Differential Geometry and its Applications 30 (2012) 233–237

Contents lists available at SciVerse ScienceDirect

Differential Geometry and its Applications

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A criterion for detecting the identifiability of symmetric tensors of size three ☆

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ARTICLE INFO

Article history:
Received 23 September 2011
Available online 17 April 2012
Communicated by J.M. Landsberg

MSC:
14N05
15A69

Keywords:
Postulation of finite sets
Secant varieties of Veronese varieties

ABSTRACT

We prove a criterion for the identifiability of symmetric tensors P of type $3 \times \cdots \times 3$, d times, whose rank k is bounded by $(d^2 + 2d)/8$. The criterion is based on the study of the Hilbert function of a set of points $P_1, \ldots, P_k$ which computes the rank of the tensor P.

1. Introduction

The aim of this paper is to study criteria which can assure that an explicitly given symmetric tensor, whose rank k is known, is identifiable, i.e. it can be written *uniquely* (up to scalar multiplication and permutations), as a sum of decomposable tensors.

Recently, new methods for studying the identifiability of tensors are arising from the theory of secant varieties to projective varieties, and their tangential behavior.

In the paper, we deal with symmetric tensors (and their geometric counterpart: the space of a Veronese embedding of a projective space). Let us introduce some definition, in order to properly state the problem, along with our achievements.

Let $\mathbb{P}^n := \mathbb{P}^n_{\mathbb{C}}$ be a projective space over the complex field.

Write ν_d for the d-th Veronese map, which sends $\mathbb{P}^n$ to a space $\mathbb{P}^N$, with $N = \binom{n+d}{n} - 1$. The embedding space $\mathbb{P}^N$ can be seen as the space of symmetric tensors (up to scalar multiplication) of type $(n+1) \times (n+1) \times \cdots \times (n+1)$. We call $n+1$ the *size* of these tensors.

The image $X := \nu_d(\mathbb{P}^n)$ corresponds to the subset parameterizing decomposable symmetric tensors (as always: up to scalar multiplication). The *rank* of $P \in \mathbb{P}^N$ is the minimum k for which there exists an expression

$$P = P_1 + \cdots + P_k$$

with $P_i \in X$ for all i.

Identifiability is related with the uniqueness of the previous expression.

☆ The authors were partially supported by MIUR and GNSAGA of INdAM (Italy).
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0926-2245/$ – see front matter
http://dx.doi.org/10.1016/j.difgeo.2012.04.004

Definition 1.1. We say that $P \in \mathbb{P}^N$, of rank k, is *identifiable* if there is a unique expression of P (up to scaling and permutations) as sum of k elements in X.

In geometric terms, X is a projective variety (the *d-th Veronese variety of* $\mathbb{P}^n$) and the rank of a tensor P corresponds to the minimal k such that P belongs to the *standard open subset* $U_k(X)$ of the secant variety $S_k(X)$, formed by $(k-1)$-spaces spanned by k distinct points of X.

Following the classical Terracini's analysis of the tangent spaces to secant varieties, one obtains criteria for detecting when a *general* tensor P of rank k is identifiable. An account of how this can be done can be found in [6]. Let we recall briefly what happens for the *general* symmetric tensor of rank k.

It is a general non-sense that when the dimension of the secant variety $U_k(X)$ is not the expected value (i.e. when X is $(k-1)$-*defective*), then also identifiability fails. After the results of [1], the cases in which the Veronese variety $\nu_d(\mathbb{P}^n)$ is defective, are well known. On the other hand, there are cases in which the dimension attains the expected value, and nevertheless the general symmetric tensor is not identifiable. For $n = 2$, it is classically known that identifiability of the generic symmetric tensor fails, besides the defective cases, only when $d = 3$ and $k = 6$ (see [2]). In higher dimension, by the results of [6], and the analysis of the tangential behaviour of Veronese varieties, carried on by the first author in [3], one knows that the general symmetric tensor of rank k is identifiable, as soon as $k < (N+1)/(n+1)$, with the only possible exception $(n,d) = (3,4)$ (and the defective cases, listed in [1]).

The previous methods, however, only tell us about generic tensors, but do not apply to detect whether or not a *specific* tensor P is identifiable.

Remark 1.2. Indeed, if we know that the general tensor of rank k is identifiable, then we can say that every point, of rank k, in the regular locus of $S_k(X)$ is identifiable, by the Zariski Main Theorem.

On the other hand, since the equations for secant varieties are far from been known, it seems uneasy to detect directly whether or not a given P belongs to the singular locus of $S_k(X)$.

In a private conversation, Joseph Landsberg asked one of us about the chance of finding *some* criteria for the identifiability of a specific, given tensor.

Landsberg himself, with Buczyński and Ginensky, found a criterion which works for symmetric tensors of any size and dimension d, provided that the rank k is at most $(d+1)/2$ (see [5]). The criterion thus works for tensors whose rank increases linearly, with respect to d.

Landsberg's problem amounts also to determine methods for certifying that a given point of the standard open subset $U_k(X) \subset S_k(X)$, is not singular.

Following an idea developed by A. Bernardi and the first author (see [4]), we are able to produce here, for the case $n = 2$ and in some range for the rank k, a criterion for detecting identifiability.

Our method is based on the study of the Hilbert function of a set of points $Z = \{x_1, \dots, x_k\}$, such that $P = \nu_d(x_1) + \dots + \nu_d(x_k)$, i.e. such that P belongs to the linear span

$$P \in \langle \nu_d(x_1), \dots, \nu_d(x_k) \rangle.$$

Let us recall the following:

Definition 1.3. A set of k distinct points $Z \in \mathbb{P}^n$ has *general uniform position* (*GUP*) if for any $m = 1, \dots, k$, no subsets $Z' \subset Z$ of cardinality m belong to hypersurfaces of degree u, as soon as

$$m \geqslant \binom{u+n}{n}.$$

It is known that general sets of points have GUP, and the Hilbert function of points with GUP is well known, when $n = 2$, i.e. when Z sits in a plane.

With this in mind, by using standard results for the Hilbert function of points in the plane (we will refer to [8] for this), as well as by means of Lemma 8 in [4], we are able to give a criterion for the identifiability of symmetric tensors of size 3, i.e. points in the projective space of the Veronese variety $\nu_d(\mathbb{P}^2)$.

Theorem 1.4. *Consider the Veronese variety $X = \nu_d(\mathbb{P}^2)$, $n > 2$, embedded in the space $\mathbb{P}^N$, $N = d(d+3)/2$. Let $P \in \mathbb{P}^N$ be a point of rank k, $P = P_1 + \dots + P_k$, with $P_i = \nu_d(x_i)$. Assume that the subset $Z = \{x_1, \dots, x_k\} \subset \mathbb{P}^2$ has GUP, and*

$$k < \frac{d^2 + 2d}{8}.$$

Then P is identifiable.

By means of the Zariski Main Theorem, the previous theorem can be rephrased in terms of the singular locus of $S_k(X)$.

Corollary 1.5. *Let X be the Veronese surface $X = \nu_d(\mathbb{P}^2)$ and consider a subset $Z = \{x_1, \dots, x_k\} \subset \mathbb{P}^2$ with GUP. Assume $k < (d^2 + 2d)/8$ and consider the span*

$$L = \langle \nu_d(x_1), \dots, \nu_d(x_k) \rangle.$$

Then $L \cap U_k(X)$ meets the singular locus $\mathrm{Sing}(S_k(X))$ only along a subset of $S_{k-1}(X)$.

Going back to Landsberg's problem, we notice that the effectiveness of the criterion for deciding the identifiability of a given P depends on how much we know about P. In particular, we need to know:

- the rank k of P;
- one decomposition $P = P_1 + \dots + P_k$, $P_i \in X$.

Then, assuming that we are in the range $k < (d^2 + 2d)/8$ (quadratic, with respect to the dimension of the tensor), it is easy to compute the set $Z = \{x_1, \dots, x_k\}$, with $P_i = \nu_d(x_i)$, and see if it has GUP.

Although these assumptions require a certain knowledge about the tensor P, we hope that the criterion could be effective, in some interesting cases.

On the other hand, the criterion has some intriguing geometric aspects. To mention one: a link between the postulation of Z and the identifiability of points in $\langle Z \rangle$.

It is, in any event, a starting point. We cannot exclude that, on the same lines, it will be possible to produce criteria with a wider range of applicability.

2. Proof of the criterion

We keep here the notation of the Introduction, from the geometric point of view.

So, $X = \nu_d(\mathbb{P}^2)$ is the d-th Veronese embedding of the plane in $\mathbb{P}^N$, $N = (d + 3)d/2$. P is a point of $\mathbb{P}^N$, which has rank $k > 1$. Fix k points $x_1, \dots, x_k$ of $\mathbb{P}^2$ such that

$$P \in \langle \nu_d(x_1), \dots, \nu_d(x_k) \rangle.$$

Write $Z = \{x_1, \dots, x_k\}$. We make the following assumptions:

- $k < (d^2 + 2d)/8$;
- Z has GUP.

We want to prove that P is identifiable.

Assume, on the contrary, that there is another subset $Z' \subset \mathbb{P}^2$, $Z' = \{y_1, \dots, y_k\}$, of cardinality k, such that $P \in \langle \nu_d(Z') \rangle$.

Call W the union $W = Z \cup Z'$, which is a subset of cardinality $w \leqslant 2k$. We will look carefully at the Hilbert function h_W of W and at its difference function Dh_W.

Claim 2.1. *$h_W(d) < w$, so that $Dh_W(d + 1) > 0$.*

Proof. By our first assumption on P, it turns out that the linear spans of both $\nu_d(Z)$ and $\nu_d(Z')$ have dimension k. Moreover they meet in a point P which cannot lie in the linear span of the intersection $\nu_d(Z) \cap \nu_d(Z')$. It follows that $\nu_k(W)$ does not impose $w = 2k - \#(\nu_d(Z) \cap \nu_d(Z'))$ conditions to the hyperplanes of $\mathbb{P}^N$, from which the claims on h_W follow at once. □

Now, we use the numerical assumption, together with a knowledge of the main properties of Hilbert functions of subschemes of $\mathbb{P}^2$.

Claim 2.2. *Let u be the integer such that*

$$\frac{u^2 + 3u + 2}{2} \leqslant k < \frac{(u + 1)^2 + 3(u + 1) + 2}{2}.$$

Then $u + 2 \leqslant d/2$ and the function Dh_W satisfies

$$Dh_W(i) = i + 1 \quad \text{for } i = 0, \dots, u.$$

Proof. The first inequality follows immediately from the assumption $k < (d^2 + 2d)/8$. The second one follows from the fact that Z has GUP, and thus

$$h^0(I_W(u)) \leqslant h^0(I_Z(u)) = 0. \qquad \square$$

Claim 2.3. *There exists some $j \leqslant d$ with*

$$u+1 > Dh_W(j) = Dh_W(j+1) > 0.$$

Proof. First observe that the definition of u and the numerical assumption $k < (d^2+2d)/8$ imply that

$$2k \leqslant (u+1)d - u^2 + u + 2. \tag{2.1}$$

On the other hand, if the quoted j does not exist, we must have

$$Dh_W(d+1-i) \geqslant i+1 \quad \text{for } i = 0, \dots, u$$

which gives, after a short computation, that the cardinality of W satisfies

$$\#(W) \geqslant (u+1)d - u^2 + u + 2,$$

a contradiction. □

Define the number m as

$$m = \min\bigl\{Dh_W(j)\colon\ j \leqslant d \text{ and } Dh_W(j) = Dh_W(j+1) > 0\bigr\}.$$

By the previous claim, we know that $m \leqslant u$.

Claim 2.4. *There exists a curve $M \subset \mathbb{P}^2$, of degree m, such that M intersects W in a subset A of cardinality $a \geqslant (m+1)d - m^2 + m + 2$.*

Proof. This is an easy consequence of well-known facts on sets of points W in the plane, whose function Dh_W has the behaviour described in Claim 2.3.

Namely (see e.g. [7] or [8], proposition at p. 112), since

$$\max\bigl\{Dh_W(i)\bigr\} > m = Dh_W(j) = Dh_W(j+1) > 0,$$

we know that there exists a curve M of degree m which meets W in a subset A whose Hilbert function is defined as

$$Dh_A = \min\{m, Dh_W\}.$$

It follows immediately that the cardinality of A is at least:

$$a = \frac{m(m+1)}{2} + (m+1)(d+2-2m) + \frac{m(m+1)}{2},$$

which gives the claim. □

Now we have all the ingredients for the:

Proof of the Main Theorem. Define $B = W \setminus M$. By [8, p. 112], we know that the function Dh_B satisfies, for all i: $Dh_A(i) + Dh_B(i-m) = Dh_W(i)$.

Since $Dh_W(d) \leqslant m$, then $Dh_A(d) = \min\{m, Dh_W(d)\} = Dh_W(d)$, so that one has $Dh_B(d-m) = 0$. It follows from [4, Lemma 8], that $Z - M = Z' - M$, which implies that $Z \cap M$ has the same cardinality as $Z' \cap M$.

Thus $Z \cap M$ has cardinality:

$$\frac{\#(A)}{2} \geqslant \frac{(m+1)d - m^2 + m + 2}{2},$$

and sits in a curve of degree M. Since Z has GUP, we get that

$$\frac{(m+1)d - m^2 + m + 2}{2} < \frac{m^2 + 3m + 2}{2}.$$

This implies $d \leqslant 2m$, which is impossible, since $m < u$ and $u < d/2$, by Claim 2.2. □

Remark 2.5. Let us notice that, with the same method, one can prove a slightly stronger condition. Namely, with the previous assumptions, it follows that P cannot belong to the linear span of another subscheme $\nu_d(Z')$, with $\deg(Z') = k$, dropping the assumption that Z' is reduced.

Indeed, in this case, we may define $W = Z' \cup (Z \setminus Z')$. The results on the Hilbert function of 0-dimensional subschemes of $\mathbb{P}^2$, obtained in [8], remain true even if W is not reduced, as well as Lemma 8 of [4]. So, the previous arguments work verbatim.

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