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Prime Rings Having One-Sided Ideal with Polynomial Identity Coincide with Special Johnson Rings*

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Throughout R is a prime ring and is regarded as an algebra over its centroid. Let $R^{\Delta}({}^{\Delta}R)$ denote the right (left) singular ideal of R. R is called Johnson ring if it satisfies any of the two equivalent conditions: (a) $R^{\Delta} = 0 = {}^{\Delta}R$ and R possesses uniform right and left ideals, (b) the right (left) quotient ring of R (in the sense of Utumi) is $\mathrm{Hom}_D(V, V)$ where V is a vector space over a division ring D. In addition if D is finite dimensional over its center then R is called a special Johnson ring. Denote by C the center of Utumi's right quotient ring of R. The results shown are: (1) R is a special Johnson ring iff there exists a nonzero one-sided ideal with polynomial identity (*PI*), (2) R has generalized polynomial identity (*GPI*) nontrivial over C iff each nonzero right (left) ideal of R contains a nonzero right(left) ideal with *PI*, (3) If R has *GPI* nontrivial over C then R cannot have nonzero nil one-sided ideals, (4) If R is integral domain the R has *GPI* nontrivial over C iff R has *PI*, (5) There does not exist a simple radical ring R satisfying a generalized polynomial identity nontrivial over the center of $\mathrm{Hom}_R(R, R)$, (6) R is a special Johnson ring with nonzero socle iff each nonnil right (left) contains an idempotent ($\neq 0$) and there exists a nonzero one-sided ideal with *PI*.

1. Amitsur showed in [1] that if R is a primitive algebra satisfying a generalized polynomial identity over its centroid then R has a minimal left ideal Re and eRe is finite dimensional over its center. Martindale proved [6] that if R is a prime ring with a generalized polynomial identity (nontrivial) over the center C of the Utumi's right ring of quotients Q_r of R then $S = RC$ is a primitive ring with minimal left ideal Se and eSe is finite dimensional over its center. The object of this paper is first to observe that a certain class of algebras have a generalized polynomial identity iff there exists a nonzero one-sided ideal with a polynomial identity. Among other results it is shown

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(i) an integral domain R satisfies a generalized polynomial identity nontrivial over the center of (Utumi's) ring of quotients of R iff R satisfies a polynomial identity, (ii) there does not exist a simple radical ring R with a generalized polynomial identity nontrivial over the center of $\mathrm{Hom}_R(R, R)$.

Throughout R is a prime ring and is regarded as an algebra over its centroid. Let $R^{\Delta}({}^{\Delta}R)$ denote the right (left) singular ideal of R. R is called a Johnson ring if it satisfies any of the two equivalent conditions: (a) $R^{\Delta} = 0 = {}^{\Delta}R$ and R possesses uniform right as well as left ideals, (b) the right (or left) quotient ring of R (in the sense of Utumi) is $\mathrm{Hom}_D(V, V)$ where V is a vector space over a division ring D. In addition if D is finite dimensional over its center then R is called a special Johnson ring. Denote by C the center of the Utumi's right ring of quotients Q_r of R. For definitions refer [1], [4], and [5].

2. We first give another version of the Martindale Theorem.

THEOREM 2.1. *If R satisfies a generalized polynomial identity (nontrivial) over C then R is a special Johnson ring.*

Proof. By Martindale, $S = RC$ is a primitive ring with nonzero socle, and eSe is finite dimensional over its center where eS is a minimal right ideal. So that $S^{\Delta} = 0 = {}^{\Delta}S$. We assert that if E is a large right ideal of R then EC is a large right ideal of S. Let K be a nonzero right ideal of S and $0 \neq k \in K$. Then there exists a dense right ideal D of R such that kD is a nonzero right ideal of R. This implies $0 \neq kD \cap E \subset K \cap EC$, and hence EC is a large right ideal in S. From this it is immediate that $R^{\Delta} = 0$, and Q_r is also the Utumi's right ring of quotients of S. Since Q_r is the ring of quotients of a primitive ring S which has nonzero socle, $Q_r = \mathrm{Hom}_{eSe}(Se, Se)$ where Se is a minimal left ideal of S. Similarly we can show that the left quotient ring of S (and also of R) is the full ring of $\ell.t.$ of a vector space over a division ring. Hence R is a special Johnson ring.

3. Let A be a nonzero one-sided ideal, say, for definiteness a right ideal which satisfies a polynomial identity over the centroid of R. Denote by $\ell(A)$ the left annihilator of A in A. Clearly $\ell(A) \neq A$. Suppose $xAy \subset \ell(A)$. Then $xAyA = 0$ and hence $xA = 0$ or $yA = 0$ since R is prime. This shows that $A/\ell(A)$ is a prime ring and being a homomorphic image of A, has a polynomial identity. Since $A/\ell(A)$ is prime, $A/\ell(A)$ satisfies some standard identity $S_m(x) \equiv \Sigma \pm x_i x_{i_2} \cdots x_{i_m} = 0$. This implies A satisfies $[S_m(x)]^2 = 0$ proving that AC also satisfies $[S_m(x)]^2 = 0$. Hence $S = RC$ satisfies a generalized polynomial identity. Conversely let S satisfy a generalised polynomial identity nontrivial over C. If eS is a minimal right ideal of S, then eS satisfies some polynomial identity since eSe is a finite dimensional division algebra (and hence satisfies a standard identity). Let fQr be a minimal right ideal of Qr. Then $fQrf$ is isomorphic to eSe, and thus fQr also satisfies a polynomial identity. This implies that any minimal closed right ideal B of R satisfies a polynomial identity since it is wellknown that $B = fQr \cap R$, where fQr is a minimal right ideal of Qr. Next, let A be an arbitrary nonzero right ideal of R and A^s be its unique maximal essential extension in R. By the lattice isomorphism between the closed right ideals of R and of Qr, A^s contains a minimal closed right ideal B which satisfies a polynomial identity by the previous remark. Also, A^s being essential extension of A, $B \cap A \neq 0$. Hence we have shown the following:

THEOREM 3.1. *Let R be a prime ring. Then R has generalized polynomial identity nontrivial over C iff each nonzero right ideal of R contains a nonzero right ideal of R satisfying a polynomial identity.*

Remark. Theorems 2.1 and 3.1 show that the Utumi's ring of quotients of R is $\mathrm{Hom}_D(V, V)$ where D is finite-dimensional over its center if R has a nonzero one-sided ideal with a polynomial identity. It is interesting to compare this result with the Posner's Theorem that the classical quotient ring of R is $\mathrm{Hom}_D(V, V)$ where V is a finite-dimensional vector space over D and D is finite-dimensional over its center if R satisfies a polynomial identity. We add that the classical quotient ring of R in the Posner's Theorem coincides with the Utumi's ring of quotients of R.

Theorem 3.1 along with a result of Belluce and Jain ([3, Theorem 1]) yield.

THEOREM 3.2. *If R is an integral domain then R has generalized polynomial identity over C iff R has a polynomial identity.*

Another proof of Theorem 3.2 can be given if we first prove a lemma. Let Q denote the right quotient ring of R. The lemma proved below shows that Q is simple if R is an integral domain. Similar arguments will show that the left quotient ring is also simple.

LEMMA. *If R is an integral domain then Q is simple.*

Proof. Since $R^\Delta = 0$, Q is known to be Von Neumann regular. Let A be a nonzero ideal in Q. Then $A \cap R \neq 0$. Let $0 \neq a \in A \cap R$. There exists $x \in Q$ such that $a = axa$. But then $1 - xa \in r(a)$, where $r(a)$ is the right annihilator of a in Q. If $r(a) \neq 0$, then $r(a) \cap R \neq 0$. But R being integral domain, we must say $r(a) = 0$. Then $1 - xa = 0$ and hence $A = Q$, showing that Q is simple.

Another proof of the Theorem 3.2. By the lemma, $Q = \mathrm{Hom}_D(V, V)$ is simple. So Q is simple Artinian. Also D is finite dimensional over its center (and so has polynomial identity). Hence Q satisfies a polynomial identity and so does R.

Remark. The condition that a generalized polynomial identity is non-trivial over a certain field is important. There exists an integral domain with a generalized polynomial identity which does not satisfy any polynomial identity as the following example shows.

EXAMPLE. Let F be a field, x, y be two indeterminates and σ be any automorphism of F different from the identity. Let $xa = a^\sigma x$, $ya = a^\sigma y$ where a is any element of F and a^σ denotes the image of a by the mapping σ.

Then $R = F[x, y]$ satisfies generalized polynomial identity $xry = yrx$, $r \in R$. But R cannot satisfy any polynomial identity since R is not an Ore-domain. It can be shown that there exists λ in the center C of the quotient ring of R such that $x = \lambda y$ and thus $xry = yrx$ is trivial over C.

We proceed to give another case where generalized polynomial identity implies polynomial identity. Recall that a right ideal of A of a ring R is called closed if A as a right R-module has no essential extension in R_R excepting A_R itself. It is well-known that if $R^\Delta = 0$ then the lattices of closed right ideals of R is isomorphic to the lattice of closed right ideals of Q and these lattices are complemented modular. An example of closed right ideal of a ring R with $R^\Delta = 0$ is any annihilator right ideal. Of course, any closed right ideal is not necessarily annihilator right ideal. One necessary and sufficient condition for the set of closed right ideals to coincide with the set of annihilator right ideals is that R is prime Goldie ring. We now prove

THEOREM 3.3. *If the lattice of closed right (or left) ideals of R has either chain condition then R has generalized polynomial identity over C iff R has a polynomial identity.*

Proof. Assume that R has a generalized polynomial identity over C. Since e_iQ, $e_i^2 = e_i$ are obviously closed right ideals of Q, we get by our hypothesis that $Q = \mathrm{Hom}_D(V, V)$ can have at most a finite set of orthogonal idempotents and hence Q is simple Artinian. But we also know that D is finite-dimensional over its center (and hence has polynomial identity). So Q satisfies a polynomial identity. This proves the theorem.

4. We now show that a prime ring with a generalized polynomial identity does not possess nil one-sided ideals. It is appropriate to mention here, however, that there may exist nil subrings which are not even locally nilpotent as against the result for prime rings with polynomial identities that nil subrings are nilpotent.

THEOREM 4.1. *If R has a generalized polynomial identity over C then R has no nonzero nil one-sided ideals.*

Proof. Let A be a nil right ideal in R. If $A \neq 0$, A contains a uniform right ideal U, since R is a Johnson ring. It is well known that $K = \operatorname{Hom}_R(U, U)$ is an integral domain. Furthermore the mapping $\sigma : U \to K$ in which $u \to \ell_u$ where ℓ_u is a left multiplication by u is a ring homomorphism. Since σU is a nil left ideal in the integral domain K, $\sigma U = 0$. But then $U^2 = 0$, a contradiction. Hence $A = 0$, proving the theorem.

COROLLARY 4.2. *There does not exist a simple nil ring $R(R^2 \neq 0)$ satisfying a generalized polynomial identity over the center of* $\operatorname{Hom}_R(R, R)$.

After having known Corollary 4.2, a natural question is: Does there exist a nontrivial simple radical ring R satisfying a generalized polynomial identity (nontrivial) over the center of $\operatorname{Hom}_R(R, R)$? The answer is in the negative as the following argument shows. If $R^2 \neq 0$, R is a prime ring and Martindale theorem gives RC has nonzero socle, where C is the center of $\operatorname{Hom}_R(R, R)$. But RC is clearly a nonzero ideal of R and hence $RC = R$. Hence R cannot be a radical ring. We may restate this in

THEOREM 4.3. *There does not exist a nontrivial simple radical ring R satisfying a generalized polynomial identity over the center of* $\operatorname{Hom}_R(R, R)$.

5. It is known [2] that a primitive ring has a nonzero socle if there exists a nonzero one-sided ideal with J-pivotal monomial, in particular, a polynomial identity in which case this result is also a consequence of a more general theorem of Amitsur [1] for primitive algebras with a generalized polynomial identity. In this section we prove a similar result for prime rings.

THEOREM 5.1. *R is a special Johnson ring with a nonzero socle iff each nonnil right (or left) ideal contains a nonzero idempotent and there exists a nonzero one-sided ideal with a polynomial identity.*

Proof. Let A be a nonzero right ideal of R with a polynomial identity and $\ell(A)$ be the left annihilator of A in A. Then $A/\ell(A)$ is a prime ring with a polynomial identity. This shows that A cannot be nil. Otherwise $A = \ell(A)$ and hence $A^2 = 0$, a contradiction. The argument, in fact, proves that if B is any right ideal contained in A then B cannot be nil. Then by hypothesis we can construct a descending chain of annihilator right ideals of R contained in A, namely,

$$e_1R \supset e_2R \supset e_3R \supset \cdots \supset \cdots .$$

This chain of right ideals modulo $\ell(A)$ should terminate since $A/\ell(A)$ is a prime ring with a polynomial identity. But then there exists idempotents

e_i, e_{i+1} such that $e_iR \supset e_{i+1}R$ and $(e_i - e_{i+1})^2 = 0$. This yields $e_i = e_{i+1}e_i$. Hence $e_iR = e_{i+1}e_iR \subset e_{i+1}R$, giving that $e_iR = e_{i+1}R$. This proves that A contains a minimal right ideal of R, completing the proof of sufficiency. The necessity follows from Theorems 2.1 and 3.1 and the fact that in a prime ring with nonzero socle each nonzero right ideal contains a nonzero idempotent.

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