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When does P-localization preserve homotopy pushouts or pullbacks?

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Abstract

We give conditions under which localization at a set of primes P in the sense of Casacuberta and Peschke [Trans. Amer. Math. Soc. 339 (1993) 117–140] preserves homotopy pushouts and homotopy pullbacks. We then apply these results to infer conditions under which P-localization preserves homotopy epimorphisms and homotopy monomorphisms. We also obtain conditions under which P-localization of non-nilpotent spaces induces P-localization of its homotopy groups.

Keywords: Localization; Homotopy-pushout; Homotopy-pullback; Homotopy-epimorphism; Homotopy-monomorphism

Introduction

Localizing spaces at a set of primes in the sense of Casacuberta and Peschke [3] is one of the possible ways of extending P-localization of nilpotent spaces [13,5] over all spaces. On nilpotent spaces the effect of P-localization is quite transparent, while various degrees of mystery surround its effect on non-nilpotent spaces and maps between them. This constitutes an unwelcome obstacle in the use of localization methods.

Here we give conditions under which P-localization preserves fundamental constructs of homotopy theory, like homotopy pushouts/pullbacks, and homotopy epimorphism/ monomorphisms. The main results are 2.1 preservation of homotopy pushouts, 3.2 preservation of homotopy epimorphisms, 4.2 preservation of homotopy fibrations, 4.3

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preservation of homotopy pullbacks, and 5.3. We comment briefly on what 'preserves' means in each case. Further details can be found in the introductions to the various sections.

Let ho$\mathcal{CW}$ denote the homotopy category of based connected CW-complexes, and ho$\mathcal{CW}_P$ the full subcategory of P-local CW-complexes. The P-localization of a homotopy pushout diagram in ho$\mathcal{CW}$ always deserves to be regarded as a homotopy pushout in ho$\mathcal{CW}_P$. Thus 2.1 expresses conditions under which P-localization preserves a homotopy pushout diagram in ho$\mathcal{CW}$ in the stronger sense that the result is again a homotopy pushout in ho$\mathcal{CW}$. Likewise, P-localizing an epimorphism f in ho$\mathcal{CW}$ always yields an epimorphism in ho$\mathcal{CW}_P$. So 3.1, and 3.2 express conditions under which f_P is an epimorphism in ho$\mathcal{CW}$. This extends earlier work of Lin and Shen [7] who answered a question posed by Hilton and Roitberg in [6] by showing that, in the homotopy category of nilpotent spaces, localization at a prime p always preserves epimorphisms strongly; see Sections 2 and 3.

P-localization need not turn a homotopy pullback diagram in ho$\mathcal{CW}$ into a homotopy pullback diagram in ho$\mathcal{CW}_P$. However, if it does, then the P-localized homotopy pullback in ho$\mathcal{CW}_P$ is automatically a homotopy pullback in ho$\mathcal{CW}$. The situation for homotopy monomorphisms is similar. These results depend upon the new concept of a P-torsion action space: the fundamental group acts on higher homotopy groups through automorphisms whose order is finite and is divisible by primes in P. For such spaces we show that P-localization induces P-localization of all homotopy groups 4.2; see Sections 4 and 5.

Notation and conventions

Throughout P denotes a set of prime numbers, and P' the multiplicative closure of the set of primes not in P. Localization at P of a group G will be as in [11,10]. Let $P[G]$ denote the ring localization of the group ring $\mathbb{Z}G$ obtained by inverting all of the elements $1 + g + \cdots + g^{n-1}$, where $g \in G$, and $n \in P'$. See [3, Section 2] for details on this construction and its role in the P-localization of spaces.

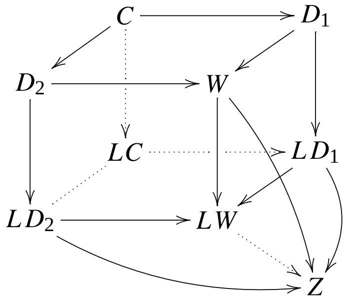
Localization at P of a connected based space will be as in [3]. We denote by ho$\mathcal{CW}$ the homotopy category of based CW-spaces, and by ho$\mathcal{CW}_P$ the full subcategory of P-local spaces. For a space Z and a right module M over its fundamental group, $\mathcal{H}_*(Z; M)$ denotes homology with twisted coefficients in M.

1. Categorical facts

For the reader's convenience we collect here some basic facts about the interaction between localizing functors and the concepts of pushouts, homotopy pushouts, epimorphisms, and pullbacks, homotopy pullbacks and monomorphisms. As a reference for category theoretical terms, we recommend [8]. The classical reference for homotopy (co-)limits is [1]. For specific properties of homotopy pushouts and pullbacks we recommend [9,12]. Finally, we point out that a homotopy pushout is a weak pushout in ho$\mathcal{CW}$, the homotopy category of based CW-spaces, and that a homotopy pullback is a weak pullback in ho$\mathcal{CW}$.

Proposition 1.1. *A localizing functor $L:\mathcal{C}\to\mathcal{D}$ on a category $\mathcal{C}$ turns weak pushouts into weak pushouts in $\mathcal{D}$, the full subcategory of $\mathcal{C}$ consisting of L-local objects.*

Proof. In the commutative box below, suppose the top is a weak pushout in $\mathcal{C}$ and the bottom results from applying L to the top.



We need to show that the bottom square has the weak pushout property in $\mathcal{D}$. Now, if Z is L-local, we first obtain a map $W\to Z$ with the required commuting properties. Second, this map factors through LW, and the expected commuting properties follow from the universal property of L. □

Turning to epimorphisms and monomorphisms, we begin with the observation:

Lemma 1.2. *If $f:A\to B$ in a category $\mathcal{C}$ has a right inverse* (*respectively a left inverse*), *then f is an epimorphism* (*respectively a monomorphism*) *of $\mathcal{C}$. Moreover, every functor $\mathcal{C}\to\mathcal{D}$ sends f to an epimorphism* (*respectively a monomorphism*) *of $\mathcal{D}$.*

In the presence of weak pushouts (respectively weak pullbacks), we recognize epimorphisms (respectively monomorphisms) via the following lemma. We omit its straightforward proof.

Lemma 1.3. *Suppose the category $\mathcal{C}$ has weak pushouts* (*respectively weak pullbacks*). *Then a morphism $f:A\to B$ in $\mathcal{C}$ is*

(1) *an epimorphism if and only if $u=v$, for any weak pushout diagram,*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle v} \\ B & \xrightarrow[u]{} & W \end{array}$$

(2) *a monomorphism if and only if $i=j$, for any weak pullback diagram,*

$$\begin{array}{ccc} U & \xrightarrow{i} & Y \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[f]{} & Z \end{array}$$

2. Preserving homotopy pushouts

A general theorem [4, 1.7] suggests that P-localization will, in general, not commute with homotopy colimits: there is a natural isomorphism

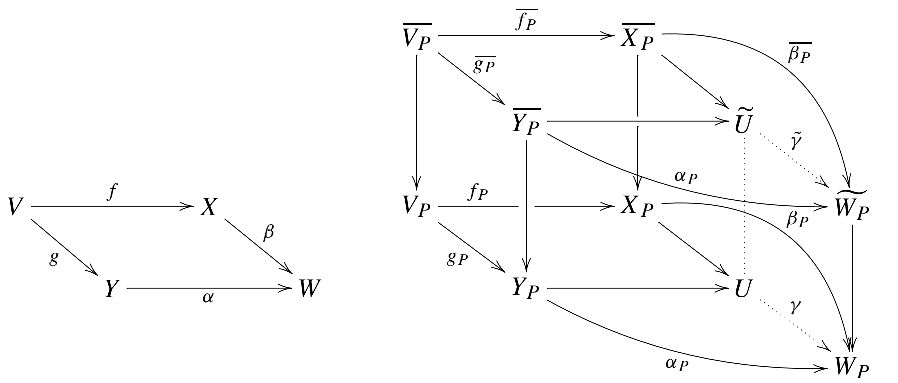
$$(\mathrm{ho}\varinjlim \underline{X})_P \simeq \big(\mathrm{ho}\varinjlim \underline{X_P}\big)_P.$$

For a specific example in the case of homotopy pushouts, consider the universal map $S^1_P \vee S^1_P \to (S^1 \vee S^1)_P$: Inspection of fundamental groups shows that it is not a homotopy equivalence. The following theorem gives conditions under which the canonical map

$$\mathrm{ho}\varinjlim(Y_P \leftarrow V_P \to X_P) \longrightarrow \big(\mathrm{ho}\varinjlim(Y \leftarrow V \to X)\big)_P$$

is a homotopy equivalence

Theorem 2.1. *Let $f:V \to X$ and $g:V \to Y$ be maps inducing epimorphisms of fundamental groups. Form the homotopy pushout diagram on the left below.*



Assume that the following conditions are satisfied:

- (i) *Each of the group homologies $\widetilde{H}_*(\ker \pi_1(\beta f)_P; \mathbb{Z})$, $\widetilde{H}_*(\ker \pi_1(\alpha_P); \mathbb{Z})$, and $\widetilde{H}_*(\ker \pi_1(\beta_P); \mathbb{Z})$ are graded $\mathbb{Z}_P$-modules;*
- (ii) *For $G := \pi_1 W_P$, the coefficient map $\mathbb{Z}_P G \to P[G]$ induces homology isomorphisms $\mathcal{H}_*(-; \mathbb{Z}_P G) \to \mathcal{H}_*(-; P[G])$.*

Then P-localization turns the left homotopy pushout into a homotopy pushout in $\mathrm{ho}\mathcal{CW}$.

Proof. For the argument P-localize the left pushout diagram to obtain the outer part of the right floor. The bottom square is a homotopy pushout, and γ is some map rendering the floor homotopy commutative. We claim that γ is a homotopy equivalence. To see this build the top floor by taking homotopy fibers over BG. Thus '$\widetilde{\ }$' denotes 'universal cover'. Consequently [12, 7.6.1], all sides of the right diagram are homotopy pullbacks. By Mather's 2nd cube theorem [9, p. 240], the top square is a homotopy pushout diagram. We show that $\tilde{\gamma}$ is a homotopy equivalence. The claim then follows because, with the long exact homotopy sequences of the relevant fibrations, we conclude that γ is a weak homotopy equivalence and, hence, a homotopy equivalence.

We first establish that the simply connected spaces $\widetilde{U}$ and $\widetilde{W_P}$ are P-local: $\widetilde{W}_P$ because its homotopy groups are P-local; $\widetilde{U}$ because $\widetilde{H}_*(\widetilde{U};\mathbb{Z})$ is a graded $\mathbb{Z}_P$-module; see [5, Theorem 3B]. We infer this latter property from: (1) $\overline{V_P}$, $\overline{X_P}$, $\overline{Y_P}$ have P-local fundamental groups which act P-locally on their higher homotopy groups. Therefore these spaces are P-local. (2) $\widetilde{H}_*(\overline{V_P};\mathbb{Z})$, $\widetilde{H}_*(\overline{X_P};\mathbb{Z})$, and $\widetilde{H}_*(\overline{Y_P};\mathbb{Z})$ are $\mathbb{Z}_P$-modules using hypothesis (i) and 2.2. The $H_*(-;\mathbb{Z})$-Mayer–Vietoris sequence of the top square shows that $\widetilde{H}_*(\widetilde{U};\mathbb{Z})$ is a $\mathbb{Z}_P$-module as well.

It follows that $\tilde{\gamma}$ is a homotopy equivalence if $H_*(\tilde{\gamma};\mathbb{Z}_P)$ is an isomorphism. To key to verifying this property are the isomorphisms below: for $A = V, X, Y$,

$$\mathcal{H}_*\big(W;\,P[G]\big) \cong \mathcal{H}_*\big(W_P;\,P[G]\big) \cong \mathcal{H}_*(W_P;\mathbb{Z}_P G) \cong H_*\big(\widetilde{W_P};\mathbb{Z}_P\big),$$
$$\mathcal{H}_*\big(A;\,P[G]\big) \cong \mathcal{H}_*\big(A_P;\,P[G]\big) \cong \mathcal{H}_*(A_P;\mathbb{Z}_P G) \cong H_*\big(\overline{A_P};\mathbb{Z}_P\big).$$

The last isomorphism holds because $\ker(\pi_1 A_P \to \pi_1 W_P)$ acts trivially on $\mathbb{Z}_P G$.

Thus the $\mathcal{H}_*(-;\,P[G])$-Mayer–Vietoris sequence of the diagram on the left turns into an $H_*(-;\mathbb{Z}_P)$-Mayer–Vietoris sequence of the outer rectangle of the top floor. Now $\tilde{\gamma}$ establishes a morphism of $H_*(-;\mathbb{Z}_P)$-Mayer–Vietoris sequences, and the 5-lemma shows that $H_*(\tilde{\gamma};\mathbb{Z}_P)$ is an isomorphism as claimed. □

The following lemma is needed in the proof of 2.1.

Lemma 2.2. *Let X be a P-local space such that the homology $\widetilde{H}_*(G;\mathbb{Z})$ of its fundamental group G is a graded $\mathbb{Z}_P$-module. Then $\widetilde{H}_*(X;\mathbb{Z})$ is a graded $\mathbb{Z}_P$-module.*

Proof. The Serre spectral sequence abutting to $H_*(X;\mathbb{Z})$ associated to the fibration $\widetilde{X} \to X \to BG$ has

$$E^2_{r,s} \cong \mathcal{H}_r\big(BG;\,H_s\big(\widetilde{X};\mathbb{Z}\big)\big).$$

From the assumptions we see that, for $(r,s) \neq (0,0)$, $E^2_{r,s}$ is a $\mathbb{Z}_P$-module. This implies the claim. □

Remark 2.3. The class of P-local groups G for which $\widetilde{H}_*(G;\mathbb{Z})$ is a $\mathbb{Z}_P$-module contains all finite P-groups [2, III.10], all nilpotent P-local groups, and is closed under directed colimits.

3. Strongly preserving homotopy epimorphisms

From 1.1 and 1.3 we deduce that P-localizing an epimorphism in ho$\mathcal{CW}$ yields an epimorphism in ho$\mathcal{CW}_P$. Here we address the question: for which epimorphisms f in ho$\mathcal{CW}$, apart from those of 1.2, is f_P also an epimorphism in ho$\mathcal{CW}$?

A homotopy epimorphism in ho$\mathcal{CW}$ necessarily induces an epimorphism of fundamental groups (consider maps into Eilenberg-Mac Lane spaces $K(\pi,1)$). Moreover,

P-localization of an epimorphism of groups always yields an epimorphism of groups. This is so because, in $\mathcal{G}rp$ categorical epimorphisms and surjective homomorphisms coincide. But in $\mathcal{G}rp$, P-localization turns surjective maps into surjective maps; [10, 2.2].

We give two complementary criteria for P-localization to turn an epimorphism in ho$\mathcal{CW}$ into an epimorphism in ho$\mathcal{CW}$. We make frequent and tacit use of 1.3.

Proposition 3.1. *Associated with an epimorphism f in* ho$\mathcal{CW}$ *consider the homotopy pushout diagrams below*:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle u} \\ Y & \xrightarrow[u]{} & W \end{array} \qquad\qquad \begin{array}{ccc} X_P & \xrightarrow{f_P} & Y_P \\ {\scriptstyle f_P}\downarrow & & \downarrow{\scriptstyle \beta} \\ Y_P & \xrightarrow[\alpha]{} & U \end{array}$$

If $\widetilde{H}_*(\pi_1 X_P; \mathbb{Z})$ is a $\mathbb{Z}_P$-module and if U is a nilpotent space, then f_P is an epimorphism in ho$\mathcal{CW}$.

Proof. First note that $\pi_1 Y_P \cong \pi_1 U$ is nilpotent. Therefore $\widetilde{H}_*(U; \mathbb{Z})$ is a graded $\mathbb{Z}_P$-module. This follows with Lemma 2.2 using the Mayer–Vietoris sequence of the diagram on the right. Thus the nilpotent space U is P-local, by [5, Theorem 3B]. On the other hand, P-localizing the diagram on the left yields a weak pushout diagram in the homotopy category of P-local spaces; see 1.1. Therefore $\alpha = \beta$, implying that f_P is a homotopy epimorphism in ho$\mathcal{CW}$. □

Theorem 3.2. *Let $f : X \to Y$ be an epimorphism in* ho$\mathcal{CW}$. *Assume that:*

(i) *the group homology $\widetilde{H}_*(\ker(\pi_1 f)_P; \mathbb{Z})$ is a graded $\mathbb{Z}_P$-module*;
(ii) *for $G := \pi_1 Y_P$, the coefficient map $\mathbb{Z}_P G \to P[G]$ induces homology isomorphisms $\mathcal{H}_*(-; \mathbb{Z}_P G) \to \mathcal{H}_*(-; P[G])$.*

Then f_P is an epimorphism in ho$\mathcal{CW}$.

Proof. It follows from 2.1 that P-localization preserves the homotopy pushout of $Y \xleftarrow{f} X \xrightarrow{f} Y$. With 1.3 we conclude that f_P is an epimorphism in ho$\mathcal{CW}$. □

Corollary 3.3. *Let $f : X \to Y$ be an epimorphism in* ho$\mathcal{CW}$ *such that $\pi_1 Y_P$ is a P-torsion group. If, in addition,* $\ker((\pi_1 X)_P \to (\pi_1 Y)_P)$ *is a directed colimit of nilpotent and P-torsion groups, then f_P is an epimorphism in* ho$\mathcal{CW}$.

Proof. This follows from 3.2: hypothesis 3.2(i) is satisfied by 2.3. Hypothesis 3.2(ii) is satisfied by [3, 2.18]. □

4. Preserving homotopy pullbacks

P-localization of spaces relates less directly to homotopy pullbacks and homotopy monomorphisms than to homotopy pushouts and homotopy epimorphisms. Various facets of this phenomenon are: (1) there is no pullback analogue of 1.1; (2) a homotopy monomorphism necessarily induces a monomorphism of homotopy groups (consider maps with a sphere as its domain); (3) P-localization of groups fails to preserve monomorphisms.

Here we give conditions under which P-localization preserves homotopy pullbacks. Our main result here depends upon two auxiliary results each of which has some interest in its own right: (1) Under which conditions does P-localization induce P-localization of all homotopy groups? (2) under which conditions does P-localization preserve a homotopy fibration? Our approach to answering these questions relies upon the concept of a P-torsion action space:

Definition 4.1. A space X is a P-torsion-action space (PTA-space) if there is a P-torsion group Q, called an acting torsion group, and an epimorphism $\pi_1 X \twoheadrightarrow Q$, such that, for each $k \geqslant 2$, the action homomorphism $\pi_1 X \to \mathrm{Aut}(\pi_k X)$ factors through Q.

Theorem 4.2. *Suppose X is a PTA-space with P-local fundamental group and acting torsion group Q. Then the following hold*:

(i) *For $k \geqslant 2$, $\pi_k X \to \pi_k X_P$ is P-localization.*
(ii) *X_P is a PTA-space with acting torsion group Q.*
(iii) *P-localization preserves the homotopy fiber sequence $X' \to X \to BQ$.*

Proof. First consider the case where $Q = 1$ is the 1-element group; i.e., where $G := \pi_1 X$ acts trivially on the higher homotopy groups of X. In the diagram below set $H := G$. So X' is the universal cover of X. Apply P-localization fiberwise [4, 1.F] to the fibration on the left to obtain the commutative diagram below.

$$\begin{array}{ccc} X' & \xrightarrow{u'} & X'_P \\ \downarrow & & \downarrow \\ X & \xrightarrow{u} & \overline{X} \\ \downarrow & & \downarrow \\ BH & = & BH \end{array}$$

The map u is a P-equivalence, and we claim that $\overline{X}$ is P-local. We know that u' P-localizes the homotopy groups of X'. Therefore the same applies to u. Further, the action of G on $\pi_k \overline{X}$ is trivial for $k \geqslant 2$. This follows from the fact that u induces G-module morphisms of higher homotopy groups: If $b \in \pi_k \overline{X}$, then $b = u(a)/n$ for some $a \in \pi_k X$ and $n \in P'$. If $g \in G$, it follows that

$$n(g \,.\, b) = g \,.\, u(a) = u(a).$$

As divisibility by n is unique in $\pi_k\overline{X}$, $g\,.\,b = b$. Thus $\overline{X}$ is P-local, and the claim follows in the case where $Q = 1$.

As for the general case, set $H := Q$ in the diagram above. The map u is a P-equivalence, and we claim that $\overline{X}$ is P-local. We know that $\pi_1 X \cong \pi_1\overline{X}$ is P-local. Further, we just saw that u' and, hence u, induce P-localization of higher homotopy groups. The action of $\pi_1 X'_P$ on the higher homotopy groups is trivial, hence factors through Q. Therefore $\pi_1\overline{X}$ is a PTA-space with acting torsion group Q, implying [3, 2.18] that $\overline{X}$ is P-local. Thus u P-localizes X. □

Theorem 4.3. *From the homotopy pullback diagram on the left*

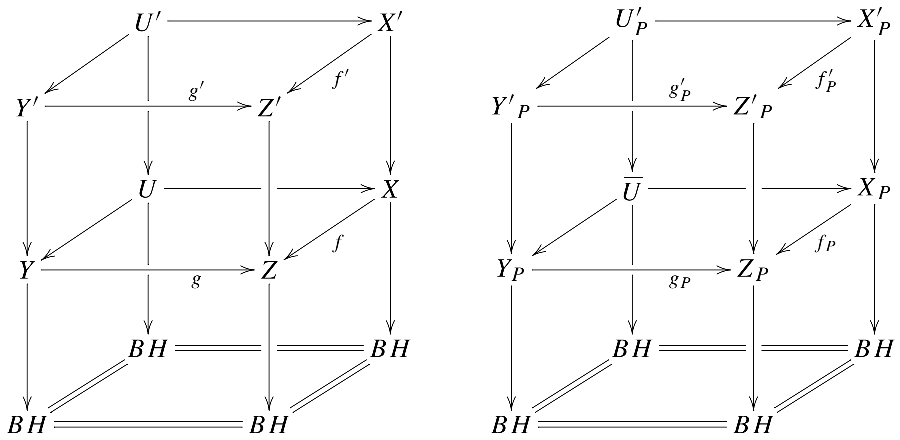
$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Y & \underset{g}{\longrightarrow} & Z \end{array} \qquad\qquad \begin{array}{ccc} U_P & \longrightarrow & X_P \\ \downarrow & & \downarrow f_P \\ Y_P & \underset{g_P}{\longrightarrow} & Z_P \end{array}$$

obtain the diagram on the right by P-localization. Assume the following:

(i) *The fundamental groups of X, Y, Z are P-local.*
(ii) *f and g induce isomorphisms of fundamental groups.*
(iii) *X, Y, Z are PTA-spaces with common acting torsion group Q.*

Then the diagram on the right is a homotopy pullback, and $U \to U_P$ P-localizes higher homotopy groups.

Proof. First consider the case where $Q = 1$. To build the left commutative diagram below set $H := G := \pi_1 Z$, and form the vertical fiber sequences; i.e., the top floor consists of universal covers of the middle floor.



All of the 6 faces of the top left cube are homotopy pullbacks. Apply P-localization fiberwise to obtain the homotopy commutative diagram on the right. Then all the sides of the right top cube are homotopy pullbacks. The top face on the right is also a homotopy pullback. To see this, note that tensoring the homotopy Mayer–Vietoris sequence of the top face on the left with $\mathbb{Z}_P$ yields a homotopy Mayer–Vietoris sequence of the top right square. The latter sequence maps into the homotopy Mayer–Vietoris sequence of the homotopy pullback diagram of $\widetilde{Y}_P \to \widetilde{Z}_P \leftarrow \widetilde{X}_P$, whose limit space we call W. With the 5-lemma we conclude that $\widetilde{U}_P \to W$ is a homotopy equivalence. It follows that the right middle floor is a homotopy pullback as well. Consequently [4, 1.10] $\overline{U}$ is also a P-local space, implying that the P-equivalence $U \to \overline{U}$ P-localizes. This implies the claim in the case where $Q = 1$.

For the general case, set $H := Q$ in the diagram above. Again, all of the 6 faces of the top left cube are homotopy pullbacks. Apply P-localization fiberwise to obtain the homotopy commutative diagram on the right. Then all the sides of the right top cube are homotopy pullbacks. We just saw that the top face on the right is also a homotopy pullback. It follows that the right middle floor is a homotopy pullback as well. Consequently [4, 1.10] $\overline{U}$ is also a P-local space, implying that the P-equivalence $U \to \overline{U}$ P-localizes. That $U \to U_P$ P-localizes higher homotopy groups follows with a homotopy Mayer–Vietoris argument. □

5. Preserving homotopy monomorphisms

A homotopy monomorphism $f : Y \to Z$ induces monomorphisms in homotopy groups (consider maps $\alpha, \beta : S^n \to Y$). This simple criterion allows us to see that localization in $\mathrm{ho}\mathcal{CW}$ at a set of primes P does not preserve every homotopy monomorphism:

Example 5.1. The map $f : BC_3 \to B\Sigma_3$, associated to the inclusion of the cyclic group C_3 of order 3 into the symmetric group Σ_3, is a monomorphism in $\mathrm{ho}\mathcal{CW}$. Localizing it at $P := \{3\}$ yields $(C_3)_P = C_3$, while $(\Sigma_3)_P = 1$. Therefore f_P cannot be a homotopy monomorphism.

On the other hand:

Lemma 5.2. *If $f : Y \to Z$ is a monomorphism in the homotopy category of P-local CW-spaces, it is automatically a monomorphism in* $\mathrm{ho}\mathcal{CW}$.

Proof. Consider the homotopy pullback diagram below.

$$\begin{array}{ccc} U & \xrightarrow{i} & Y \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[f]{} & Z \end{array}$$

By [4, 1.10], U is P-local. It follows that $i = j$ because f is a monomorphism in the homotopy category of P-local CW-spaces 1.3. However, the diagram is a weak pullback in $\mathrm{ho}\mathcal{CW}$. So f is a monomorphism in $\mathrm{ho}\mathcal{CW}$. □

Theorem 5.3. *Let $f:Y \to Z$ be a homotopy monomorphism. Suppose $\pi_1 f$ is an isomorphism of P-local fundamental groups and that Y and Z are PTA-spaces with common acting torsion group Q. Then $f_P:Y_P \to Z_P$ is a homotopy monomorphism.*

Proof. Consider the commutative diagrams below:

$$\begin{array}{ccc} U & \xrightarrow{i} & Y \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[f]{} & Z \end{array} \qquad\qquad \begin{array}{ccc} U_P & \xrightarrow{i_P} & Y_P \\ {\scriptstyle j_P}\downarrow & & \downarrow{\scriptstyle f_P} \\ Y_P & \xrightarrow[f_P]{} & Z_P \end{array}$$

If the diagram on the left is a homotopy pullback, we conclude $i = j$ from 1.3. By 4.3 the right diagram is a homotopy pullback, and it satisfies $i_P = j_P$. So f_P is a homotopy monomorphism. □

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