V(\phi)$, $L[f](t) = (c \ast \psi)(t)$ converges absolutely and uniformly on $\mathbb{R}$. Hence $L[f](t) \in C(\mathbb{R})$.

**Proof.** First assume $l(t) \in L^2(\mathbb{R})$. Then $\psi(t) \in C_\infty(\mathbb{R})$ by the Riemann–Lebesgue lemma since $\hat{\psi}(\xi) = \hat{\phi}(\xi)\hat{l}(\xi) \in L^1(\mathbb{R})$. Since $\sum_{n \in \mathbb{Z}} |\hat{\psi}(\xi + 2n\pi)| \leq \int_0^{2\pi} C_\phi(\xi) G_\xi(\xi) d\xi \leq 2\pi \|C_\phi(\xi)\|_{L^\infty(\mathbb{R})} \|l\|_{L^2(\mathbb{R})}^2$.

Thus for any $t \in \mathbb{R}$, we have by the Poisson summation formula (cf. Lemma 5.1 in [13])
\[
\sum_{n \in \mathbb{Z}} \hat{\psi}(\xi + 2n\pi)e^{it(\xi + 2n\pi)} = \sum_{n \in \mathbb{Z}} \psi(t + n)e^{-in\xi} \quad \text{in } L^2[0, 2\pi].
\]

Therefore for any \( t \) in \( \mathbb{R} \)

\[
C_\psi(t) = \sum_{n \in \mathbb{Z}} |\psi(t + n)|^2 = \frac{1}{2\pi} \left\| \sum_{n \in \mathbb{Z}} \psi(t + n)e^{-in\xi} \right\|_{L^2[0,2\pi]}^2
\]

\[
= \frac{1}{2\pi} \left\| \sum_{n \in \mathbb{Z}} \hat{\psi}(\xi + 2n\pi)e^{it(\xi + 2n\pi)} \right\|_{L^2[0,2\pi]}^2 \leq \|\mathcal{G}_\phi(\xi)\|_{L^\infty(\mathbb{R})} \|f\|_{L^2(\mathbb{R})}^2.
\]

By Young’s inequality on the convolution product, \( \|L[f]\|_{L^\infty(\mathbb{R})} \leq \|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})} \) so that \( L[\cdot]: L^2(\mathbb{R}) \to L^\infty(\mathbb{R}) \) is a bounded linear operator. Hence for any \( f(t) = (c \ast \phi)(t) = \sum_{n \in \mathbb{Z}} c(n)\phi(t - n) \in V(\phi) \),

\[
L[f](t) = \sum_{n \in \mathbb{Z}} c(n)L[\phi(t - n)] = \sum_{n \in \mathbb{Z}} c(n)\psi(t - n),
\]

which converges absolutely and uniformly on \( \mathbb{R} \) by (b). Now assume that \( H_\phi(\xi) \in L^2[0, 2\pi]. \) The case \( \hat{\psi}(\xi) \in L^2(\mathbb{R}) \) is reduced to type (ii). So let \( \hat{\psi}(\xi) \in L^2(\mathbb{R}) \). Then \( \hat{\psi}(\xi) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \) so that \( \hat{\psi}(\xi) = \hat{\phi}(\xi)\hat{\psi}(\xi) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \) and so \( \psi(\xi) \in \mathcal{C}_c(\mathbb{R}) \cap L^2(\mathbb{R}). \) Since \( \sum_{n \in \mathbb{Z}} |\psi(\xi + 2n\pi)| \leq \|\hat{\phi}(\xi)\|_{L^2(\mathbb{R})} \|\psi(\xi)\|_{L^2(\mathbb{R})} \), we have again by the Poisson summation formula

\[
C_\psi(t) = \frac{1}{2\pi} \left\| \sum_{n \in \mathbb{Z}} \hat{\psi}(\xi + 2n\pi)e^{it(\xi + 2n\pi)} \right\|_{L^2[0,2\pi]}^2 \leq \frac{1}{2\pi} \|\hat{\phi}(\xi)\|_{L^2[0,2\pi]}^2
\]

so that \( \sup_{\mathbb{R}} C_\psi(t) < \infty. \) For any \( f \in L^2(\mathbb{R}) \),

\[
\|L[f]\|_{L^2(\mathbb{R})} = \|f * g\|_{L^2(\mathbb{R})} = \frac{1}{\sqrt{2\pi}} \|\hat{\phi}(\xi)\|_{L^2(\mathbb{R})} \leq \|\hat{\phi}(\xi)\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}.
\]

Hence \( L[\cdot]: L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) is a bounded linear operator so that for any \( f(t) = \sum_{n \in \mathbb{Z}} c(n)\psi(t - n) \in V(\psi) \), \( L[f](t) = \sum_{n \in \mathbb{Z}} c(n)\psi(t - n) \) converges in \( L^2(\mathbb{R}) \). By (b), \( (c \ast \psi)(t) \) also converges absolutely and uniformly on \( \mathbb{R} \).

By Lemma 3.1(b), \( \psi(t) \in L^2(\mathbb{R}) \). However, \( (c \ast \psi)(t) \) may not converge in \( L^2(\mathbb{R}) \) unless \( \{\psi(t - n); \ n \in \mathbb{Z}\} \) is a Bessel sequence. Lemma 3.1(b) improves Lemma 1 in [9], in which the proof uses \( l(t) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \), \( \sup_{\mathbb{R}} C_\psi(t) < \infty \), and the integral version of Minkowski inequality. Note that the condition \( H_\phi(\xi) \in L^2[0, 2\pi] \) implies \( \phi(t) \in L^2(\mathbb{R}) \cap C_\mathbb{c}(\mathbb{R}) \) and \( \sup_{\mathbb{R}} C_\psi(t) < \infty \) (see Proposition 2.4 in [13]). Note also that \( H_\phi(\xi) \in L^2[0, 2\pi] \) if \( \phi(t) = O((1 + |t|)^{-r}) \), \( r > 1 \), which holds e.g. for \( \phi(t) := (\phi_0 \ast \phi_{n-1})(t) \) the cardinal B-spline of degree \( n \) \((\geq 1)\), where \( \phi_0 := \chi_{[0,1)}(t) \).

We have as a consequence of Lemma 3.1: Let \( L[\cdot] \) be an LTI system with impulse response \( l(t) \) of type (i) or (ii) or (iii) as above and \( \psi(t) := \phi(t) \). Then for any \( f(t) = (\mathcal{J}F)(t) \in V(\phi) \), \( F(\xi) \in L^2[0, 2\pi] \)

\[
L[f](t) = \left\{ \begin{array}{ll}
F(\xi), & 0 \leq |t| \leq \frac{1}{2\pi} Z\psi, \\
\frac{1}{2\pi} Z\psi(\xi, t), & \frac{1}{2\pi} Z\psi(\xi, t) \end{array} \right. \quad \text{in } L^2[0,2\pi]
\]

since \( L[\cdot] \) is a bounded linear operator from \( L^2(\mathbb{R}) \) into \( L^2(\mathbb{R}) \) or \( L^\infty(\mathbb{R}) \) and \( \{\psi(t - n); \ n \in \mathbb{Z}\} \in \ell^2, \ t \in \mathbb{R} \). Let

\[
\psi_j(t) := L_j[\phi](t) \quad \text{and} \quad g_j(\xi) := \frac{1}{2\pi} Z\psi_j(\sigma_j, \xi), \quad 1 \leq j \leq N.
\]

Then by (2)

\[
L_j[f]\sigma_j + j\pi = \left\{ \begin{array}{ll}
F(\xi), & 0 \leq |t| \leq \frac{1}{2\pi} Z\psi_j(\sigma_j + j\pi, \xi) \\text{in } L^2[0,2\pi] \end{array} \right. = \left\{ \begin{array}{ll}
F(\xi), & 0 \leq |t| \leq \frac{1}{2\pi} Z\psi_j(\xi, t) \\text{in } L^2[0,2\pi] \end{array} \right.
\]

for any \( f(t) = (\mathcal{J}F)(t) \in V(\phi) \) and \( 1 \leq j \leq N \). Then by (3) and the isomorphism \( \mathcal{J} \) from \( L^2[0, 2\pi] \) onto \( V(\phi) \), the sampling expansion (1) is equivalent to

\[
F(\xi) = \sum_{j=1}^{N} \sum_{n \in \mathbb{Z}} \{ F(\xi), \ \overline{g_j(\xi)}e^{-in\xi} \}_{L^2[0,2\pi]} S_{j,n}(\xi), \quad F(\xi) \in L^2[0, 2\pi],
\]

where \( \{ S_{j,n}(\xi); 1 \leq j \leq N, \ n \in \mathbb{Z}\} \) is a frame or a Riesz basis of \( L^2[0, 2\pi] \). This observation leads us to consider the problem: when is \( \{ \overline{g_j(\xi)}e^{-in\xi}; 1 \leq j \leq N, \ n \in \mathbb{Z}\} \) a frame or a Riesz basis of \( L^2[0, 2\pi] \)?
Note that
\[
\{\tilde{g}_j(\xi)e^{-ir_j\tilde{\xi}} : 1 \leq j \leq N, \ n \in \mathbb{Z}\} = \left\{g_{j,m_j}(\xi)e^{-ir_j\tilde{\xi}} : 1 \leq j \leq N, \ 1 \leq m_j \leq \frac{r}{r_j}, \ n \in \mathbb{Z}\right\}
\]
where \( r := \text{lcm}(r_j : 1 \leq j \leq N) \) and \( g_{j,m_j}(\xi) := g_j(\xi)e^{i(m_j-1)r_j} \) for \( 1 \leq j \leq N \).

Let \( D \) be the unitary operator from \( L^2[0,2\pi] \) onto \( L^2(I)^r \), where \( I = [0, \frac{2\pi}{r}] \), defined by
\[
DF := \left[F\left(\xi + (k-1)\frac{2\pi}{r}\right)\right]_{k=1}^T, \quad F(\xi) \in L^2[0,2\pi].
\]
We also let
\[
G(\xi) := \left[DG_{1,1}(\xi), \ldots, DG_{1,\frac{r}{r_j}}(\xi), \ldots, DG_{N,1}(\xi), \ldots, DG_{N,\frac{r}{r_j}}(\xi)\right]^T
\]
be a \((\sum_{j=1}^N \frac{r}{r_j}) \times r\) matrix on \( I \) and \( \lambda_m(\xi), \lambda_M(\xi) \) be the smallest and the largest eigenvalues of the positive semi-definite \( r \times r \) matrix \( G(\xi)^*G(\xi) \), respectively.

**Lemma 3.2.** Let \( \alpha_G := \|\lambda_m(\xi)\|_0 \) and \( \beta_G := \|\lambda_M(\xi)\|_\infty \) be the essential infimum of \( \lambda_m(\xi) \) and the essential supremum of \( \lambda_M(\xi) \) respectively. Then \( \{\tilde{g}_j(\xi)e^{-ir_j\tilde{\xi}} : 1 \leq j \leq N, \ n \in \mathbb{Z}\} \)

(a) a Bessel sequence in \( L^2[0,2\pi] \) if and only if \( \beta_G < \infty \) or equivalently \( \{Z_{\psi_j}(\sigma_j, \xi) : 1 \leq j \leq N\} \in L^\infty[0,2\pi] \);
(b) a frame of \( L^2[0,2\pi] \) if and only if \( 0 < \alpha_G \leq \beta_G < \infty \);
(c) a Riesz basis of \( L^2[0,2\pi] \) if and only if \( 0 < \alpha_G \leq \beta_G < \infty \) and \( \sum_{j=1}^N \frac{1}{j} = 1 \).

**Proof.** Since \( \{\tilde{g}_j(\xi)e^{-ir_j\tilde{\xi}} : 1 \leq j \leq N, \ n \in \mathbb{Z}\} \) is a Bessel sequence or a frame or a Riesz basis of \( L^2[0,2\pi] \) if and only if \( \{\tilde{g}_j(\xi)e^{-ir_j\tilde{\xi}} : 1 \leq j \leq N, \ 1 \leq m_j \leq \frac{r}{r_j}, \ n \in \mathbb{Z}\} \) is a Bessel sequence or a frame or a Riesz basis of \( L^2[0,2\pi] \) respectively, all of the conclusions follow from Lemma 3 in [9].

Note that in [9], the authors use the Fourier transform \( \hat{f}(\xi) = \int_{-\infty}^{\infty} f(t)e^{-2\pi it\xi} \frac{dt}{2\pi} \) so that they use \( L^2[0,1] \) instead of \( L^2[0,2\pi] \).

Assume that \( 0 < \alpha_G \leq \beta_G < \infty \) so that \( \{\tilde{g}_j(\xi)e^{-ir_j\tilde{\xi}} : 1 \leq j \leq N, \ n \in \mathbb{Z}\} \) or equivalently \( \{\tilde{g}_j, m_j(\xi)e^{-ir_j\tilde{\xi}} : 1 \leq j \leq N, \ 1 \leq m_j \leq \frac{r}{r_j}, \ n \in \mathbb{Z}\} \) is a frame of \( L^2[0,2\pi] \). Then we can show easily (see the arguments in the proof of Theorem 2 in [9]) that \( \{\tilde{g}_j, m_j(\xi)e^{-ir_j\tilde{\xi}} : 1 \leq j \leq N, \ 1 \leq m_j \leq \frac{r}{r_j}, \ n \in \mathbb{Z}\} \) has a dual frame of the form \( \{S_j, m_j(\xi)e^{-ir_j\tilde{\xi}} : 1 \leq j \leq N, \ 1 \leq m_j \leq \frac{r}{r_j}, \ n \in \mathbb{Z}\} \) with \( S_j, m_j(\xi) = \int L^\infty[0,2\pi] \) for \( 1 \leq j \leq N \) and \( 1 \leq m_j \leq \frac{r}{r_j} \) satisfying
\[
[DS_{1,1}(\xi), \ldots, DS_{1,\frac{r}{r_j}}(\xi), \ldots, DS_{N,1}(\xi), \ldots, DS_{N,\frac{r}{r_j}}(\xi)]^T = \frac{r}{2\pi} \left[G(\xi)^\dagger + B(\xi)(I - G(\xi)G(\xi)^\dagger)\right],
\]
where \( G(\xi)^\dagger := (G(\xi)^*G(\xi))^{-1}G(\xi)^* \) is the pseudo-inverse of \( G(\xi) \), \( B(\xi) \) is any \( r \times \sum_{j=1}^N \frac{r}{r_j} \) matrix with entries in \( L^\infty(I) \), and \( I \) is the \((\sum_{j=1}^N \frac{r}{r_j}) \times (\sum_{j=1}^N \frac{r}{r_j}) \) identity matrix. In particular, when we choose \( B(\xi) = 0 \) in (5), we have the canonical dual frame of the frame \( \{\tilde{g}_j, m_j(\xi)e^{-ir_j\tilde{\xi}} : 1 \leq j \leq N, \ 1 \leq m_j \leq \frac{r}{r_j}, \ n \in \mathbb{Z}\} \).

We are now ready to give the main results of this paper. We first discuss the sampling expansion (1), which is a frame expansion in \( V(\phi) \).

**Theorem 3.3.** Let \( \alpha_G \) and \( \beta_G \) be the same as in Lemma 3.2. Assume \( \beta_G < \infty \). Then the following are all equivalent.

(a) There is a frame \( \{s_{j,m_j}(t - m) : 1 \leq j \leq N, \ 1 \leq m_j \leq \frac{r}{r_j}, \ n \in \mathbb{Z}\} \) of \( V(\phi) \) for which
\[
f(t) = \sum_{j=1}^N \sum_{m_j=1}^N \sum_{n \in \mathbb{Z}} L_j[f](\sigma_j + r_j(m_j - 1) + rm)s_{j,m_j}(t - m), \quad f(t) \in V(\phi).
\]
(b) There is a frame \( \{s_{j,n}(t) : 1 \leq j \leq N, \ n \in \mathbb{Z}\} \) of \( V(\phi) \) for which
\[
f(t) = \sum_{j=1}^N \sum_{n \in \mathbb{Z}} L_j[f](\sigma_j + r_jn)s_{j,n}(t), \quad f(t) \in V(\phi).
\]
(c) $0 < \alpha_G$.

Proof. Assume $\beta_G < \infty$. Then by Lemma 3.2, \( \{ \overline{g_j(\xi)} e^{-ir\xi} \}: 1 \leq j \leq N, \ n \in \mathbb{Z} \) is a Bessel sequence in $L^2[0,2\pi]$. First (a) implies (b) trivially. Assume (b). Applying the isomorphism $\mathcal{J}^{-1}$ to (7) gives (3)

\[
F(\xi) = \sum_{j=1}^{N} \sum_{n \in \mathbb{Z}} \left\langle F(\xi), \overline{g_j(\xi)} e^{-ir\xi} \right\rangle_{L^2[0,2\pi]} S_{j,n}(\xi), \quad F(\xi) \in L^2[0,2\pi],
\]

where $\{S_{j,n}(\xi)\}: 1 \leq j \leq N, \ n \in \mathbb{Z}$ is a frame of $L^2[0,2\pi]$. Then the Bessel sequence $\{\overline{g_j(\xi)} e^{-ir\xi} \}: 1 \leq j \leq N, \ n \in \mathbb{Z}$ is in fact a dual frame of $\{S_{j,n}(\xi)\}: 1 \leq j \leq N, \ n \in \mathbb{Z}$ (cf. Lemma 5.6.2 in [2]). Hence (c) must hold by Lemma 3.2. Finally assume (c). Then $0 < \alpha_G < \beta_G < \infty$ so that $\{\overline{g_j(\xi)} e^{-ir\xi} \}: 1 \leq j \leq N, \ 1 \leq m_j \leq \frac{\xi}{r_j}, \ n \in \mathbb{Z}$ is a frame of $L^2[0,2\pi]$. Then we have a frame expansion on $L^2[0,2\pi]$:

\[
F(\xi) = \sum_{j=1}^{N} \sum_{m_j=1}^{r_j} \sum_{n \in \mathbb{Z}} \left\langle F(\xi), \overline{g_{j,m_j}(\xi)} e^{-ir\xi} \right\rangle_{L^2[0,2\pi]} S_{j,m_j}(\xi) e^{-ir\xi}, \quad F(\xi) \in L^2[0,2\pi],
\]

where $S_{j,m_j}(\xi)$’s are given by (5). Then the sampling expansion (6) comes from (8) by applying the isomorphism $\mathcal{J}$ since

\[
\left\langle F(\xi), \overline{g_{j,m_j}(\xi)} e^{-ir\xi} \right\rangle_{L^2[0,2\pi]} = \left( F(\xi), \frac{1}{2\pi} Z_{\psi}(\sigma_j + r_j(m_j - 1) + rn, \xi) \right)_{L^2[0,2\pi]},
\]

for $(\mathcal{J}F)(t) = f(t)$. □

Note that when $0 < \alpha_G \leq \beta_G < \infty$, the sampling series (6) converges not only in $L^2(\mathbb{R})$ but also uniformly on any subset of $\mathbb{R}$, on which $C_\theta(t)$ is bounded. Moreover since $\alpha_G > 0$, the rank of $G(\xi)$ is $r$ a.e. so that $1 \leq \sum_{j=1}^{N} \frac{1}{r_j}$, which means that the total sampling rate $\sum_{j=1}^{N} \frac{1}{r_j}$ of the sampling expansion (6) must be at least 1, the Nyquist sampling rate for signals in $V(\phi)$.

In the extreme case we have:

**Theorem 3.4.** Let $\alpha_G$ and $\beta_G$ be the same as in Lemma 3.2. Then there is a Riesz basis $\{s_{j,m_j,n}(t)\}: 1 \leq j \leq N, \ 1 \leq m_j \leq \frac{\xi}{r_j}, \ n \in \mathbb{Z}$ of $V(\phi)$ for which

\[
f(t) = \sum_{j=1}^{N} \sum_{m_j=1}^{r_j} \sum_{n \in \mathbb{Z}} L_{j}[f](\sigma_j + r_j(m_j - 1) + rn)s_{j,m_j,n}(t), \quad f(t) \in V(\phi)
\]

if and only if $0 < \alpha_G \leq \beta_G < \infty$ and $\sum_{j=1}^{N} \frac{1}{r_j} = 1$.

**Proof.** Assume $0 < \alpha_G \leq \beta_G < \infty$ and $\sum_{j=1}^{N} \frac{1}{r_j} = 1$. Then by Lemma 3.2, $\{\overline{g_{j,m_j}(\xi)} e^{-ir\xi} \}: 1 \leq j \leq N, \ 1 \leq m_j \leq \frac{\xi}{r_j}, \ n \in \mathbb{Z}$ is a Riesz basis of $L^2[0,2\pi]$. Then we have

\[
F(\xi) = \sum_{j=1}^{N} \sum_{m_j=1}^{r_j} \sum_{n \in \mathbb{Z}} \left\langle F(\xi), \overline{g_{j,m_j}(\xi)} e^{-ir\xi} \right\rangle_{L^2[0,2\pi]} S_{j,m_j}(\xi) e^{-ir\xi}, \quad F(\xi) \in L^2[0,2\pi],
\]

where $\{S_{j,m_j}(\xi) e^{-ir\xi} \}: 1 \leq j \leq N, \ 1 \leq m_j \leq \frac{\xi}{r_j}, \ n \in \mathbb{Z}$ is the dual of $\{\overline{g_{j,m_j}(\xi)} e^{-ir\xi} \}: 1 \leq j \leq N, \ 1 \leq m_j \leq \frac{\xi}{r_j}, \ n \in \mathbb{Z}$. Applying the isomorphism $\mathcal{J}$ to (10) gives (9), where $s_{j,m_j,n}(t) = \mathcal{J}(S_{j,m_j}(\xi) e^{-ir\xi}) = s_{j,m_j}(t - rn)$ and $\mathcal{J}(S_{j,m_j}(\xi)) = s_{j,m_j}(t)$. Conversely assume that the Riesz basis expansion (9) holds on $V(\phi)$. Applying the isomorphism $\mathcal{J}^{-1}$ to (9) gives

\[
F(\xi) = \sum_{j=1}^{N} \sum_{m_j=1}^{r_j} \sum_{n \in \mathbb{Z}} \left\langle F(\xi), \overline{g_{j,m_j}(\xi)} e^{-ir\xi} \right\rangle_{L^2[0,2\pi]} \mathcal{J}^{-1}(s_{j,m_j,n}(\xi)), \quad F(\xi) \in L^2[0,2\pi]
\]
which is a Riesz basis expansion on $L^2[0, 2\pi]$. Then \( \{g_{j,m}(\xi)e^{-irm\xi}: 1 \leq j \leq N, 1 \leq m_j \leq \frac{1}{r_j}, n \in \mathbb{Z} \} \) must be a Riesz basis of $L^2[0, 2\pi]$ so that $0 < \alpha_C \leq \beta_C < \infty$ and $\sum_{j=1}^{N} \frac{1}{r_j} = 1$ by Lemma 3.2. As the dual Riesz basis of $\{g_{j,m}(\xi)e^{-irm\xi}: 1 \leq j \leq N, 1 \leq m_j \leq \frac{1}{r_j}, n \in \mathbb{Z} \}$, \( \{J^{-1}(s_{j,m,n}(t)): 1 \leq j \leq N, 1 \leq m_j \leq \frac{1}{r_j}, n \in \mathbb{Z} \} \) must be of the form $\{S_{j,m}(\xi)e^{irm\xi}: 1 \leq j \leq N, 1 \leq m_j \leq \frac{1}{r_j}\}$ satisfy Eq. (5) with $B(\xi) = 0$. Hence

\[
\begin{align*}
s_{j,m,n}(t) &= J(S_{j,m}(\xi)e^{irm\xi}) = s_{j,m}(t-rn), \quad 1 \leq j \leq N \text{ and } n \in \mathbb{Z}.
\end{align*}
\]

Finally, we have

\[
\begin{align*}
s_k,m_n(t) &= \sum_{j=1}^{N} \sum_{m_j=1}^{r_j} \sum_{n \in \mathbb{Z}} L_j[s_k,m_n](\sigma_j + r_j(m_j - 1) + rn)s_{j,m}(t-rn)
\end{align*}
\]

so that $L_j[s_k,m_n](\sigma_j + r_j(m_j - 1) + rn) = \delta_{j,k}\delta_{n,0}$. \(\square\)

When $N = 1$, write $L_1[\cdot], I_1(\cdot), \sigma_1, r_1,$ and $\psi_1(\cdot)$ as $L[\cdot], I(\cdot), \sigma, r,$ and $\psi(\cdot)$.

**Corollary 3.5.** (Cf. Theorem 3.1 in [11]) Let $N = 1$. Then there is a Riesz basis $\{s_n(t): n \in \mathbb{Z}\}$ of $V(\phi)$ such that

\[
\begin{align*}
\text{if and only if } r &= 1 \text{ and }
0 < \|Z_\psi(\sigma, \xi)\|_0 < \|Z_\psi(\sigma, \xi)\|_\infty \leq \infty.
\end{align*}
\]

In this case, we also have:

- $s_n(t) = s(t-n), \quad n \in \mathbb{Z};$
- $\hat{s}(\xi) = \frac{\hat{\phi}(\xi)}{Z_\psi(\sigma, \xi)};
- L[s](\sigma + n) = \delta_{n,0}, \quad n \in \mathbb{Z}.

**Proof.** Note that for $r = 1$, $G(\xi) = \frac{1}{2\pi}Z_\psi(\sigma, \xi)$ and $\lambda_m(\xi) = \lambda_M(\xi) = (\frac{1}{2\pi})^2|Z_\psi(\sigma, \xi)|^2$ so that $0 < \alpha_C \leq \beta_C < \infty$ if and only if (12) holds. Therefore, everything except (13) follows from Theorem 3.4. Finally applying (11) to $\phi(t)$ gives

\[
\begin{align*}
\phi(t) &= \sum_{n \in \mathbb{Z}} \psi(\sigma + n)s(t-n)
\end{align*}
\]

from which we have (13) by taking the Fourier transform. \(\square\)

When $l(t) = \delta(t)$ so that $L[\cdot]$ is the identity operator, Corollary 3.5 reduces to a regular shifted sampling on $V(\phi)$ (see Theorem 3.3 in [13]).

**Remark 3.6.** In (1), we may allow rational sampling periods. If $r_j = \frac{p_j}{q_j}$, where $p_j$ and $q_j$ are coprime positive integers, then

\[
\begin{align*}
\{L_j[f](\sigma_j + r_j n): n \in \mathbb{Z}\} &= \{L_j[f](\sigma_j + r_j(k-1) + p_j n): 1 \leq k \leq \frac{q_j}{r_j}, \quad n \in \mathbb{Z}\}.
\end{align*}
\]

Hence the case of rational sampling periods $\{r_j\}_{j=1}^{N}$ can be reduced to the case of integer sampling periods $\{p_j\}_{j=1}^{N}$ by extending the number of LTI systems involved.

For example when $N = 1$, we have:

**Corollary 3.7.** Let $N = 1$ and $q$ be an integer. Assume $Z_\psi(\sigma_j, \xi) \in L^\infty[0, 2\pi], 1 \leq j \leq q$, where $\sigma_j := \sigma + \frac{1}{q}(j-1)$. Then the following are all equivalent.

(a) There is a frame $\{s_n(t): n \in \mathbb{Z}\}$ of $V(\phi)$ for which

\[
\begin{align*}
f(t) = \sum_{n \in \mathbb{Z}} L[f]\left(\sigma + \frac{1}{q}n\right)s_n(t), \quad f(t) \in V(\phi).
\end{align*}
\]
(b) There is a frame \( \{s_j(t-n) : 1 \leq j \leq q, n \in \mathbb{Z}\} \) of \( V(\phi) \) for which
\[
f(t) = \sum_{j=1}^{q} \sum_{n \in \mathbb{Z}} L(f)(\sigma_j + n)s_j(t-n), \quad f(t) \in V(\phi).
\]

(c) \( \| \sum_{j=1}^{\infty} |Z_\psi(\sigma_j, \xi)| \|_0 > 0 \).

**Proof.** Since \( \{L_1(f)[\sigma + \frac{1}{q} n] : n \in \mathbb{Z}\} = \{L_1(f)[\sigma + n] : 1 \leq j \leq q, n \in \mathbb{Z}\} \), we have a shifted symmetric multi-channel sampling for \( q \) LTI systems \( \{L_j[\cdot] : 1 \leq j \leq q\} \) with \( L_j[\cdot] = L_1[\cdot], 1 \leq j \leq q \). Then \( g_j(\xi) = \frac{1}{2\pi} Z_\psi(\sigma_j, \xi), 1 \leq j \leq q \) and \( G(\xi)^* G(\xi) = \frac{1}{(2\pi)^q} \sum_{j=1}^{q} |Z_\psi(\sigma_j, \xi)|^2 \). Hence \( \alpha_G > 0 \) if and only if \( \| \sum_{j=1}^{\infty} |Z_\psi(\sigma_j, \xi)| \|_0 > 0 \). Therefore, Corollary 3.7 is a consequence of Theorem 3.3. \( \Box \)

4. Example

Let \( \phi_0 := \chi_{(0,1)}(t) \) be the Haar scaling function and
\[
\phi_1(t) = (\phi_0 \ast \phi_0)(t) = \chi_{(0,1)}(t) + (2-t)\chi_{(1,2)}(t)
\]
a B-spline of degree 1. Then \( \phi_1(t) \) is a continuous Riesz generator \[3\] and \( \sup_{\mathbb{R}} C_{\phi_1}(t) = \sup_{\mathbb{R}} \sum_{n \in \mathbb{Z}} |\phi_1(t+n)|^2 < \infty \). First we take \( N = 2, \sigma_1 = \sigma_2 = 0, r_1 = 1, r_2 = 2, \) and two LTI systems \( L_1[\cdot] \) and \( L_2[\cdot] \) with impulse responses \( l_1(t) = \chi_{[-\frac{1}{2}, 0]}(t) \) and \( l_2(t) = \chi_{[-1, -\frac{1}{2}]}(t) \). Then it's easy to see that
\[
g_1(\xi) = \frac{1}{2\pi} Z_{\psi_1}(0, \xi) = \frac{1}{2\pi} \sum_{n \in \mathbb{R}} \psi_1(n) e^{-i n \xi} = \frac{1}{16\pi} (1 + 3e^{-i\xi}),
\]
\[
g_2(\xi) = \frac{1}{2\pi} Z_{\psi_2}(0, \xi) = \frac{1}{2\pi} \sum_{n \in \mathbb{R}} \psi_2(n) e^{-i n \xi} = \frac{1}{16\pi} (3 + e^{-i\xi}),
\]
where \( \psi_j(t) = L_j[\phi](t) \). Hence \( g_{1,1}(\xi) = g_1(\xi), g_{1,2}(\xi) = g_1(\xi) e^{i\xi}, g_{2,1}(\xi) = g_2(\xi) \) so that (cf. (4))
\[
G(\xi) = \left[ Dg_{1,1}, Dg_{1,2}, Dg_{2,1} \right]^T = \frac{1}{16\pi} \begin{bmatrix} 1 + 3e^{-i\xi} & 1 - 3e^{-i\xi} \\ 3 + e^{i\xi} & 3 - e^{i\xi} \\ 3 + e^{-i\xi} & 3 - e^{-i\xi} \end{bmatrix}
\]
and
\[
G(\xi)^* G(\xi) = \frac{1}{(16\pi)^2} \begin{bmatrix} 30 + 18 \cos \xi & 8 + 6i \sin \xi \\ 8 - 6i \sin \xi & 30 + 18 \cos \xi \end{bmatrix}
\]
The eigenvalues of \( G(\xi)^* G(\xi) \) are \( \frac{1}{(16\pi)^2}[30 + 18 \cos \xi \pm \sqrt{100 - 36 \cos^2 \xi}] \) so that \( \frac{2}{(16\pi)^2} \leq \alpha_G = \|\lambda_m(\xi)\|_0 < \beta_G = \|\lambda_M(\xi)\|_\infty \leq \frac{58}{16\pi^2} \). Hence by Theorem 3.3, there is a frame \( \{s_j(t-2n) : j = 1, 2, 3 \text{ and } n \in \mathbb{Z}\} \) of the space of linear splines \( V(\phi_1) \) for which the following asymmetric multi-channel sampling expansion holds:
\[
f(t) = \sum_{n \in \mathbb{Z}} \left[ L_1[f](2n)s_1(t-2n) + L_1[f](2n+1)s_2(t-2n) + L_2[f](2n)s_3(t-2n) \right], \quad f(t) \in V(\phi_1),
\]
which converges in \( L^2(\mathbb{R}) \) and absolutely and uniformly on \( \mathbb{R} \).

We now take \( N = 1 \) and \( l(t) = \delta(t) \) so that \( L_1[\cdot] \) is the identity operator. Let \( q (\geq 1) \) be an integer and \( 0 \leq \sigma < \frac{1}{q} \).

Note first that for any fixed \( t \in \mathbb{R} \), \( Z_{\phi_1}(t, \xi) = \sum_{n \in \mathbb{Z}} \phi_1(t+n)e^{-i n \xi} \in C[0, 2\pi] \) since \( \phi_1(t) \) has compact support. Hence \( \|Z_{\phi_1}(t, \cdot)\|_{L^1[0, 2\pi]} < \infty \) for each \( t \in \mathbb{R} \). Since \( Z_{\phi_1}(\sigma, \xi) = \sigma + (1 - \sigma) e^{-i\xi} \) for \( 0 \leq \sigma < 1 \), \( \|Z_{\phi_1}(\sigma, \xi)\|_0 = 2|\sigma - \frac{1}{2}| \) and \( \|Z_{\phi_1}(\sigma, \xi)\|_{L^\infty} = 1 \). Therefore, by Corollary 3.5, for any \( \sigma \) with \( 0 \leq \sigma < 1 \), there is a Riesz basis \( \{s(t-n) : n \in \mathbb{Z}\} \) of \( V(\phi) \) such that
\[
f(t) = \sum_{n \in \mathbb{Z}} f(\sigma+n)s(t-n), \quad f(t) \in V(\phi_1)
\]
if and only if \( \sigma \neq \frac{1}{2} \). On the other hand, by Corollary 3.7, for any \( q (\geq 2) \) and any \( \sigma \) with \( 0 \leq \sigma < \frac{1}{q} \), there is a frame \( \{s_j(t-n) : 1 \leq j \leq q, n \in \mathbb{Z}\} \) such that
\[
f(t) = \sum_{j=1}^{q} \sum_{n \in \mathbb{Z}} f \left( \sigma + \frac{1}{q}(j-1)+n \right) s_j(t-n), \quad f(t) \in V(\phi_1).
\]
Acknowledgments

This work is partially supported by Korea Research Foundation Grant No. 2009-0084583. Authors thank the referee for many valuable comments, which improve the paper.

References