Available online at www.sciencedirect.com

SciVerse ScienceDirect

Stochastic Processes and their Applications 122 (2012) 42–67

stochastic processes and their applications

www.elsevier.com/locate/spa

# Asymptotics of stationary solutions of multivariate stochastic recursions with heavy tailed inputs and related limit theorems

Dariusz Buraczewski, Ewa Damek, Mariusz Mirek*

*Uniwersytet Wroclawski, Instytut Matematyczny, pl. Grunwaldzki 2/4, 50-384 Wroclaw, Poland*

Received 19 October 2010; received in revised form 17 October 2011; accepted 18 October 2011
Available online 28 October 2011

---

**Abstract**

Let $\Phi_n$ be an i.i.d. sequence of Lipschitz mappings of $\mathbb{R}^d$. We study the Markov chain $\{X_n^x\}_{n=0}^{\infty}$ on $\mathbb{R}^d$ defined by the recursion $X_n^x = \Phi_n(X_{n-1}^x), n \in \mathbb{N}, X_0^x = x \in \mathbb{R}^d$. We assume that $\Phi_n(x) = \Phi(A_n x, B_n(x))$ for a fixed continuous function $\Phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, commuting with dilations and i.i.d random pairs $(A_n, B_n)$, where $A_n \in \mathrm{End}(\mathbb{R}^d)$ and $B_n$ is a continuous mapping of $\mathbb{R}^d$. Moreover, $B_n$ is $\alpha$-regularly varying and $A_n$ has a faster decay at infinity than $B_n$. We prove that the stationary measure $\nu$ of the Markov chain $\{X_n^x\}$ is $\alpha$-regularly varying. Using this result we show that, if $\alpha < 2$, the partial sums $S_n^x = \sum_{k=1}^n X_k^x$, appropriately normalized, converge to an $\alpha$-stable random variable. In particular, we obtain new results concerning the random coefficient autoregressive process $X_n = A_n X_{n-1} + B_n$.

*MSC:* 60J10; 60F05; 60B15

*Keywords:* Markov chains; Stationary measures; Heavy tailed random variables; Limit theorems

---

## 1. Introduction and main results

We consider the vector space $\mathbb{R}^d$ endowed with an arbitrary norm $|\cdot|$. We fix once and for all a continuous mapping $\Phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, commuting with dilations, i.e. $\Phi(tx, ty) = t\Phi(x, y)$ for every $t > 0$. Let $(A, B)$ be a random pair, where $A \in \mathrm{End}(\mathbb{R}^d)$ and $B$ is a continuous

---

* Corresponding author.

*E-mail addresses:* dbura@math.uni.wroc.pl (D. Buraczewski), edamek@math.uni.wroc.pl (E. Damek), mirek@math.uni.wroc.pl (M. Mirek).

doi:10.1016/j.spa.2011.10.010

mapping of $\mathbb{R}^d$. We assume that $B$ is of the form $B(x) = B^1 + B^2(x)$, where $B^1$ is a random vector in $\mathbb{R}^d$ and $B^2$ is a random mapping of $\mathbb{R}^d$ such that $|B^2(x)| \le B^3|x|^{\delta_0}$ for every $x \in \mathbb{R}^d$, where $\delta_0 \in [0, 1)$ is a fixed number and $B^3 \ge 0$ is random. Given a sequence $(A_n, B_n)_{n\in\mathbb{N}}$ of independent random copies of the generic pair $(A, B)$ and a starting point $x \in \mathbb{R}^d$, we define the Markov chain by

$$\begin{aligned} &X_0^x = x, \\ &X_n^x = \Phi(A_n X_{n-1}^x, B_n(X_{n-1}^x)), \quad \text{for } n \in \mathbb{N}. \end{aligned} \tag{1.1}$$

If $x = 0$ we just write for simplicity $X_n$ instead of $X_n^0$. Also, to simplify the notation, let $\Phi_n(x) = \Phi(A_n x, B_n(x))$. Then the definition above can be expressed in a more concise way: $X_n^x = \Phi_n(X_{n-1}^x)$.

The main example we have in mind is a random coefficient autoregressive process on $\mathbb{R}^d$, called also a random difference equation or an affine stochastic recursion. This process is defined by

$$X_{1,n}^x = A_n X_{1,n-1}^x + B_n. \tag{1.2}$$

And as one can easily see it is a particular example of (1.1), just by taking $\Phi(x, y) = x + y$ and $B_n^2 \equiv 0$.

For an another example take $d = 1$, $\Phi(x, y) = \max(x, y)$ and $B_n^2 \equiv 0$. Then we obtain the random extremal equation

$$X_{2,n}^x = \max(A_n X_{2,n-1}^x, B_n), \tag{1.3}$$

studied e.g. by Goldie [15].

In this paper we assume that the Markov chain $\{X_n^x\}$ is $\gamma$-geometric. This means that there are constants $0 < C < \infty$ and $0 < \rho < 1$ such that the moment of order $\gamma > 0$ of the Lipschitz coefficient of $\Phi_n \circ \cdots \circ \Phi_1$ decreases exponentially fast as $n$ goes to infinity, i.e.

$$\mathbb{E}\left[\left|X_n^x - X_n^y\right|^\gamma\right] \le C\rho^n |x - y|^\gamma, \quad n \in \mathbb{N}, x, y \in \mathbb{R}^d. \tag{1.4}$$

We say that a random vector $W \in \mathbb{R}^d$ is regularly varying with index $\alpha > 0$ (or $\alpha$-regularly varying) if there is a slowly varying function $L$ such that the limit

$$\lim_{t\to\infty} t^\alpha L(t)\mathbb{E}\left[f(t^{-1}W)\right] = \int_{\mathbb{R}^d\setminus\{0\}} f(x)\Lambda(dx) =: \langle f, \Lambda\rangle, \tag{1.5}$$

exists for every $f \in C_c(\mathbb{R}^d \setminus \{0\})$ and thus defines a Radon measure $\Lambda$ on $\mathbb{R}^d \setminus \{0\}$. The measure $\Lambda$ will be called the tail measure. It can be easily checked that $\int_{\mathbb{R}^d\setminus\{0\}} f(rx)\Lambda(dx) = r^\alpha\langle f, \Lambda\rangle$ for every $r > 0$, and so the tail measure $\Lambda$ is $\alpha$-homogeneous, i.e. in radial coordinates we have

$$\langle f, \Lambda\rangle = \int_0^\infty \int_{\mathbb{S}^{d-1}} f(r\omega)\, \sigma_\Lambda(d\omega)\frac{dr}{r^{1+\alpha}}, \tag{1.6}$$

for some measure $\sigma_\Lambda$ on the unit sphere $\mathbb{S}^{d-1} \subseteq \mathbb{R}^d$. The measure $\sigma_\Lambda$ will we called the spherical measure of $\Lambda$. Observe that $\sigma_\Lambda$ is nonzero if and only if $\Lambda$ is nonzero.

Under mild assumptions there exists a unique stationary distribution $\nu$ of $\{X_n^x\}$ (see Lemma 2.2). The main purpose of this paper is to prove, under some further hypotheses, that the distribution $\nu$ is $\alpha$-regularly varying and next to obtain a limit theorem for partial sums $S_n^x = \sum_{k=1}^n X_k^x$.

Our first main result is the following:

**Theorem 1.7.** *Let* $\{X_n^x\}$ *be the Markov chain defined by* (1.1). *Assume that*

- $B^1$ *is* $\alpha$*-regularly varying with the nonzero tail measure* $\Lambda_b$ *and the corresponding slowly varying function* $L_b$ *is bounded away from zero and infinity on any compact set;*
- *the Markov chain* $\{X_n^x\}$ *is* $\gamma$*-geometric for some* $\gamma > \alpha$*;*
- *there exists* $\beta > \alpha$ *such that* $\mathbb{E}\|A\|^\beta < \infty$*;*
- *there exists* $\varepsilon_0 > 0$ *such that* $\mathbb{E}\left[(B^3)^{\frac{\alpha}{\delta_0}+\varepsilon_0}\right] < \infty$, *if* $0 < \delta_0 < 1$ *and* $\mathbb{E}\left[(B^3)^{\alpha+\varepsilon_0}\right] < \infty$, *if* $\delta_0 = 0$*;*
- $\mathbb{P}[B^1 : \Phi(0, B^1) \neq 0] > 0$.

*Then the Markov chain* $\{X_n^x\}$ *has a unique stationary measure* $\nu$. *If* $X$ *is a random variable distributed according to* $\nu$, *then* $X$ *is* $\alpha$*-regularly varying with a nonzero tail measure* $\Lambda^1$, *i.e. for every* $f \in C_c(\mathbb{R}^d \setminus \{0\})$

$$\lim_{t\to\infty} t^\alpha L_b(t)\mathbb{E}\left[f(t^{-1}X)\right] = \langle f, \Lambda^1\rangle. \tag{1.8}$$

*Moreover, the above convergence holds for every bounded function* $f$ *such that* $0 \notin \operatorname{supp} f$ *and* $\Lambda^1(\operatorname{Dis}(f)) = 0$ *(*$\operatorname{Dis}(f)$ *is the set of all discontinuities of the function* $f$*). In particular*

$$\lim_{t\to\infty} t^\alpha L_b(t)\mathbb{P}\left[|X| > t\right] = \langle \mathbf{1}_{\{|\cdot|>1\}}, \Lambda^1\rangle.$$

There are many results describing existence of stationary measures of Markov chains and their tails, especially in the context of general stochastic recursions (see e.g. [11,15] for the one-dimensional case and [27] for the multidimensional one). Let us return for a moment to the example of the autoregressive process (1.2). It is well-known that if $\mathbb{E}\log^+\|A_1\| < \infty$, then the Lyapunov exponent $\lambda = \lim_{n\to\infty}\frac{1}{n}\log\|A_1\cdot\cdots\cdot A_n\|$ exists and it is constant a.s. [14]. Moreover, if $\lambda < 0$ and $\mathbb{E}\log^+|B_1| < \infty$, then the process $X_n$ converges in distribution to the random vector

$$X = \sum_{n=1}^{\infty} A_1\cdot\cdots\cdot A_{n-1}B_n, \tag{1.9}$$

whose law $\nu_1$ is the unique stationary measure of the process $\{X_{1,n}\}$. Properties of the measure $\nu_1$ are well described. The most significant result is due to Kesten [22], who proved, under a number of hypotheses, the main ones being $\lim_{n\to\infty}(\mathbb{E}\|A_1\cdot\cdots\cdot A_n\|^\alpha)^{\frac{1}{n}} = 1$ and $\mathbb{E}|B|^\alpha < \infty$, for some $\alpha > 0$, that the measure $\nu_1$ of $\{X_{1,n}^x\}$ is $\alpha$-regularly varying at infinity (indeed, Kesten proved weaker convergence; however in this context it turns out to be equivalent to the definition of $\alpha$-regularly varying measures—see [3,5]). A short and elegant proof of this result in one-dimensional settings was given by Goldie [15]. Other multidimensional results were obtained in [1,8,18,24,25].

However, the theorem above concerns a slightly different situation. For the autoregressive process, Theorem 1.7 deals with the case where the $B$-part is dominating. If we assume that $B_1$ is $\alpha$-regularly varying, $\lim_{n\to\infty}(\mathbb{E}\|A_1\cdot\cdots\cdot A_n\|^\alpha)^{\frac{1}{n}} < 1$ (then the Markov chain $X_{1,n}$ is $\alpha$-geometric) and $\mathbb{E}\|A_1\|^\beta < \infty$ for some $\beta > \alpha$, then the hypotheses of Theorem 1.7 are satisfied and we conclude that $\nu_1$ is $\alpha$-regularly varying. In this particular case similar results were proved in one dimension by Grey [16] and Grincevicius [17] and in the multivariate setting in [21,29]. However, [29] deals with the situation of independent $A_n$ and $B_n$ and in [21] a particular norm $|\sum_{i=1}^{d} x_i e_i| = \max_{i=1}^{d} |x_i|$ is considered. Theorem 1.7 holds for an arbitrary norm and so it provides a new result even for the recursion (1.2).

Our approach is more general and it may be applied to a larger class of Lipschitz recursions. It is valid for multidimensional generalizations of the autoregressive process, e.g. for recursions: $X_{2,n} = A_n X_{2,n-1} + B_n + C_n(x)$, $X_{3,n} = \max\{A_n X_{3,n-1}, B_n\}$, $X_{4,n} = \max\{A_n X_{4,n-1}, B_n\} + C_n$, where $\max\{x, y\} = (\max\{x_1, y_1\}, \dots, \max\{x_d, y_d\})$, for $x, y \in \mathbb{R}^d$. Some of these processes were studied in a similar context in one dimension in [15,16,27]. Under appropriate assumptions, each of these recursions possesses a unique stationary measure and its tail is described by Theorem 1.7.

Let us explain the $\gamma$-geometricity assumption (1.4), which ensures contractivity of the system. The standard approach to stochastic recursions is to assume that the consecutive random mappings are contractive on average, i.e. $\mathbb{E}\left[\log \mathrm{Lip}(\Phi_n)\right] < 0$, where $\mathrm{Lip}(\Phi_n)$ denotes the Lipschitz coefficient of $\Phi_n$ (see e.g. [11]). However, in higher dimensions this approach does not provide sufficiently exact information. One can easily construct a stochastic recursion where Lipschitz coefficients of random mappings are larger than 1, but the system still possesses some contractivity properties. For example, consider on $\mathbb{R}^2$ the autoregressive process, where $A$ is a random diagonal matrix with entries on the diagonal $(2, 1/3)$ and $(1/3, 2)$ both with probability $1/2$. Then the Lipschitz coefficient of $A$ is always 2, but since $X_n^x - X_n^y = A_n \cdot \dots \cdot A_1(x - y)$, the corresponding Markov chain is $\gamma$-geometric for small values of $\gamma$; thus this is a contractive system. This is why to study the autoregressive process in higher dimensions one has to consider the Lyapunov exponents, not Lipschitz coefficients. And, this is also why we introduce in more general settings the concept of $\gamma$-geometric random processes.

Let $\mu$ be the law of $A$ and $[\mathrm{supp}\,\mu] \subseteq \mathrm{End}(\mathbb{R}^d)$ be the semigroup generated by the support of $\mu$. It turns out that in a sense formula (1.9) is universal and, even in the general settings, the tail measures can be described by similar expressions. Our next theorem is mainly a consequence of the previous one, but provides a precise description of the tail measure $\Lambda^1$. This result is interesting in its own right, but will play also a crucial role in the proof of the limit theorem.

Before stating the theorem let us define a sequence $(\Gamma_n)$ of Radon measures on $\mathbb{R}^d \setminus \{0\}$ as follows. Let $\Gamma_1$ be the tail measure of $\Phi(0, B^1)$ (we will prove in Lemma 2.6 that $\Phi(0, B^1)$ is $\alpha$-regularly varying). For $n \geq 2$, we define $\langle f, \Gamma_n \rangle = \mathbb{E}\left[\langle f \circ A_2 \circ \dots \circ A_n, \Gamma_1 \rangle\right]$.

**Theorem 1.10.** *Suppose the assumptions of Theorem 1.7 are satisfied. If* $\Phi(x, 0) = x$ *for every* $x \in \overline{[\mathrm{supp}\,\mu] \cdot \Phi[\{0\} \times \mathrm{supp}\,\Lambda_b]}$, *and* $\lim_{n\to\infty} \left(\mathbb{E}\|A_1 \cdot \dots \cdot A_n\|^{\alpha}\right)^{\frac{1}{n}} < 1$, *then the tail measure* $\Lambda^1$ *defined in* (1.8) *can be expressed as*

$$\langle f, \Lambda^1 \rangle = \sum_{k=1}^{\infty} \langle f, \Gamma_k \rangle = \langle f, \Gamma_1 \rangle + \mathbb{E}\left[\sum_{k=2}^{\infty} \langle f \circ A_2 \circ \dots \circ A_k, \Gamma_1 \rangle\right]. \tag{1.11}$$

*Furthermore, the measures* $\Gamma_n$ *are* $\alpha$*-homogeneous and their spherical measures satisfy*

$$\mathbb{E}\left[\int_{\mathbb{S}^{d-1}} f(A * \omega)\, |A\omega|^{\alpha} \sigma_{\Gamma_n}(d\omega)\right] = \int_{\mathbb{S}^{d-1}} f(\omega) \sigma_{\Gamma_{n+1}}(d\omega), \tag{1.12}$$

*for every* $n \in \mathbb{N}$ *and* $f \in C(\mathbb{S}^{d-1})$, *where* $A * \omega = \frac{A\omega}{|A\omega|}$. *In particular, the spherical measure of* $\Lambda^1$ *is given by*

$$\sigma_{\Lambda^1}(d\omega) = \sum_{n=1}^{\infty} \sigma_{\Gamma_n}(d\omega). \tag{1.13}$$

**Remark 1.14.** The condition: $\Phi(x,0)=x$ for every $x\in\overline{[\operatorname{supp}\mu]\cdot\Phi[\{0\}\times\operatorname{supp}\Lambda_b]}\subseteq\mathbb{R}^d$ is only a technical assumption which can be easily verified in many cases. Indeed, in the case of the recursion (1.2), we know that $\Phi(x,y)=x+y$ and then one has nothing to check. In the case of the recursion (1.3), $\Phi(x,y)=\max\{x,y\}$ and then $\Phi(x,0)=x$ holds only for $x\in[0,\infty)$, so we need to know whether $\overline{[\operatorname{supp}\mu]\cdot\Phi[\{0\}\times\operatorname{supp}\Lambda_b]}\subseteq[0,\infty)$. It is clear that the inclusion depends on the underlying random variables $A$ and $B^1$, and the sufficient assumptions are $\mathbb{P}[A\geq 0]=1$ and $\lim_{t\to\infty}t^\alpha\mathbb{P}[B^1>t]=c>0$.

In the second part of the paper we study the behavior of the Birkhoff sums $S_n^x$. We prove that if $\alpha\in(0,2)$ then there are constants $d_n,a_n$ such that $a_n^{-1}S_n^x-d_n$ converges in law to an $\alpha$-stable random variable. In order to state our results we need some further hypotheses and definitions.

The normalization of partial sums will be given by the sequence of numbers $a_n$ defined by the formula

$$a_n=\inf\left\{t>0:\nu\{x\in\mathbb{R}^d:|x|>t\}\leq 1/n\right\},$$

where $\nu$ is the stationary distribution of $\{X_n^x\}$. One can easily prove that (see Theorem 7.7 in [12], page 151)

$$\lim_{n\to\infty}n\mathbb{P}(|X|>a_n)=1\quad\text{and}\quad\lim_{n\to\infty}\frac{a_n^\alpha L_b(a_n)}{n}=\langle\mathbf{1}_{\{|\cdot|>1\}},\Lambda^1\rangle=c>0,\tag{1.15}$$

for $\Lambda^1$ being the tail measure of the stationary solution $X$ as in Theorem 1.7.

The characteristic functions of limiting random variables depend on the measure $\Lambda^1$. However, in their description another Markov chain will play a significant role. Let $W_n^x=\overline{\Phi}_n(W_{n-1}^x)$, where $W_0^x=x\in\mathbb{R}^d$, $\overline{\Phi}_n(x)=\Phi(A_nx,0)$ and let $W(x)=\sum_{k=1}^\infty W_k^x$. Then $W_n^x$ is a particular case of recursion (1.1), with $B_n=0$. Given $v\in\mathbb{R}^d$ we define $h_v(x)=\mathbb{E}\left[e^{i\langle v,W(x)\rangle}\right]$.

Our next result is:

**Theorem 1.16.** *Suppose that the assumptions of Theorem 1.7 are satisfied for some $\alpha\in(0,2)$. Assume additionally that $\Phi$ is a Lipschitz mapping and that there is a finite constant $C>0$ such that $|B^2|\leq C$ a.e. Then the sequence $a_n^{-1}S_n^x-d_n$ converges in law to an $\alpha$-stable random variable with the Fourier transform $\Upsilon_\alpha(tv)=\exp C_\alpha(tv)$, for*

$$C_\alpha(tv)=\frac{t^\alpha}{c}\int_{\mathbb{R}^d}\left(\left(e^{i\langle v,x\rangle}-1\right)h_v(x)\right)\Lambda^1(dx),\qquad\text{if }\alpha\in(0,1);$$

$$C_1(tv)=\frac{t}{c}\int_{\mathbb{R}^d}\left(\left(e^{i\langle v,x\rangle}-1\right)h_v(x)-\frac{i\langle v,x\rangle}{1+|x|^2}\right)\Lambda^1(dx)-\frac{it\log t\langle v,m_{\sigma_{\Lambda^1}}\rangle}{c},$$
*if $\alpha=1$;*
$$C_\alpha(tv)=\frac{t^\alpha}{c}\int_{\mathbb{R}^d}\left(\left(e^{i\langle v,x\rangle}-1\right)h_v(x)-i\langle v,x\rangle\right)\Lambda^1(dx),\qquad\text{if }\alpha\in(1,2);$$

*where $t>0$, $v\in\mathbb{S}^{d-1}$, $c$ is the constant defined in (1.15) and $m_{\sigma_{\Lambda^1}}=\int_{\mathbb{S}^{d-1}}\omega\sigma_{\Lambda^1}(d\omega)$ and $\sigma_{\Lambda^1}$ is the spherical measure of the tail measure $\Lambda^1$ defined in Theorem 1.7,*

- *if $\alpha\in(0,1)$, $d_n=0$;*
- *if $\alpha=1$, $d_n=n\xi(a_n^{-1})$, $\xi(t)=\int_{\mathbb{R}^d}\frac{tx}{1+|tx|^2}\nu(dx)$;*
- *if $\alpha\in(1,2)$, $d_n=a_n^{-1}nm$, for $m=\int_{\mathbb{R}^d}x\nu(dx)$.*

*The functions* $C_\alpha$ *satisfy* $C_\alpha(tv) = t^\alpha C_\alpha(v)$ *for* $\alpha \in (0,1) \cup (1,2)$.

*Moreover, if* $\lim_{n\to\infty} (\mathbb{E}\|A_1 \cdot \cdots \cdot A_n\|^\alpha)^{\frac{1}{n}} < 1$, $\Phi(x,0) = x$ *for every* $x \in \overline{[\operatorname{supp}\mu] \cdot \operatorname{supp}\nu}$, *and* $\Phi[\{0\} \times \operatorname{supp}\sigma_{\Lambda_b}]$ *is not contained in any proper subspace of* $\mathbb{R}^d$, *then the limit laws are fully nondegenerate, i.e.* $\Re C_\alpha(tv) < 0$ *for every* $t > 0$ *and* $v \in \mathbb{S}^{d-1}$ *and* $\alpha \in (0,2)$.

**Remark 1.17.** The condition: $\Phi(x,0) = x$ for every $x \in \overline{[\operatorname{supp}\mu] \cdot \operatorname{supp}\nu}$, requires an explanation as in Remark 1.14. It is obvious if $\Phi(x,y) = x + y$. For instance, if $\Phi(x,y) = \max\{x,y\}$, then $\Phi(x,0) = x$ for $x \in [0,\infty)$, it is sufficient to assume $\mathbb{P}[A \ge 0] = 1$, $\mathbb{E}[A^\alpha] < 1$ and $\lim_{t\to\infty} t^\alpha \mathbb{P}[B^1 > t] = c > 0$.

If $\alpha > 2$ then $\frac{S_n^x - nm}{\sqrt{n}}$ converges to a normal law which is a straightforward application of the martingale method; see [4,28,30] and the references given there. Let us underline that the theorem above concerns dependent random variables with infinite variance. In the context of stochastic recursions similar problems were studied in e.g. [2,7,19,27]. Our proof of Theorem 1.16 is based on the spectral method, introduced by Nagaev in 50's to prove limit theorems for Markov chains. This method has been strongly developed recently and it has been used in the context of limit theorems related to stochastic recursions; see e.g. [7,19,20,27].

Throughout the whole paper, unless otherwise stated, we will use the convention that $C > 0$ stands for a large positive constant whose value varies from occurrence to occurrence.

## 2. Tails of random recursions

First we will prove existence and uniqueness of the stationary measure for the Markov chain $\{X_n^x\}$ defined in (1.1) as well as some further properties of $\gamma$-geometric Markov chains that will be used in the sequel. Following classical ideas, going back to Furstenberg [13] (see also [11]), we consider the backward process $Y_n^x = \Phi_1 \circ \cdots \circ \Phi_n(x)$, which has the same law as $X_n^x$. The process $\{Y_n^x\}$ is not a Markov chain; however sometimes it is more comfortable to use than $\{X_n^x\}$, e.g. it allows us conveniently to construct the stationary distribution of $\{X_n^x\}$. Notice that since $X_n^x$ is $\gamma$-geometric, then $Y_n^x$ is as well, i.e.

$$\mathbb{E}\left[|Y_n^x - Y_n^y|^\gamma\right] \le C\rho^n |x-y|^\gamma, \quad x, y \in \mathbb{R}^d, n \in \mathbb{N}, \tag{2.1}$$

for $C$ and $\rho$ being as in (1.4).

If $x = 0$ we write for simplicity $Y_n$ instead of $Y_n^x$. To emphasize the role of the starting point, which can be sometimes a random variable $X_0$, we write $X_n^{X_0} = \Phi_n \circ \cdots \circ \Phi_1(X_0)$ and $Y_n^{X_0} = \Phi_1 \circ \cdots \circ \Phi_n(X_0)$, where $X_0$ is an arbitrary initial random variable.

**Lemma 2.2.** *Let* $\{X_n^x\}$ *be a Markov chain generated by a system of random functions, which is* $\gamma$*-geometric and satisfies* $\mathbb{E}|X_1|^\delta < \infty$, *for some positive constants* $\gamma, \delta > 0$. *Then there exists a unique stationary measure* $\nu$ *of* $\{X_n^x\}$ *and for any initial random variable* $X_0$, *the process* $\{X_n^{X_0}\}$ *converges in distribution to* $X$ *with the law* $\nu$.

*Moreover, if additionally* $\mathbb{E}|X_0|^\beta < \infty$ *and* $\mathbb{E}\left|X_1^{X_0}\right|^\beta < \infty$ *for some* $\beta < \gamma$, *then*

$$\sup_{n\in\mathbb{N}} \mathbb{E}|X_n^{X_0}|^\beta < \infty. \tag{2.3}$$

**Proof.** Take $\varepsilon = \min\{1, \delta, \gamma\}$; then the Markov chain $X_n = X_n^0$ is $\varepsilon$-geometric. To prove convergence in distribution of $X_n$ it is sufficient to show that $Y_n$ converges in $L^\varepsilon$. For this purpose we prove that $\{Y_n\}$ is a Cauchy sequence in $L^\varepsilon$. Fix $n \in \mathbb{N}$; then for any $m > n$ we have

$$\begin{aligned}\mathbb{E}\left[|Y_m - Y_n|^\varepsilon\right] &\le \sum_{k=n}^{m-1} \mathbb{E}\left[|Y_{k+1} - Y_k|^\varepsilon\right] = \sum_{k=n}^{m-1} \mathbb{E}\left[|Y_k^{\Phi_{k+1}(0)} - Y_k|^\varepsilon\right] \\ &\le C \sum_{k=n}^{m-1} \rho^k \mathbb{E}|\Phi_{k+1}(0)|^\varepsilon \le \frac{C\mathbb{E}|X_1|^\varepsilon}{1-\rho} \cdot \rho^n.\end{aligned}$$

This proves that $Y_n$ converges in $L^\varepsilon$, and hence also in distribution, to a random variable $X$. Therefore, $X_n^x$ converges in distribution to the same random variable $X$, for every $x \in \mathbb{R}^d$.

To prove uniqueness of the stationary measure assume that there is another stationary measure $\nu'$. Then, by the Lebesgue theorem, for every bounded continuous function $f$,

$$\nu'(f) = \int_{\mathbb{R}^d} \mathbb{E}\left[f(X_n^x)\right] \nu'(dx) \xrightarrow[n\to\infty]{} \int_{\mathbb{R}^d} \mathbb{E}\left[f(X)\right] \nu'(dx) = \nu(f),$$

and hence $\nu = \nu'$. The same arguments prove that the sequence $X_n^Z$ converges in distribution to $X$ for any initial random variable $Z$ on $\mathbb{R}^d$.

To prove the second part of the lemma, let us consider two cases. Assume that $\beta < \gamma \le 1$; then we write

$$\begin{aligned}\mathbb{E}\left|Y_n^{X_0}\right|^\beta &\le \sum_{k=0}^{n-1} \mathbb{E}\left|Y_k^{X_0} - Y_{k+1}^{X_0}\right|^\beta + \mathbb{E}\,|X_0|^\beta \\ &\le \sum_{k=0}^{n-1} \rho^k \mathbb{E}\left|X_1^{X_0} - X_0\right|^\beta + \mathbb{E}\,|X_0|^\beta \le C < \infty.\end{aligned}$$

If $\gamma > 1$, it is enough to take $1 \le \beta < \gamma$ and apply the Hölder inequality, i.e.

$$\begin{aligned}\left(\mathbb{E}\left|Y_n^{X_0}\right|^\beta\right)^{\frac{1}{\beta}} &\le \sum_{k=0}^{n-1} \left(\mathbb{E}\left|Y_k^{X_0} - Y_{k+1}^{X_0}\right|^\beta\right)^{\frac{1}{\beta}} + \left(\mathbb{E}\,|X_0|^\beta\right)^{\frac{1}{\beta}} \\ &\le \sum_{k=0}^{n-1} \rho^k \left(\mathbb{E}\left|X_1^{X_0} - X_0\right|^\beta\right)^{\frac{1}{\beta}} + \left(\mathbb{E}\,|X_0|^\beta\right)^{\frac{1}{\beta}} \le C < \infty. \quad \square\end{aligned}$$

Before we formulate the next lemma, notice that if a random variable $W$ is regularly varying, then

$$\sup_{t>0} \left\{t^\alpha L(t) \mathbb{P}\left[|W| > t\right]\right\} < \infty. \tag{2.4}$$

Moreover, if $L$ is a slowly varying function which is bounded away from zero and infinity on any compact interval then, by Potter's Theorem [9, p. 25], given $\delta > 0$ there is a finite constant $C > 0$ such that

$$\sup_{t>0} \frac{L(t)}{L(\lambda t)} \le C \max\left\{\lambda^\delta, \lambda^{-\delta}\right\}, \tag{2.5}$$

for every $\lambda > 0$.

The following lemma is a multidimensional generalization of Lemma 2.1 in [10].

**Lemma 2.6.** *Let $Z_1, Z_2 \in \mathbb{R}^d$ be $\alpha$-regularly varying random variables with the tail measures $\Lambda_1$, $\Lambda_2$, respectively (with the same slowly varying function $L_b$ which is bounded away from zero and infinity on any compact interval), such that*

$$\lim_{t\to\infty} t^\alpha L_b(t)\mathbb{P}[|Z_1| > t, |Z_2| > t] = 0. \tag{2.7}$$

*Then the random variable $(Z_1, Z_2)$ valued in $\mathbb{R}^d \times \mathbb{R}^d$ is regularly varying with index $\alpha$ and its tail measure $\Lambda$ is defined by*

$$\langle F, \Lambda\rangle = \langle F(\cdot, 0), \Lambda_1\rangle + \langle F(0, \cdot), \Lambda_2\rangle,$$

*i.e. for every $F \in C_c((\mathbb{R}^d \times \mathbb{R}^d) \setminus \{0\})$,*

$$\lim_{t\to\infty} t^\alpha L_b(t)\mathbb{E}\left[F\left(t^{-1}Z_1, t^{-1}Z_2\right)\right] = \langle F, \Lambda\rangle. \tag{2.8}$$

*Moreover, the formula above is valid for every bounded continuous function $F$ supported outside* 0.

**Proof.** Since every $F \in C_c((\mathbb{R}^d \times \mathbb{R}^d) \setminus \{0\})$ may be written as a sum of two functions with supports in $\left(\mathbb{R}^d \setminus B_\eta(0)\right) \times \mathbb{R}^d$ and $\mathbb{R}^d \times \left(\mathbb{R}^d \setminus B_\eta(0)\right)$ respectively, for some $\eta > 0$, it is enough to consider only one factor of this decomposition. We assume that we are in the first case, i.e. $\mathrm{supp} F \subseteq \left(\mathbb{R}^d \setminus B_\eta(0)\right) \times \mathbb{R}^d$. Then to obtain the result for such a function it is enough to justify that

$$\lim_{t\to\infty} t^\alpha L_b(t)\mathbb{E}\left[F\left(t^{-1}Z_1, t^{-1}Z_2\right) - F\left(t^{-1}Z_1, 0\right)\right] = 0. \tag{2.9}$$

Fix $\varepsilon > 0$ and write

$$\begin{aligned} &t^\alpha L_b(t)\left|\mathbb{E}\left[F\left(t^{-1}Z_1, t^{-1}Z_2\right) - F\left(t^{-1}Z_1, 0\right)\right]\right| \\ &\quad\le t^\alpha L_b(t)\mathbb{E}\left[\left|F\left(t^{-1}Z_1, t^{-1}Z_2\right)\right| \mathbf{1}_{\{|Z_2|>\varepsilon t\}}\right] + t^\alpha L_b(t)\mathbb{E}\left[\left|F\left(t^{-1}Z_1, 0\right)\right| \mathbf{1}_{\{|Z_2|>\varepsilon t\}}\right] \\ &\qquad + t^\alpha L_b(t)\mathbb{E}\left[\left|F\left(t^{-1}Z_1, t^{-1}Z_2\right) - F\left(t^{-1}Z_1, 0\right)\right| \mathbf{1}_{\{|Z_2|\le\varepsilon t\}}\right]. \end{aligned}$$

We denote the consecutive expressions in the sum above by $g_1(t), g_2(t), g_3(t)$, respectively. Taking $\lambda = \min\{\eta, \varepsilon\}$, by (2.5) and (2.7) we obtain

$$\begin{aligned} 0 &\le \lim_{t\to\infty} g_1(t) \le \lim_{t\to\infty} t^\alpha L_b(t)\|F\|_\infty \mathbb{P}[|Z_1| > \eta t, |Z_2| > \varepsilon t] \\ &\le \|F\|_\infty \cdot \sup_{t>0} \frac{L_b(t)}{L_b(\lambda t)} \cdot \lim_{t\to\infty} \left(t^\alpha L_b(\lambda t)\mathbb{P}[|Z_1| > \lambda t, |Z_2| > \lambda t]\right) = 0. \end{aligned}$$

Arguing in a similar way to above we deduce that $\lim_{t\to\infty} g_2(t) = 0$. Finally, to prove that $g_3$ converges to 0, assume first that $F$ is a Lipschitz function with the Lipschitz coefficient $\mathrm{Lip}(F)$. Then by (2.4)

$$\begin{aligned} g_3(t) &\le \mathrm{Lip}(F) t^\alpha L_b(t)\mathbb{E}\left[|t^{-1}Z_2| \mathbf{1}_{\{|t^{-1}Z_1|>\eta\}} \mathbf{1}_{\{|t^{-1}Z_2|\le\varepsilon\}}\right] \\ &\le \varepsilon \cdot \mathrm{Lip}(F) \sup_{t>0}\left\{t^\alpha L_b(t)\mathbb{P}[|t^{-1}Z_1| > \eta]\right\} \le C\varepsilon. \end{aligned}$$

Passing with $\varepsilon$ to 0, we obtain (2.9) for Lipschitz functions.

To prove the result for arbitrary functions, notice first that (2.4) implies

$$\sup_{t>0}\left\{t^\alpha L_b(t)\mathbb{P}[\eta t < |Z_1|+|Z_2| < Mt]\right\} < \infty.$$

Now we approximate $F \in C_c\left(\left(\mathbb{R}^d \setminus B_\eta(0)\right) \times \mathbb{R}^d\right)$ by a Lipschitz function $G \in C_c\left(\left(\mathbb{R}^d \setminus B_\eta(0)\right) \times \mathbb{R}^d\right)$ such that $\|F-G\|_\infty < \varepsilon$. Then

$$\begin{aligned}
&t^\alpha L_b(t)\left|\mathbb{E}\left[F(t^{-1}Z_1,t^{-1}Z_2)-F(t^{-1}Z_1,0)\right]\right|\\
&\quad\le t^\alpha L_b(t)\mathbb{E}\left[\left|F(t^{-1}Z_1,t^{-1}Z_2)-G(t^{-1}Z_1,t^{-1}Z_2)\right|\right]\\
&\qquad+t^\alpha L_b(t)\left|\mathbb{E}\left[G(t^{-1}Z_1,t^{-1}Z_2)-G(t^{-1}Z_1,0)\right]\right|\\
&\qquad+t^\alpha L_b(t)\mathbb{E}\left[\left|F(t^{-1}Z_1,0)-G(t^{-1}Z_1,0)\right|\right]\\
&\quad\le \varepsilon t^\alpha L_b(t)\mathbb{P}[\eta t<|Z_1|+|Z_2|<Mt]\\
&\qquad+t^\alpha L_b(t)\left|\mathbb{E}\left[G(t^{-1}Z_1,t^{-1}Z_2)-G(t^{-1}Z_1,0)\right]\right|\\
&\qquad+\varepsilon t^\alpha L_b(t)\mathbb{P}[\eta t<|Z_1|<Mt],
\end{aligned}$$

and hence passing with $t$ to infinity and then with $\varepsilon$ to zero we obtain (2.9) and so also (2.8).

To prove the second part of the lemma, let $F$ be an arbitrary bounded continuous function on $\mathbb{R}^d \times \mathbb{R}^d$ supported outside 0. Assume $\|F\|_\infty = 1$. Take $r > 0$ and let $\phi_1, \phi_2$ be nonzero functions on $\mathbb{R}^d \times \mathbb{R}^d$ such that $\phi_1+\phi_2 = 1$, $\mathrm{supp}\phi_1 \subseteq B_{2r}(0)$ and $\mathrm{supp}\phi_2 \subseteq B_r(0)^c$. Then by (2.4) and (2.5)

$$\begin{aligned}
&\lim_{r\to\infty}\sup_{t>0} t^\alpha L_b(t)\mathbb{E}\left[(\phi_2 F)(t^{-1}Z_1,t^{-1}Z_2)\right]\\
&\quad\le \lim_{r\to\infty}\sup_{t>0} t^\alpha L_b(t)\left(\mathbb{P}[|Z_1|>rt]+\mathbb{P}[|Z_2|>rt]\right)\\
&\quad\le \lim_{r\to\infty}\sup_{t>0} r^{-\alpha}\frac{L_b(t)}{L_b(rt)}(rt)^\alpha L_b(rt)\left(\mathbb{P}[|Z_1|>rt]+\mathbb{P}[|Z_2|>rt]\right)=0.
\end{aligned}$$

By (2.8)

$$\lim_{t\to\infty} t^\alpha L_b(t)\mathbb{E}\left[(\phi_1 F)(t^{-1}Z_1,t^{-1}Z_2)\right] = \langle \phi_1 F, \Lambda\rangle.$$

Therefore, passing with $r$ to infinity, we obtain (2.8) for non-compactly supported functions $F$. $\square$

The next lemma when considered for the one-dimensional recursion (1.2) is known as Breiman's lemma [6]. In the multidimensional affine settings the lemma was proved in [21] (Lemma 2.1). Here we write it in the generality corresponding to our framework and, at the same time, we present a simpler proof than in [21].

**Lemma 2.10.** *Assume that*

- *random variables $(A, B)$ and $X \in \mathbb{R}^d$ are independent;*
- *$X$ and $B^1$ are $\alpha$-regularly varying with the tail measures $\Lambda$, $\Lambda_b$, respectively (with the same slowly varying function $L_b$ which is bounded away from zero and infinity on any compact interval);*
- *$\mathbb{E}\|A\|^\beta < \infty$ for some $\beta > \alpha$;*

- *there is* $\varepsilon_0 > 0$ *such that* $\mathbb{E}\left[(B^3)^{\frac{\alpha}{\delta_0}+\varepsilon_0}\right] < \infty$, *if* $0 < \delta_0 < 1$ *and* $\mathbb{E}\left[(B^3)^{\alpha+\varepsilon_0}\right] < \infty$, *if* $\delta_0 = 0$.

*Then both* $AX$ *and* $\Phi(AX, B(X))$ *are* $\alpha$*-regularly varying with the tail measures* $\widetilde{\Lambda}$ *and* $\Lambda_1$ *respectively, where* $\langle f, \widetilde{\Lambda}\rangle = \mathbb{E}\left[\langle f \circ A, \Lambda\rangle\right]$ *and*

$$\langle f, \Lambda_1\rangle = \langle f \circ \Phi(\cdot, 0), \widetilde{\Lambda}\rangle + \langle f \circ \Phi(0, \cdot), \Lambda_b\rangle. \tag{2.11}$$

**Proof.** First, conditioning on $A$, we will prove that for any bounded function $f$ supported in $\mathbb{R}^d \setminus B_\eta(0)$ for some $\eta > 0$, there exists a function $g$ such that

$$\sup_{t>0}\left\{t^\alpha L_b(t)\mathbb{E}\left[f\left(t^{-1}AX\right)|A\right]\right\} \le g(A), \quad \text{and} \quad \mathbb{E}[g(A)] < \infty. \tag{2.12}$$

Observe that $\sup_{t>0} t^\alpha L_b(t)\mathbb{P}\left[|X| > t\right] = C < \infty$ and assume that $\operatorname{supp} f \subseteq \mathbb{R}^d \setminus B_\eta(0)$, $\eta < 1$, and fix $\delta < \beta - \alpha$. If $\|A\| \le 1$ then, by (2.5), for every $t > 0$,

$$t^\alpha L_b(t)\mathbb{E}\left[f\left(t^{-1}AX\right)|A\right] \le \|f\|_\infty t^\alpha L_b(t)\mathbb{P}\left[|X| > t\eta\right] \le C\eta^{-\alpha-\delta}\|f\|_\infty = C_1 < \infty.$$

If $2^n \le \|A\| \le 2^{n+1}$ for $n \in \mathbb{N}$ then, again by (2.5), for every $t > 0$,

$$\begin{aligned} t^\alpha L_b(t)\mathbb{E}\left[f\left(t^{-1}AX\right)|A\right] &\le \|f\|_\infty t^\alpha L_b(t)\mathbb{P}\left[2^{n+1}|X| > t\eta\right] \\ &\le C2^{(n+1)(\alpha+\delta)}\eta^{-\alpha-\delta}\|f\|_\infty = C_2 2^{n(\alpha+\delta)}. \end{aligned}$$

Finally, notice that

$$\begin{aligned} \mathbb{E}[g(A)] &\le C_1\mathbb{P}[\|A\| \le 1] + C_2\sum_{n=1}^{\infty} 2^{n(\alpha+\delta)}\mathbb{P}\left[\|A\| \ge 2^n\right] \\ &\le C_1 + C_2\mathbb{E}\|A\|^\beta \cdot \sum_{n=1}^{\infty} 2^{n(\alpha+\delta-\beta)} < \infty, \end{aligned}$$

and the proof of (2.12) is completed. Now in view of (2.12) we can easily prove that $AX$ is regularly varying with index $\alpha$. Indeed, taking $f \in C_c(\mathbb{R}^d \setminus B_\eta(0))$, conditioning on $A$, and using dominated convergence theorem we have

$$\begin{aligned} \lim_{t\to\infty} t^\alpha L_b(t)\mathbb{E}\left[f(t^{-1}AX)\right] &= \mathbb{E}\left[\lim_{t\to\infty} t^\alpha L_b(t)\mathbb{E}\left[(f \circ A)(t^{-1}X)|A\right]\right] \\ &= \mathbb{E}\left[\langle f \circ A, \Lambda\rangle\right] = \langle f, \widetilde{\Lambda}\rangle, \end{aligned}$$

and hence $AX$ is $\alpha$-regularly varying as desired.

For the second part of the lemma, we are going to apply Lemma 2.6, with $Z_1 = AX$, $Z_2 = B(X)$ and the function $f \circ \Phi$. Notice, that since $\Phi(0, 0) = 0$ the function $f \circ \Phi$ is supported outside 0. It may happen (e.g. when $\Phi(x, y) = x + y$) that $f \circ \Phi$ is not compactly supported; however it is still a bounded function. Therefore, we have to prove that $B(X)$ is $\alpha$-regularly varying with the tail measure $\Lambda_b$ and (2.7) is satisfied, i.e.

$$\lim_{t\to\infty} t^\alpha L_b(t)\mathbb{P}\left[|AX| > t, |B(X)| > t\right] = 0. \tag{2.13}$$

To prove that $B(X)$ is $\alpha$-regularly varying notice that from the first part of the lemma with $B^3$ instead of $A$ we know that if $\delta_0 > 0$, then $(B^3)^{\frac{1}{\delta_0}} X$ is $\alpha$-regular. Therefore,

$$\lim_{t\to\infty} t^\alpha L_b(t)\mathbb{P}\left[B^2(X) > t\right] \le \lim_{t\to\infty} t^\alpha L_b(t)\mathbb{P}\left[(B^3)^{\frac{1}{\delta_0}}|X| > t^{\frac{1}{\delta_0}}\right] = 0,$$

so $B^2(X)$ is $\alpha$-regularly varying with the tail measure 0. If $\delta_0 = 0$, then $\lim_{t\to\infty} t^\alpha L_b(t)\, \mathbb{P}\left[B^2(X) > t\right] = 0$ can be easily established. Hence applying Lemma 2.6 for $Z_1 = B_1$, $Z_2 = B^2(X)$ and $f \circ \widetilde{\Phi}$, where $\widetilde{\Phi}(x, y) = x + y$, we deduce

$$\begin{aligned}\lim_{t\to\infty} t^\alpha L_b(t)\mathbb{E}\left[f(t^{-1}B(X))\right] &= \lim_{t\to\infty} t^\alpha L_b(t)\mathbb{E}\left[(f\circ\widetilde{\Phi})\left(t^{-1}B^1, t^{-1}B^2(X)\right)\right]\\ &= \langle (f\circ\widetilde{\Phi})(\cdot,0), \Lambda_b\rangle + \langle (f\circ\widetilde{\Phi})(0,\cdot), 0\rangle = \langle f, \Lambda_b\rangle.\end{aligned}$$

In order to prove (2.13) take $f(x) = \mathbf{1}_{\{|\cdot|>1\}}(x)$; then applying (2.12) and conditioning on $(A, B^1)$ we obtain

$$\begin{aligned}& t^\alpha L_b(t)\mathbb{P}\left[|AX| > t, |B(X)| > t\right]\\ &\quad\le t^\alpha L_b(t)\mathbb{E}\left[f(t^{-1}AX)\mathbf{1}_{\{|B^1|>t/2\}}\right] + t^\alpha L_b(t)\mathbb{P}\left[|B^2(X)| > t/2\right]\\ &\quad\le \mathbb{E}\left[\mathbf{1}_{\{|B^1|>t/2\}}\cdot\sup_{t>0} t^\alpha L_b(t)\mathbb{E}\left[f(t^{-1}AX)|(A,B^1)\right]\right] + t^\alpha L_b(t)\mathbb{P}\left[|B^2(X)| > t/2\right]\\ &\quad\le \mathbb{E}\left[\mathbf{1}_{\{|B^1|>t/2\}} g(A)\right] + t^\alpha L_b(t)\mathbb{P}\left[|B^2(X)| > t/2\right].\end{aligned}$$

The last expression converges to 0 as $t$ goes to infinity. Finally, from Lemma 2.6 we obtain that $\Phi(A, B)(X)$ is $\alpha$-regular:

$$\begin{aligned}\lim_{t\to\infty} t^\alpha L_b(t)\mathbb{E}\left[f\left(t^{-1}\Phi(A,B)(X)\right)\right] &= \lim_{t\to\infty} t^\alpha L_b(t)\mathbb{E}\left[(f\circ\Phi)\left(t^{-1}AX, t^{-1}B(X)\right)\right]\\ &= \langle f, \Lambda_1\rangle.\end{aligned}$$

This proves (2.11) and completes the proof of the lemma. □

**Proof of Theorem 1.7.** Since the stationary solution $X$ does not depend on the choice of the initial random variable $X_0$, without any loss of generality, we may assume that $X_0$ is $\alpha$-regularly varying with some nonzero tail measure $\Lambda_0$. Then by Lemma 2.10, for every $n \in \mathbb{N}$, $X_n^{X_0}$ is $\alpha$-regularly varying with the tail measure $\Lambda_n$ satisfying (2.11) with $\widetilde{\Lambda}_{n-1}$, being the tail measure of $A_n X_{n-1}^{X_0}$, instead of $\widetilde{\Lambda}$. So, we have to prove that $\Lambda_n$ converges weakly to some measure $\Lambda^1$, which we can identify as the tail measure of $X$. This measure will be nonzero, since for every $n \in \mathbb{N}$ and positive $f$, $\langle f, \Lambda_n\rangle \ge \langle f\circ\Phi(0,\cdot), \Lambda_b\rangle$. From now we will consider the backward process $\{Y_n^x\}$. We may assume that $\delta > 0$ in (2.5) is sufficiently small, i.e. $\delta < \min\{\alpha, \gamma - \alpha\}$. Suppose first that $f$ is an $\varepsilon$-Hölder function for $0 < \varepsilon < \delta$ and $\operatorname{supp} f \subseteq \mathbb{R}^d \setminus B_\eta(0)$. By (2.1) there exist constants $0 < C_0 < \infty$ and $0 < \rho_0 < 1$ such that

$$\mathbb{E}\left[\left|Y_n^x - Y_n^y\right|^s\right] \le C_0\rho_0^n|x-y|^s$$
$$\text{for } s \in \{\gamma, \alpha-\delta, \alpha+\delta\}, n \in \mathbb{N}, \text{ and } x, y \in \mathbb{R}^d. \tag{2.14}$$

We will prove that there are constants $0 < C < \infty$ and $0 < \rho < 1$ such that for every $m > n$

$$\sup_{t>0}\left\{t^\alpha L_b(t)\mathbb{E}\left|f(t^{-1}Y_m^{X_0}) - f(t^{-1}Y_n^{X_0})\right|\right\} \le C\rho^n. \tag{2.15}$$

We begin by showing that

$$\sup_{t>0}\left\{t^{\alpha}L_b(t)\mathbb{E}\left|f\left(t^{-1}Y_k^{X_0}\right)-f\left(t^{-1}Y_k\right)\right|\right\}\le C\rho^k, \tag{2.16}$$

for $k\in\mathbb{N}$. We have

$$\begin{aligned}\mathbb{E}\left[f(t^{-1}Y_k^{X_0})-f(t^{-1}Y_k)\right]&=\mathbb{E}\left[\left(f(t^{-1}Y_k^{X_0})-f(t^{-1}Y_k)\right)\mathbf{1}_{\{|t^{-1}Y_k|>\frac{\eta}{2}\}}\right]\\&\quad+\mathbb{E}\left[\left(f(t^{-1}Y_k^{X_0})-f(t^{-1}Y_k)\right)\mathbf{1}_{\{|t^{-1}Y_k^{X_0}|>\eta\}}\mathbf{1}_{\{|t^{-1}Y_k|<\frac{\eta}{2}\}}\right]=I_1+I_2.\end{aligned}$$

Notice that $\mathbb{E}|\Phi_1(0)|^{\beta}<\infty$ for every $\beta<\alpha$ ; hence by (2.3), $\sup_{k\in\mathbb{N}}\mathbb{E}|Y_k|^{\beta}\le C<\infty$. Therefore, on the one hand, we have an estimate for small $t>0$:

$$\begin{aligned}t^{\alpha}L_b(t)|I_1|&\le Ct^{\alpha-\varepsilon}L_b(t)\mathbb{E}\left[\mathbb{E}\left[\left|Y_k^{X_0}-Y_k\right|^{\varepsilon}\mathbf{1}_{\{|Y_k|>t\eta/2\}}|X_0\right]\right]\\&\le Ct^{\alpha-\varepsilon}L_b(t)\mathbb{E}\left[|X_0|^{\varepsilon}\right]\rho_0^k.\end{aligned}$$

On the other hand, by the Hölder inequality with $p=\frac{\gamma}{\varepsilon}$, $q=\frac{\gamma}{\gamma-\varepsilon}$, conditioning on $X_0$ we have an estimate for sufficiently large $t>0$:

$$\begin{aligned}t^{\alpha}L_b(t)|I_1|&\le Ct^{\alpha-\varepsilon}L_b(t)\mathbb{E}\left[\mathbb{E}\left[\left|Y_k^{X_0}-Y_k\right|^{\varepsilon}\mathbf{1}_{\{|Y_k|>t\eta/2\}}\middle|X_0\right]\right]\\&\le Ct^{\alpha-\varepsilon}L_b(t)\mathbb{E}\left[\mathbb{E}\left[\left|Y_k^{X_0}-Y_k\right|^{p\varepsilon}\middle|X_0\right]^{\frac{1}{p}}\mathbb{E}\left[\mathbf{1}_{\{|Y_k|>t\eta/2\}}\middle|X_0\right]^{\frac{1}{q}}\right]\\&\le Ct^{\alpha-\varepsilon}L_b(t)\mathbb{E}\left[\mathbb{E}\left[\left|Y_k^{X_0}-Y_k\right|^{\gamma}\middle|X_0\right]^{\frac{1}{p}}\right]\mathbb{P}\left[|Y_k|>t\eta/2\right]^{\frac{1}{q}}\\&\le Ct^{\alpha-\varepsilon}L_b(t)\rho_0^{\frac{k}{p}}\mathbb{E}|X_0|^{\varepsilon}\cdot t^{-\left(\alpha-\frac{\varepsilon\delta}{\gamma-\varepsilon}\right)\frac{1}{q}}\mathbb{E}\left[|Y_k|^{\alpha-\frac{\varepsilon\delta}{\gamma-\varepsilon}}\right]^{\frac{1}{q}}\\&\le CL_b(t)t^{\frac{1}{p}(\alpha+\delta-\gamma)}\rho_0^{\frac{k}{p}}.\end{aligned}$$

Finally, we have obtained

$$t^{\alpha}L_b(t)|I_1|\le CL_b(t)\min\left\{t^{\alpha-\varepsilon},t^{\frac{1}{p}(\alpha+\delta-\gamma)}\right\}\rho_0^{\frac{k}{p}}.$$

Denote by $\widetilde{L}_n$ the Lipschitz coefficient of $\Phi_1\circ\cdots\circ\Phi_n$. Since $X_0$ is $\alpha$-regularly varying, by (2.4) and (2.5) we obtain

$$\begin{aligned}t^{\alpha}L_b(t)|I_2|&\le 2\|f\|_{\infty}t^{\alpha}L_b(t)\mathbb{P}\left[\left|Y_k^{X_0}-Y_k\right|>t\eta/2\right]\\&\le 2\|f\|_{\infty}t^{\alpha}L_b(t)\mathbb{P}\left[\widetilde{L}_k|X_0|>t\eta/2\right]\\&\le C\|f\|_{\infty}\mathbb{E}\left[\widetilde{L}_k^{\alpha}\frac{L_b(t)}{L_b\left(\frac{t\eta}{2\widetilde{L}_k}\right)}\mathbb{E}\left[\left(\frac{t\eta}{2\widetilde{L}_k}\right)^{\alpha}L_b\left(\frac{t\eta}{2\widetilde{L}_k}\right)\mathbf{1}_{\left\{|X_0|>\frac{t\eta}{2\widetilde{L}_k}\right\}}\middle|\widetilde{L}_k\right]\right]\\&\le C\|f\|_{\infty}\mathbb{E}\left[\widetilde{L}_k^{\alpha+\delta}+\widetilde{L}_k^{\alpha-\delta}\right]\le C\|f\|_{\infty}\rho_0^k.\end{aligned}$$

Hence, we deduce (2.16) and in order to prove (2.15) it is enough to justify

$$\sup_{t>0}\left\{t^{\alpha}L_b(t)\mathbb{E}\left[\left|f(t^{-1}Y_m)-f(t^{-1}Y_n)\right|\right]\right\}\le C\rho^n,\quad m>n. \tag{2.17}$$

For this purpose we carry out the decomposition

$$f(t^{-1}Y_m) - f(t^{-1}Y_n) = \sum_{k=n}^{m-1} \Big( f(t^{-1}Y_{k+1}) - f(t^{-1}Y_k) \Big),$$

and next we estimate $\mathbb{E}[f(t^{-1}Y_{k+1}) - f(t^{-1}Y_k)]$ using exactly the same arguments as in (2.16), with $Y_{k+1} = Y_k \circ \Phi_{k+1}$ instead of $Y_k^{X_0}$ and $\Phi_{k+1}(0)$ instead of $X_0$. Thus we obtain that

$$\sup_{t>0} \left\{ t^\alpha L_b(t) \mathbb{E} \left| f\left(t^{-1}Y_{k+1}\right) - f\left(t^{-1}Y_k\right) \right| \right\} \le C\rho^k, \tag{2.18}$$

which in turn implies (2.17) and hence (2.15). Now letting $m \to \infty$ we have

$$\sup_{t>0} \left\{ t^\alpha L_b(t) \mathbb{E} \left[ |f(t^{-1}X) - f(t^{-1}Y_n^{X_0})| \right] \right\} \le C\rho^n. \tag{2.19}$$

We know that, for every $n \in \mathbb{N}$, $Y_n^{X_0}$ is $\alpha$-regularly varying with the tail measure $\Lambda_n$. Moreover, in view of (2.15), the sequence $\Lambda_n(f)$ is a Cauchy sequence, and hence it converges. Let $\Lambda^1(f)$ denote the limit of $\Lambda_n(f)$. In view of (2.19), for every $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\limsup_{t\to\infty} \left| t^\alpha L_b(f) \mathbb{E}\left[ f(t^{-1}X) \right] - \Lambda^1(f) \right| \\
&\quad \le \limsup_{t\to\infty} t^\alpha L_b(f) \mathbb{E}\left[ |f(t^{-1}X) - f(t^{-1}Y_n^{X_0})| \right] \\
&\qquad + \lim_{t\to\infty} \left| t^\alpha L_b(f) \mathbb{E}\left[ f(t^{-1}Y_n^{X_0}) \right] - \Lambda_n(f) \right| \\
&\qquad + \left| \Lambda_n(f) - \Lambda^1(f) \right| \le C\rho^n + \left| \Lambda_n(f) - \Lambda^1(f) \right|,
\end{aligned}$$

and so letting $n \to \infty$,

$$\lim_{t\to\infty} t^\alpha L_b(f) \mathbb{E}\left[ f(t^{-1}X) \right] = \Lambda^1(f), \tag{2.20}$$

for any $\varepsilon$-Hölder function.

Finally, take a continuous function $f$ compactly supported in $\mathbb{R}^d \setminus B_\eta(0)$ for some $\eta > 0$, and fix $\delta > 0$. Then there exists an $\varepsilon$-Hölder function $g$ supported in $\mathbb{R}^d \setminus B_\eta(0)$ such that $\|f - g\|_\infty \le \delta$. Moreover, let $h$ be an $\varepsilon$-Hölder function, supported in $\mathbb{R}^d \setminus B_{\eta/2}(0)$, such that $\delta h \ge |f - g|$. To define $\Lambda^1(f)$ we will first prove an inequality similar to (2.15). Notice that

$$\begin{aligned}
\sup_{t>0} \left\{ t^\alpha L_b(t) \mathbb{E} \left| f(t^{-1}Y_m) - f(t^{-1}Y_n) \right| \right\} &\le \sup_{t>0} \left\{ t^\alpha L_b(t) \mathbb{E} \left| f(t^{-1}Y_m) - g(t^{-1}Y_m) \right| \right\} \\
&\quad + \sup_{t>0} \left\{ t^\alpha L_b(t) \mathbb{E} \left| g(t^{-1}Y_m) - g(t^{-1}Y_n) \right| \right\} \\
&\quad + \sup_{t>0} \left\{ t^\alpha L_b(t) \mathbb{E} \left| g(t^{-1}Y_n) - f(t^{-1}Y_n) \right| \right\} \\
&\le \delta \Lambda_m(h) + C\rho^n + \delta \Lambda_n(h),
\end{aligned}$$

and hence $\Lambda_n(f)$ is a Cauchy sequence, since $\delta > 0$ is arbitrary. Denote its limit by $\Lambda^1(f)$. Then $\Lambda^1$ is a well defined Radon measure on $\mathbb{R}^d \setminus \{0\}$.

To prove the second part of the theorem we proceed as at the end of the proof of Lemma 2.6, obtaining (2.20) for bounded continuous functions supported outside 0. By the Portmanteau theorem we have also (2.20) for every bounded function $f$ supported outside 0 and such that $\Lambda^1(\mathrm{Dis}(f)) = 0$. Finally, since $\Lambda^1$ is $\alpha$-homogeneous, it can be written in the form (1.6); hence we have $\Lambda^1\left(\mathrm{Dis}(\mathbf{1}_{\{|\cdot|>1\}})\right) = 0$, and the proof of Theorem 1.7 is completed. □

**Proof of Theorem 1.10.** Since the stationary solution $X$ does not depend on the choice of the starting point we may assume, without any loss of generality, that $X_0 = 0$ a.s.; then in view of Lemma 2.10 we know that $X_1 = \Phi(A_1X_0, B_1(X_0)) = \Phi(0, B_1^1)$ is $\alpha$-regularly varying with the tail measure $\Lambda_1$ (notice $\Lambda_1 = \Gamma_1$). Applying Lemma 2.10 to the random variable $X_2 = \Phi(A_2X_1, B_2(X_1))$, we can express its tail measure $\Lambda_2$ in the terms of $\Lambda_1$. Indeed,

$$\begin{aligned}\langle f, \Lambda_2\rangle &= \langle f\circ\Phi(\cdot,0), \widetilde{\Lambda}_1\rangle + \langle f\circ\Phi(0,\cdot), \Lambda_b\rangle\\ &= \mathbb{E}\left[\langle f\circ\Phi(A_2(\cdot),0), \Lambda_1\rangle\right] + \langle f, \Lambda_1\rangle = \mathbb{E}\left[\langle f\circ A_2, \Lambda_1\rangle\right] + \langle f, \Lambda_1\rangle,\end{aligned}$$

since $\Phi(x,0) = x$ for every $x \in \overline{[\operatorname{supp}\mu]\cdot\Phi[\{0\}\times\operatorname{supp}\Lambda_b]} \subseteq \mathbb{R}^d$ and by the definition $\langle f\circ\Phi(0,\cdot), \Lambda_b\rangle = \langle f, \Lambda_1\rangle$. If $\Lambda_n$ denotes the tail measure of $X_n$, then an easy induction argument proves

$$\langle f, \Lambda_n\rangle = \mathbb{E}\left[\sum_{k=2}^{n}\langle f\circ A_n\circ\cdots\circ A_k, \Lambda_1\rangle\right] + \langle f, \Lambda_1\rangle, \quad n\in\mathbb{N}.$$

To prove (1.11), notice that $X_n$ has the same law as $Y_n$ and hence

$$\mathbb{E}\left[\sum_{k=2}^{n}\langle f\circ A_n\circ\cdots\circ A_k, \Lambda_1\rangle\right] = \mathbb{E}\left[\sum_{k=2}^{n}\langle f\circ A_2\circ\cdots\circ A_k, \Lambda_1\rangle\right] = \mathbb{E}\left[\sum_{k=2}^{n}\langle f, \Gamma_k\rangle\right],$$

for every $n\in\mathbb{N}$. Therefore, we have

$$\begin{aligned}&t^\alpha L_b(t)\mathbb{E}\left[f(t^{-1}X)\right] - \left(\langle f, \Gamma_1\rangle + \mathbb{E}\left[\sum_{k=2}^{\infty}\langle f, \Gamma_k\rangle\right]\right)\\ &\quad = t^\alpha L_b(t)\mathbb{E}\left[f(t^{-1}X)\right] - t^\alpha L_b(t)\mathbb{E}\left[f(t^{-1}Y_n)\right]\\ &\qquad + t^\alpha L_b(t)\mathbb{E}\left[f(t^{-1}X_n)\right] - \left(\langle f, \Gamma_1\rangle + \mathbb{E}\left[\sum_{k=2}^{n}\langle f, \Gamma_k\rangle\right]\right)\\ &\qquad + \mathbb{E}\left[\sum_{k=n+1}^{\infty}\langle f, \Gamma_k\rangle\right].\end{aligned} \tag{2.21}$$

By (2.19) there exist constants $0 < C < \infty$ and $0 < \rho < 1$ such that for every $n\in\mathbb{N}$

$$\sup_{t>0}\left|t^\alpha L_b(t)\mathbb{E}\left[f(t^{-1}X)\right] - t^\alpha L_b(t)\mathbb{E}\left[f(t^{-1}Y_n)\right]\right| \le C\rho^n. \tag{2.22}$$

Reasoning as in the first part of the proof of Theorem 1.7 one can prove that for every $\varepsilon > 0$ there is $t_\varepsilon > 0$ such that for every $t \ge t_\varepsilon$

$$\left|t^\alpha L_b(t)\mathbb{E}\left[f(t^{-1}X_n)\right] - \left(\langle f, \Gamma_1\rangle + \mathbb{E}\left[\sum_{k=2}^{n}\langle f, \Gamma_k\rangle\right]\right)\right| < \varepsilon. \tag{2.23}$$

Finally assume that supp $f \subseteq \mathbb{R}^d \setminus B_\eta(0)$ for some $\eta > 0$; then

$$\left|\mathbb{E}\left[\sum_{k=n+1}^{\infty}\langle f, \Gamma_k\rangle\right]\right| \le \|f\|_\infty \mathbb{E}\left[\sum_{k=n+1}^{\infty}\int_{\mathbb{R}^d\setminus\{0\}} \mathbf{1}_{\{y\in\mathbb{R}^d:|y|>\eta\|A_2\circ\cdots\circ A_k\|^{-1}\}}(x)\Gamma_1(dx)\right]$$
$$\le \eta^{-\alpha}\|f\|_\infty \mathbb{E}\left[\sum_{k=n+1}^{\infty}\|A_2\circ\cdots\circ A_k\|^\alpha\right] \xrightarrow[n\to\infty]{} 0, \qquad (2.24)$$

since $\lim_{n\to\infty}(\mathbb{E}\|A_1\circ\cdots\circ A_n\|^\alpha)^{\frac{1}{n}} < 1$. Combining (2.21) with (2.22)–(2.24) we obtain (1.11).

Now take $f \in C_c(\mathbb{R}^d\setminus\{0\})$ of the form $f(r\omega) = f_1(r)f_2(\omega)$, where $r > 0$, $\omega \in \mathbb{S}^{d-1}$, $f_1 \in C_c((0,\infty))$ and $f_2 \in C(\mathbb{S}^{d-1})$. In view of Lemma 2.10 we obtain

$$\left\langle f_1, \frac{dr}{r^{\alpha+1}}\right\rangle\langle f_2, \sigma_{\Gamma_n}\rangle = \langle f, \Gamma_n\rangle = \mathbb{E}\left[\int_{\mathbb{R}^d\setminus\{0\}} f(A_2\circ\cdots\circ A_n x)\Gamma_1(dx)\right]$$
$$= \mathbb{E}\left[\int_0^\infty\int_{\mathbb{S}^{d-1}} f_1(|A_2\circ\cdots\circ A_n\omega|r)f_2((A_2\circ\cdots\circ A_n)*\omega)\sigma_{\Gamma_1}(d\omega)\frac{dr}{r^{\alpha+1}}\right]$$
$$= \left\langle f_1, \frac{dr}{r^{\alpha+1}}\right\rangle\mathbb{E}\left[\int_{\mathbb{S}^{d-1}}|A_2\circ\cdots\circ A_n\omega|^\alpha f_2((A_2\circ\cdots\circ A_n)*\omega)\sigma_{\Gamma_1}(d\omega)\right],$$

where $A*\omega = \frac{A\omega}{|A\omega|}$; hence we have proved

$$\langle f_2, \sigma_{\Gamma_n}\rangle = \mathbb{E}\left[\int_{\mathbb{S}^{d-1}}|A_2\circ\cdots\circ A_n\omega|^\alpha f_2\left((A_2\circ\cdots\circ A_n)*\omega\right)\sigma_{\Gamma_1}(d\omega)\right].$$

Finally to prove (1.12) we write

$$\mathbb{E}\left[\int_{\mathbb{S}^{d-1}} f(A*\omega)|A\omega|^\alpha\sigma_{\Gamma_n}(d\omega)\right]$$
$$= \mathbb{E}\left[\int_{\mathbb{S}^{d-1}} f(A*((A_2\circ\cdots\circ A_n)*\omega))\,|A((A_2\circ\cdots\circ A_n)*\omega)|^\alpha\right.$$
$$\left.\times\ |A_2\circ\cdots\circ A_n\omega|^\alpha\sigma_{\Gamma_1}(d\omega)\right]$$
$$= \mathbb{E}\left[\int_{\mathbb{S}^{d-1}} f((A_2\circ\cdots\circ A_{n+1})*\omega)\,|A_2\circ\cdots\circ A_{n+1}\omega|^\alpha\sigma_{\Gamma_1}(d\omega)\right]$$
$$= \int_{\mathbb{S}^{d-1}} f(\omega)\sigma_{\Gamma_{n+1}}(d\omega).$$

Formula (1.13) is a simple consequence of (1.11) and the calculations stated above. This completes the proof of Theorem 1.10. □

## 3. The limit theorem

Let $\mathcal{C}(\mathbb{R}^d)$ be the space of continuous functions on $\mathbb{R}^d$. Given positive parameters $\rho, \epsilon, \lambda$ we introduce two Banach spaces $\mathcal{C}_\rho(\mathbb{R}^d)$ and $\mathcal{B}_{\rho,\epsilon,\lambda}(\mathbb{R}^d)$ defined as follows:

$$\mathcal{C}_\rho = \mathcal{C}_\rho(\mathbb{R}^d) = \left\{ f \in \mathcal{C}(\mathbb{R}^d) : |f|_\rho = \sup_{x\in\mathbb{R}^d} \frac{|f(x)|}{(1+|x|)^\rho} < \infty \right\},$$

$$\mathcal{B}_{\rho,\epsilon,\lambda} = \mathcal{B}_{\rho,\epsilon,\lambda}(\mathbb{R}^d) = \{ f \in \mathcal{C}(\mathbb{R}^d) : \|f\|_{\rho,\epsilon,\lambda} = |f|_\rho + [f]_{\epsilon,\lambda} < \infty \},$$

where

$$[f]_{\epsilon,\lambda} = \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\epsilon (1+|x|)^\lambda (1+|y|)^\lambda}.$$

On $\mathcal{C}_\rho$ and $\mathcal{B}_{\rho,\epsilon,\lambda}$ we consider the Markov operator $Pf(x) = \mathbb{E}\left[f(X_1^x)\right]$ and its Fourier perturbations

$$P_{t,v} f(x) = \mathbb{E}\left[ e^{it\langle v, X_1^x\rangle} f(X_1^x) \right],$$

where $x \in \mathbb{R}^d$, $v \in \mathbb{S}^{d-1}$ and $t > 0$. Notice that $P_{0,v} = P$. The operators will play a crucial role in the proof, since one can prove by induction that

$$P_{t,v}^n f(x) = \mathbb{E}\left[ e^{it\langle v, S_n^x\rangle} f(X_n^x) \right].$$

So, the characteristic function of $a_n^{-1} S_n - d_n$ is just

$$\mathbb{E}\left[ e^{it\langle v, a_n^{-1} S_n - d_n\rangle} \right] = P^n_{ta_n^{-1},v} \mathbf{1}(x) e^{-it\langle v, d_n\rangle}.$$

Therefore, to prove the theorem one has to consider $P_{t,v}^n$ for large $n$ and small $t$, which reduces the problem to that of describing spectral properties of the operators $P_{t,v}$ on the Banach space $\mathcal{B}_{\rho,\epsilon,\lambda}$.

Next we define another family of Fourier operators:

$$T_{t,v} f(x) = \Delta_t^{-1} P_{t,v} \Delta_t f(x), \quad t > 0,$$

where $\Delta_t$ is the dilatation operator defined by $\Delta_t f(x) = f(tx)$. This family is related to the dilated Markov chain $\{X_{n,t}^x\}_{n\in\mathbb{N}}$ defined by

$$X_{n,t}^x = t\Phi_n(t^{-1} X_{n-1,t}^x) = t\Phi(A_n t^{-1} X_{n-1,t}^x, B_n(t^{-1} X_{n-1,t}^x)).$$

Then $X_{n,t}^x = tX_n^{t^{-1}x}$ and $\lim_{t\to 0} X_{n,t}^x = W_n^x$. Moreover, if $X_n^x$ is $\gamma$-geometric then so is $X_{n,t}^x$. We can express $T_{t,v}$ in a slightly different form:

$$T_{t,v} f(x) = \mathbb{E}\left[ e^{i\langle v, X_{1,t}^x\rangle} f(X_{1,t}^x) \right].$$

For $t = 0$ we write

$$T_{0,v} f(x) = T_v f(x) = \mathbb{E}\left[ e^{i\langle v, W_1^x\rangle} f(W_1^x) \right].$$

It is not difficult to see that $h_v(x) = \mathbb{E}\left[e^{i\langle v, W(x)\rangle}\right]$ is an eigenfunction of $T_v$. If $f \in \mathcal{C}_\rho$ is an eigenfunction of operator $T_{t,v}$ with eigenvalue $k_v(t)$, then $\Delta_t f$ is an eigenfunction of the operator $P_{t,v}$ with the same eigenvalue. Moreover,

**Lemma 3.1.** *The unique eigenvalue of modulus* 1 *for operator* $P$ *acting on* $\mathcal{C}_\rho$ *is* 1 *and the eigenspace is one dimensional. The corresponding projection on* $\mathbb{C} \cdot 1$ *is given by the map* $f \mapsto \nu(f)$. *The unique eigenvalue of modulus* 1 *for operator* $T_v$ *acting on* $\mathcal{C}_\rho$, *where* $v \in \mathbb{S}^{d-1}$, *is* 1 *and the eigenspace is one dimensional. The corresponding projection on* $\mathbb{C} \cdot h_v(x)$, *is given by the map* $f \mapsto f(0) \cdot h_v(x)$.

**Proof.** For the proof, of the first part see Section 3 of [7], and of the second part see Section 5 of [27]. □

The lemma above says that 1 is the unique peripheral eigenvalue for both $P_v$ and $T_v$. Even more can be proved: the complementary part of the spectrum for both operators on $\mathcal{B}_{\rho,\epsilon,\lambda}$ is contained in a ball centered at zero and with the radius strictly smaller than 1. So, they are quasi-compact. Moreover, due to the perturbation theorem of Keller and Liverani [23] (see also [26]) for small values of $t$, spectral properties of $P_{t,v}$ (resp., $T_{t,v}$) approximate appropriate properties of $P_v$ (resp., $T_v$). The proof is based on $\gamma$-geometricity of Markov processes $X_n^x$ and $X_{n,t}^x$, and the boundedness of $B^2$ (see Theorem 1.16) which in turn allows us to show that

$$|\Phi(Ax, tB(t^{-1}x)) - \overline{\Phi}(x)| \le \mathrm{Lip}_\Phi |tB(t^{-1}x)| \le t\mathrm{Lip}_\Phi(|B^1| + C), \tag{3.2}$$

for every $x \in \mathbb{R}^d$ and $t > 0$, where $\mathrm{Lip}_\Phi$ is the Lipschitz constant of $\Phi$.

We will not present the details, since the proof is a straightforward application of the arguments presented in [7,27].

The following proposition summarizes the necessary spectral properties of operators $P_{t,v}$ and $T_{t,v}$.

**Proposition 3.3.** *Assume that* $0 < \epsilon < 1$, $\lambda > 0$, $\lambda + 2\epsilon < \rho = 2\lambda$ *and* $2\lambda + \epsilon < \alpha$; *then there exist* $\delta > 0$, $0 < \varrho < 1 - \delta$ *and* $t_0 > 0$ *such that for every* $t \in [0, t_0]$ *and every* $v \in \mathbb{S}^{d-1}$:

- $\sigma(P_{t,v})$ *and* $\sigma(T_{t,v})$ *are contained in* $\mathcal{D} = \{z \in \mathbb{C} : |z| \le \varrho\} \cup \{z \in \mathbb{C} : |z - 1| \le \delta\}$.
- *The sets* $\sigma(P_{t,v}) \cap \{z \in \mathbb{C} : |z - 1| \le \delta\}$ *and* $\sigma(T_{t,v}) \cap \{z \in \mathbb{C} : |z - 1| \le \delta\}$ *consist of exactly one eigenvalue* $k_v(t)$, *where* $\lim_{t\to 0} k_v(t) = 1$, *and the corresponding eigenspace is one dimensional.*
- *We can express operators* $P_{t,v}$ *and* $T_{t,v}$ *in the following form:*

$$P_{t,v}^n = k_v(t)^n \Pi_{P,t} + Q_{P,t}^n, \quad \text{and} \quad T_{t,v}^n = k_v(t)^n \Pi_{T,t} + Q_{T,t}^n,$$

*for every* $n \in \mathbb{N}$, $\Pi_{P,t}$ *and* $\Pi_{T,t}$ *being the projections onto the one-dimensional eigenspaces mentioned above.* $Q_{P,t}$ *and* $Q_{T,t}$ *are operators complementary to projections* $\Pi_{P,t}$ *and* $\Pi_{T,t}$ *respectively, such that* $\Pi_{P,t}Q_{P,t} = Q_{P,t}\Pi_{P,t} = 0$ *and* $\Pi_{T,t}Q_{T,t} = Q_{T,t}\Pi_{T,t} = 0$. *Furthermore* $\|Q_{P,t}^n\|_{\mathcal{B}_{\rho,\epsilon,\lambda}} = O(\varrho^n)$ *and* $\|Q_{T,t}^n\|_{\mathcal{B}_{\rho,\epsilon,\lambda}} = O(\varrho^n)$ *for every* $n \in \mathbb{N}$. *The operators* $\Pi_{P,t}$, $\Pi_{T,t}$, $Q_{P,t}$ *and* $Q_{T,t}$ *depend on* $v \in \mathbb{S}^{d-1}$, *but this is omitted for simplicity.*

The following theorem contains the basic estimate:

**Theorem 3.4.** *Let* $h_v$ *be the eigenfunction for operator* $T_v$, *and the assumptions of Proposition* 3.3 *be satisfied. Then for any* $0 < \delta \le 1$ *such that* $\epsilon < \delta < \alpha$, *there exists* $C > 0$ *such that for every* $0 < t \le t_0$ *we have*

$$\left\| \Delta_t(\Pi_{T,t} - \Pi_{T,0})h_v \right\|_{\rho,\epsilon,\lambda} \le Ct^\delta, \quad \text{and} \tag{3.5}$$

$$\nu(\Delta_t \Pi_{T,t} h_v - 1) \le Dt^\delta. \tag{3.6}$$

**Proof.** The estimate (3.5) is based on the inequality (3.2) and spectral properties of the operators $T_{t,v}$. For more details we refer the reader to Section 6 in [27]. □

The following lemma was proved in [27] as a straightforward consequence of inequality (3.5):

**Lemma 3.7.** *If* $\alpha \in (0,2)$*, the assumptions of* Proposition 3.3 *are satisfied and* $\alpha - \rho > 1$ *if* $\alpha > 1$*, then*

$$\lim_{t\to 0} \frac{L_b(t^{-1})}{t^\alpha} \int_{\mathbb{R}^d} \left(e^{it\langle v,x\rangle} - 1\right)\left(\Pi_{T,t}(h_v)(tx) - \Pi_{T,0}(h_v)(tx)\right)\nu(dx) = 0. \tag{3.8}$$

**Proof of Theorem 1.16.** Notice that $\Delta_t \Pi_{T,t}(h_v)$ is an eigenfunction of the operator $P_{t,v}$ corresponding to the eigenvalue $k_v(t)$ and we have

$$(k_v(t)-1)\cdot \nu(\Delta_t \Pi_{T,t} h_v) = \nu\left(\left(e^{it\langle v,\cdot\rangle} - 1\right)\cdot(\Delta_t \Pi_{T,t} h_v)\right). \tag{3.9}$$

We will often use Theorem 1.7, but in a stronger version. Observe that the limit

$$\lim_{t\to 0} \frac{L_b(t^{-1})}{t^\alpha} \int_{\mathbb{R}^d} f(tx)\nu(dx) = \Lambda^1(f), \tag{3.10}$$

exists for every $f \in \mathcal{F}$, where

$$\mathcal{F} = \left\{ f : \sup_{x\in\mathbb{R}^d} |x|^{-\alpha} |\log|x||^{1+\varepsilon} |f(x)| < \infty \text{ for some } \varepsilon > 0 \text{ and } \Lambda^1(\mathrm{Dis}(f)) = 0 \right\}. \tag{3.11}$$

Now we consider each case separately.

*Case* $0 < \alpha < 1$. Observe that $\lim_{t\to 0} \nu(\Delta_t \Pi_{T,t} h_v) = 1$ by (3.6); hence using (3.9) we will prove

$$\lim_{t\to 0} L_b(t^{-1}) \frac{k_v(t)-1}{t^\alpha} = \int_{\mathbb{R}^d} \left(e^{i\langle v,x\rangle} - 1\right) h_v(x) \Lambda^1(dx) =: C_\alpha(v). \tag{3.12}$$

Let us write

$$\begin{aligned} &\frac{L_b(t^{-1})}{t^\alpha} \int_{\mathbb{R}^d} \left(e^{it\langle v,x\rangle} - 1\right) \Pi_{T,t}(h_v)(tx)\nu(dx) \\ &\quad = \frac{L_b(t^{-1})}{t^\alpha} \int_{\mathbb{R}^d} \left(e^{it\langle v,x\rangle} - 1\right)\cdot\left(\Pi_{T,t}(h_v)(tx) - \Pi_{T,0}(h_v)(tx)\right)\nu(dx) \\ &\qquad + \frac{L_b(t^{-1})}{t^\alpha} \int_{\mathbb{R}^d} \left(e^{it\langle v,x\rangle} - 1\right) \Pi_{T,0}(h_v)(tx)\nu(dx). \end{aligned}$$

In view of Lemma 3.7 the first term of the sum above tends to 0. Observe that the function $f_v(x) = \left(e^{i\langle v,x\rangle} - 1\right) h_v(x)$ belongs to $\mathcal{F}$ since it is bounded and $|f_v(x)| \le 2|x|$ for $|x| < 1$. Therefore, by Lemma 3.7 and (3.10) the expression above tends to a constant as $t$ goes to 0. Thus in view of (3.9) we obtain (3.12). Now we will show that

$$\lim_{n\to\infty} \Xi_\alpha^n(tv) = \Upsilon_\alpha(tv), \tag{3.13}$$

where $\Xi_\alpha^n$ is the characteristic function of $a_n^{-1}S_n^x - d_n$. For $t_n = \frac{t}{a_n}$ notice that

$$\Xi_\alpha^n(tv) = \mathbb{E}\left(e^{it_n\langle v, S_n^x\rangle}\right) = \left(P_{t_n,v}^n 1\right)(x) = k_v^n(t_n)\left(\Pi_{P,t_n}1\right)(x) + \left(Q_{P,t_n}^n 1\right)(x).$$

Since $\lim_{n\to\infty}\|Q_{P,t_n}^n\|_{\mathcal{B}_{\rho,\epsilon,\lambda}} = 0$, by Proposition 3.3 and $\lim_{n\to\infty}\Pi_{P,t_n}1 = 1$ (see [7] or [27] for more details), we have

$$\lim_{n\to\infty}\Xi_\alpha^n(tv) = \lim_{n\to\infty}k_v^n(t_n) = e^{\lim\limits_{n\to\infty} n(k_v(t_n)-1)},$$

and finally by (1.15) and (3.12)

$$\lim_{n\to\infty} n\cdot(k_v(t_n)-1) = \lim_{n\to\infty}\frac{n\cdot t_n^\alpha}{L_b(t_n^{-1})}L_b(t_n^{-1})\frac{k_v(t_n)-1}{t_n^\alpha} = \frac{t^\alpha C_\alpha(v)}{c}.$$

This proves the pointwise convergence $\Xi_\alpha^n$ to $\Upsilon_\alpha$. Continuity of $\Upsilon_\alpha$ at 0 follows from the Lebesgue dominated convergence theorem.

*Case* $\alpha = 1$. We prove the following lemma:

**Lemma 3.14.** *For every* $0 < \delta < 1$*, there exists a constant* $C_\delta > 0$ *such that for every* $|t| \le 1$*,*

$$|\xi(t)| \le C_\delta|t|^\delta.$$

**Proof.** For $|t| \le 1$, we write

$$|\xi(t)| \le \int_{\mathbb{R}^d}\frac{|tx|}{1+|tx|^2}\nu(dx) = \left(\int_{A_1}+\int_{A_2}+\int_{A_3}\right)\left(\frac{|tx|}{1+|tx|^2}\right)\nu(dx),$$

where $A_1 = \{x\in\mathbb{R}^d : |x|\le 1\}$, $A_2 = \{x\in\mathbb{R}^d : 1<|x|\le\frac{1}{|t|}\}$ and $A_3 = \{x\in\mathbb{R}^d : |x| > \frac{1}{|t|}\}$. The first integral is dominated by $C|t|$. To estimate the third one, notice that since $\frac{|x|}{1+|x|^2}\mathbf{1}_{\{|x|>1\}}\in\mathcal{F}$, we have

$$\lim_{t\to 0}\frac{L_b(t^{-1})}{|t|}\int_{\mathbb{R}^d}\frac{|tx|}{1+|tx|^2}\mathbf{1}_{\left\{|x|\ge\frac{1}{t}\right\}}\nu(dx) = \int_{\{|x|>1\}}\frac{|x|}{1+|x|^2}\Lambda^1(dx).$$

Therefore

$$\int_{A_3}\frac{|tx|}{1+|tx|^2}\nu(dx) \le \frac{C|t|}{L_b(t^{-1})} \le C|t|^\delta.$$

Finally we estimate the second integral. Let $\delta < \delta_1 < 1$ and notice that $\frac{1}{L_b}$ is also a slowly varying function. Then

$$\begin{aligned}\sum_{k=0}^{|\log_2|t||}\int_{\mathbb{R}^d}\frac{|tx|}{1+|tx|^2}\mathbf{1}_{\{2^k<|x|\le 2^{k+1}\}}\nu(dx) &\le |t|\sum_{k=0}^{|\log_2|t||}2^{k+1}\nu(\{x\in\mathbb{R}^d : |x|>2^k\})\\ &\le C|t|\sum_{k=0}^{|\log_2|t||}\frac{1}{L_b(2^k)} \le C|t|\sum_{k=0}^{|\log_2|t||}2^{(1-\delta_1)k}\\ &\le C|t|^\delta,\end{aligned}$$

since $\frac{1}{L_b(2^k)} \le C2^{(1-\delta_1)k}$ (see [9] Proposition 1.3.6(v)). This completes the proof of the lemma. □

In order to prove

$$\lim_{t\to 0} L_b(t^{-1})\frac{k_v(t)-1-i\langle v,\xi(t)\rangle}{t} = \int_{\mathbb{R}^d}\left(\left(e^{i\langle v,x\rangle}-1\right)h_v(x)-\frac{i\langle v,x\rangle}{1+|x|^2}\right)\Lambda^1(dx) =: \widetilde{C}_1(v), \tag{3.15}$$

notice that

$$\begin{aligned}\int_{\mathbb{R}^d}&\left(e^{it\langle v,x\rangle}-1\right)\Pi_{T,t}(h_v)(tx)\nu(dx)\\ &=\int_{\mathbb{R}^d}\left(e^{it\langle v,x\rangle}-1\right)\cdot\left(\Pi_{T,t}(h_v)(tx)-\Pi_{T,0}(h_v)(tx)\right)\nu(dx)\\ &\quad+\int_{\mathbb{R}^d}\left(e^{it\langle v,x\rangle}-1\right)\cdot\left(\Pi_{T,0}(h_v)(tx)-1\right)\nu(dx)\\ &\quad+\int_{\mathbb{R}^d}\left(e^{it\langle v,x\rangle}-1-\frac{i\langle v,tx\rangle}{1+|tx|^2}\right)\nu(dx)+i\langle v,\xi(t)\rangle.\end{aligned}$$

The first term of the sum tends to 0 by Lemma 3.7. The function $f_v(x)=\left(e^{i\langle v,x\rangle}-1\right)(h_v(x)-1)$ belongs to $\mathcal{F}$. Indeed, $f_v$ is bounded and for $|x|<1$

$$|f_v(x)|\le\left|e^{i\langle x,v\rangle}-1\right||h_v(x)-1|\le 2\mathbb{E}(|W(x)|^\delta)|x|\le C|x|^{1+\delta},$$

for any $0<\delta<1$. Similarly, one can prove that $g_v(x)=e^{i\langle v,x\rangle}-1-\frac{i\langle v,x\rangle}{1+|x|^2}$ belongs to $\mathcal{F}$. Indeed, $g_v$ is bounded and for $|x|<1$

$$|g_v(x)|\le\left|e^{i\langle x,v\rangle}-1-i\langle x,v\rangle\right|+\frac{|x|^3}{1+|x|^2}\le 2|x|^{1+\delta}+\frac{|x|^3}{1+|x|^2},$$

for any $0<\delta<1$. Hence, by (3.10) we obtain

$$\lim_{t\to 0}\frac{L_b(t^{-1})}{t}\left(\int_{\mathbb{R}^d}\left(e^{it\langle v,x\rangle}-1\right)\Pi_{T,t}(h_v)(tx)\nu(dx)-i\langle v,\xi(t)\rangle\right)=\widetilde{C}_1(v). \tag{3.16}$$

Now by (3.16) we have

$$\begin{aligned}\lim_{t\to 0}&L_b(t^{-1})\frac{k_v(t)-1-i\langle v,\xi(t)\rangle}{t}\\ &=\lim_{t\to 0}L_b(t^{-1})\frac{\left(\nu\left((e^{it\langle v,\cdot\rangle}-1)(\Delta_t\Pi_{T,t}h_v)\right)-i\langle v,\xi(t)\rangle\nu(\Delta_t\Pi_{T,t}h_v)\right)}{\nu(\Delta_t\Pi_{T,t}h_v)t}\\ &=\lim_{t\to 0}L_b(t^{-1})\\ &\quad\times\left(\frac{\nu\left((e^{it\langle v,\cdot\rangle}-1)(\Delta_t\Pi_{T,t}h_v)\right)-i\langle v,\xi(t)\rangle}{\nu(\Delta_t\Pi_{T,t}h_v)t}+\frac{i\left(1-\nu(\Delta_t\Pi_{T,t}h_v)\right)\langle v,\xi(t)\rangle}{\nu(\Delta_t\Pi_{T,t}h_v)t}\right)\\ &=\widetilde{C}_1(v).\end{aligned}$$

By (3.6) and Lemma 3.14 we have

$$\lim_{t\to 0}L_b(t^{-1})\left(\frac{i\left(1-\nu(\Delta_t\Pi_{T,t}h_v)\right)\langle v,\xi(t)\rangle}{\nu(\Delta_t\Pi_{T,t}h_v)t}\right)=0,$$

and (3.15) follows. Now we need the following:

**Lemma 3.17.** *Let $m_{\sigma_{\Lambda^1}} = \int_{\mathbb{S}^{d-1}} \omega\sigma_{\Lambda^1}(d\omega)$, where $\sigma_{\Lambda^1}$ is the spherical measure associated with the tail measure $\Lambda^1$. Then for every $t \in \mathbb{R}$ and $v \in \mathbb{S}^{d-1}$*

$$\lim_{s\to 0} \frac{L_b(s^{-1})}{s} \int_{\mathbb{R}^d} \left( \frac{\langle v, stx\rangle}{1+|stx|^2} - \frac{\langle v, stx\rangle}{1+|sx|^2} \right) \nu(dx) = -t\log|t| \langle v, m_{\sigma_{\Lambda^1}}\rangle. \tag{3.18}$$

*In particular, for every $0 < \delta < 1$ there exists a constant $C_\delta > 0$ such that for every $|t| \le 1$,*

$$|t\log|t|\langle v, m_{\sigma_{\Lambda^1}}\rangle| \le C_\delta |t|^\delta. \tag{3.19}$$

**Proof.** Observe that $\frac{x}{1+|tx|^2} - \frac{x}{1+|x|^2} \in \mathcal{F}$; hence

$$\lim_{s\to 0} \frac{L_b(s^{-1})}{s} \int_{\mathbb{R}^d} \left( \frac{\langle v, stx\rangle}{1+|stx|^2} - \frac{\langle v, stx\rangle}{1+|sx|^2} \right) \nu(dx) = t\langle v, \tau(t)\rangle,$$

where $\tau(t) = \int_{\mathbb{R}^d} \left( \frac{x}{1+|tx|^2} - \frac{x}{1+|x|^2} \right) \Lambda^1(dx)$. Notice that

$$\begin{aligned}
\tau(t) &= \int_{\mathbb{R}^d} \left( \frac{x}{1+|tx|^2} - \frac{x}{1+|x|^2} \right) \Lambda^1(dx) \\
&= \int_0^\infty \int_{\mathbb{S}^{d-1}} \left( \frac{r\omega}{1+|tr\omega|^2} - \frac{r\omega}{1+|r\omega|^2} \right) \sigma_{\Lambda^1}(d\omega) \frac{dr}{r^2} \\
&= \int_{\mathbb{S}^{d-1}} \omega\sigma_{\Lambda^1}(d\omega) \cdot \int_0^\infty \left( \frac{r(1-t^2)}{(1+t^2r^2)(1+r^2)} \right) dr = -m_{\sigma_{\Lambda^1}} \log|t|.
\end{aligned}$$

The proof is completed. □

For $t_n = \frac{t}{a_n}$, $t > 0$ we have

$$\begin{aligned}
\lim_{n\to\infty} \Xi_1^n(tv) &= \lim_{n\to\infty} e^{-itn\langle v, \xi(a_n^{-1})\rangle} \mathbb{E}\left( e^{it_n\langle v, S_n^x\rangle} \right) \\
&= \lim_{n\to\infty} e^{-int\langle v, \xi(a_n^{-1})\rangle} k_v^n(t_n) = e^{\lim_{n\to\infty}\left( n(e^{-it\langle v, \xi(a_n^{-1})\rangle} k_v(t_n) - 1) \right)}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\lim_{n\to\infty} \left( n\left( e^{-it\langle v, \xi(a_n^{-1})\rangle} k_v(t_n) - 1 \right) \right) \\
&\quad = \lim_{n\to\infty} \left( \frac{nt_n}{L_b(t_n^{-1})} e^{-it\langle v, \xi(a_n^{-1})\rangle} L_b(t_n^{-1}) \frac{k_v(t_n) - 1 - i\langle v, \xi(t_n)\rangle}{t_n} \right. \\
&\qquad \left. + \, ne^{-it\langle v, \xi(a_n^{-1})\rangle} (1 + i\langle v, \xi(t_n)\rangle) - n \right) \\
&\quad = \lim_{n\to\infty} \left( \widetilde{C}_1(v) \frac{nt}{a_n L_b(a_n)} \frac{L_b(a_n)}{L_b(t^{-1}a_n)} + n\left( 1 - it\langle v, \xi(a_n^{-1})\rangle \right. \right. \\
&\qquad \left. \left. + \; O\left( t^2 \langle v, \xi(a_n^{-1})\rangle^2 \right) \right) (1 + i\langle v, \xi(t_n)\rangle) - n \right) \\
&\quad = \lim_{n\to\infty} \left( in\langle v, \xi(t_n)\rangle - \mathrm{int}\langle v, \xi(a_n^{-1})\rangle + nt\langle v, \xi(t_n)\rangle\langle v, \xi(a_n^{-1})\rangle \right. \\
&\qquad \left. + \; nO\left( t^2\langle v, \xi(a_n^{-1})\rangle^2 \right) (1 + i\langle v, \xi(t_n)\rangle) \right) + \frac{t\widetilde{C}_1(v)}{c}.
\end{aligned}$$

Notice that by (3.18) we have

$$\begin{aligned}\lim_{n\to\infty}\Big(in\langle v,\xi(t_n)\rangle-\mathrm{int}\langle v,\xi(a_n^{-1})\rangle\Big)&=\lim_{n\to\infty}it\frac{n}{a_nL_b(a_n)}\cdot a_nL_b(a_n)\\&\quad\times\int_{\mathbb{R}^d}\left(\frac{\langle v,a_n^{-1}x\rangle}{1+\left|a_n^{-1}tx\right|^2}-\frac{\langle v,a_n^{-1}x\rangle}{1+\left|a_n^{-1}x\right|^2}\right)\nu(dx)\\&=-\frac{it\log t\langle v,m_{\sigma_{\Lambda^1}}\rangle}{c},\end{aligned}$$

and the limit of two remaining factors, by Lemma 3.14, is 0. Therefore the limit of the whole expression is equal to $\varUpsilon_1(tv)=\frac{t\widetilde{C}_1(v)-it\log t\langle v,m_{\sigma_{\Lambda^1}}\rangle}{c}$. Finally, to prove continuity of $\varUpsilon_1$ at zero, it is enough to observe that for $|x|<1$,

$$\left|\left(e^{i\langle v,x\rangle}-1\right)h_v(x)-\frac{i\langle v,x\rangle}{1+|x|^2}\right|\le C|x|^{1+\delta},$$

for any $0<\delta<1$ and some $C>0$ independent of $v\in\mathbb{S}^{d-1}$.

*Case* $1<\alpha<2$. As in the previous cases we show that

$$\begin{aligned}&\lim_{t\to0}L_b(t^{-1})\frac{k_v(t)-1-i\langle v,tm\rangle}{t^\alpha}\\&\quad=\int_{\mathbb{R}^d}\left(\left(e^{i\langle v,x\rangle}-1\right)h_v(x)-i\langle v,x\rangle\right)\Lambda^1(dx)=:C_\alpha(v).\end{aligned}\tag{3.20}$$

Let us write

$$\begin{aligned}&\int_{\mathbb{R}^d}\left(e^{it\langle v,x\rangle}-1\right)\varPi_{T,t}(h_v)(tx)\nu(dx)\\&\quad=\int_{\mathbb{R}^d}\left(e^{it\langle v,x\rangle}-1\right)\cdot\left(\varPi_{T,t}(h_v)(tx)-\varPi_{T,0}(h_v)(tx)\right)\nu(dx)\\&\qquad+\int_{\mathbb{R}^d}\left(e^{it\langle v,x\rangle}-1\right)\cdot\left(\varPi_{T,0}(h_v)(tx)-1\right)\nu(dx)\\&\qquad+\int_{\mathbb{R}^d}\left(e^{it\langle v,x\rangle}-1-i\langle v,tm\rangle\right)\nu(dx)+i\langle v,tm\rangle.\end{aligned}$$

By Lemma 3.7 the first term of the sum goes to 0. Functions $f_v(x)=\left(e^{i\langle v,x\rangle}-1\right)(h_v(x)-1)$ and $g_v(x)=e^{i\langle v,x\rangle}-1-i\langle v,x\rangle$ belong to $\mathcal{F}$. Hence, by (3.10) we obtain

$$\lim_{t\to0}\frac{L_b(t^{-1})}{t^\alpha}\left(\int_{\mathbb{R}^d}\left(e^{it\langle v,x\rangle}-1\right)\varPi_{T,t}(h_v)(tx)\nu(dx)-i\langle v,tm\rangle\right)=C_\alpha(v).\tag{3.21}$$

Similarly, as in the previous case we have

$$\begin{aligned}
&\lim_{t\to 0} L_b(t^{-1})\frac{k_v(t)-1-i\langle v,tm\rangle}{t^\alpha}\\
&\quad=\lim_{t\to 0} L_b(t^{-1})\frac{\big(\nu\left((e^{it\langle v,\cdot\rangle}-1)(\Delta_t\Pi_{T,t}h_v)\right)-i\langle v,tm\rangle\nu(\Delta_t\Pi_{T,t}h_v)\big)}{\nu(\Delta_t\Pi_{T,t}h_v)t^\alpha}\\
&\lim_{t\to 0} L_b(t^{-1})\left(\frac{\nu\left((e^{it\langle v,\cdot\rangle}-1)\cdot(\Delta_t\Pi_{T,t}h_v)\right)-i\langle v,tm\rangle}{\nu(\Delta_t\Pi_{T,t}h_v)t^\alpha}+\frac{i\left(1-\nu(\Delta_t\Pi_{T,t}h_v)\right)\langle v,tm\rangle}{\nu(\Delta_t\Pi_{T,t}h_v)t^\alpha}\right)\\
&\quad= C_\alpha(v).
\end{aligned}$$

By (3.6)

$$\lim_{t\to 0} L_b(t^{-1})\left(\frac{i\left(1-\nu(\Delta_t\Pi_{T,t}h_v)\right)\langle v,tm\rangle}{\nu(\Delta_t\Pi_{T,t}h_v)t^\alpha}\right)=0,$$

and (3.20) follows.

Now we can show that

$$\lim_{n\to\infty}\Xi_\alpha^n(tv)=\Upsilon_\alpha(tv). \tag{3.22}$$

In order to prove (3.22) notice that

$$\lim_{n\to\infty}\Xi_\alpha^n(tv)=\lim_{n\to\infty}e^{-\mathrm{int}_n\langle v,m\rangle}\mathbb{E}\left(e^{it_n\langle v,S_n^x\rangle}\right)=e^{\lim_{n\to\infty}n\left(e^{-it_n\langle v,m\rangle}k_v(t_n)-1\right)}.$$

Moreover, since $\lim_{n\to\infty}nt_n^2=0$, we have

$$\begin{aligned}
&\lim_{n\to\infty}\left(n\left(e^{-it_n\langle v,m\rangle}k_v(t_n)-1\right)\right)\\
&\quad=\lim_{n\to\infty}\left(\frac{nt_n^\alpha}{L_b(t_n^{-1})}e^{-it_n\langle v,m\rangle}\cdot L_b(t_n^{-1})\frac{k_v(t_n)-1-it_n\langle v,m\rangle}{t_n^\alpha}\right.\\
&\qquad\left.+\;ne^{-it_n\langle v,m\rangle}\left(1+it_n\langle v,m\rangle\right)-n\right)\\
&\quad=\lim_{n\to\infty}\left(C_\alpha(v)\frac{nt^\alpha}{a_n^\alpha L_b(a_n)}\frac{L_b(a_n)}{L_b(t^{-1}a_n)}\right.\\
&\qquad\left.+\;\left(n\cdot\left(1-it_n\langle v,m\rangle+O\left(t_n^2\right)\right)\cdot\left(1+it_n\langle v,m\rangle\right)-n\right)\right)\\
&\quad=\frac{t^\alpha C_\alpha(v)}{c}+\lim_{n\to\infty}\left(nt_n^2\langle v,m\rangle^2+nO\left(t_n^2\right)\cdot\left(1+it_n\langle v,m\rangle\right)\right)=\frac{t^\alpha C_\alpha(v)}{c},
\end{aligned}$$

and (3.22) follows. To prove continuity of $\Upsilon_\alpha$ at zero, we proceed as in the previous cases.

Finally, under some additional assumptions, we have to prove a nondegeneracy of the limit variable $C_\alpha(v)$ for $v\in\mathbb{S}^{d-1}$. Notice first that since $\Phi(x,0)=x$ for every $x\in\overline{[\operatorname{supp}\mu]\cdot\operatorname{supp}\nu}$, $W(x)=\sum_{k=1}^\infty A_k\cdot\dots\cdot A_1x$. Let us define $W^*(x)=\sum_{k=1}^\infty A_1^*\cdot\dots\cdot A_k^*x$ and observe

$$\begin{aligned}
\Re C_\alpha(v)&=\Re\left(\int_{\mathbb{R}^d}\left(e^{i\langle v,x\rangle}-1\right)\mathbb{E}\left[e^{i\langle v,W(x)\rangle}\right]\Lambda^1(dx)\right)\\
&=\int_0^\infty\int_{\mathbb{S}^{d-1}}\mathbb{E}\left[\cos\left(t\langle W^*(v)+v,w\rangle\right)-\cos\left(t\langle W^*(v),w\rangle\right)\right]\sigma_{\Lambda^1}(dw)\frac{dt}{t^{\alpha+1}}.
\end{aligned}$$

Hence

$$\Re C_\alpha(v) = C(\alpha) \cdot \int_{\mathbb{S}^{d-1}} \mathbb{E}\left[\left|\langle W^*(v)+v, w\rangle\right|^\alpha - \left|\langle W^*(v), w\rangle\right|^\alpha\right] \sigma_{\Lambda^1}(dw),$$

for $C(\alpha) = \int_0^\infty \frac{\cos t - 1}{t^{\alpha+1}} dt < 0$. Notice that $W_v = W^*(v) + v$ is a solution of the random difference equation

$$W_v =_d A^* W_v + v. \tag{3.23}$$

Moreover, since $\lim_{n\to\infty} (\mathbb{E}\|A_1 \cdot \dots \cdot A_n\|^\alpha)^{\frac{1}{n}} < 1$, this implies that $\mathbb{E}|W_v|^\alpha < \infty$, and we have

$$\begin{aligned}\Re C_\alpha(v) &= C(\alpha) \cdot \int_{\mathbb{S}^{d-1}} \mathbb{E}\left[|\langle W_v, w\rangle|^\alpha - \left|\langle A^* W_v, w\rangle\right|^\alpha\right] \sigma_{\Lambda^1}(dw),\\ &= C(\alpha) \cdot \int_{\mathbb{S}^{d-1}} \mathbb{E}\left[|\langle W_v, w\rangle|^\alpha - |Aw|^\alpha \, |\langle W_v, A * w\rangle|^\alpha\right] \sigma_{\Lambda^1}(dw).\end{aligned}$$

Now in view of (1.12) and (1.13) we obtain

$$\int_{\mathbb{S}^{d-1}} \mathbb{E}\left[|Aw|^\alpha \, |\langle W_v, A * w\rangle|^\alpha\right] \sigma_{\Lambda^1}(dw) = \sum_{n=2}^\infty \int_{\mathbb{S}^{d-1}} \mathbb{E}\left[\, \langle W_v, w\rangle|^\alpha\right] \sigma_{\Gamma_n}(dw).$$

Therefore we can conclude that for every $v \in \mathbb{S}^{d-1}$

$$\Re C_\alpha(v) = C(\alpha) \cdot \int_{\mathbb{S}^{d-1}} \mathbb{E}\left[|\langle W_v, w\rangle|^\alpha\right] \sigma_{\Gamma_1}(dw).$$

Finally we have to prove that $\int_{\mathbb{S}^{d-1}} \mathbb{E}\left[|\langle W_v, w\rangle|^\alpha\right] \sigma_{\Gamma_1}(dw) > 0$. For this purpose, in view of (2.11), notice that for every $f \in C(\mathbb{S}^{d-1})$

$$\int_{\mathbb{S}^{d-1}} f(w) \sigma_{\Gamma_1}(dw) = \int_{\mathbb{S}^{d-1}} f\left(\frac{\Phi(0,w)}{|\Phi(0,w)|}\right) |\Phi(0,w)|^\alpha \sigma_{\Lambda_b}(dw),$$

which in turn implies

$$\begin{aligned}&\int_{\mathbb{S}^{d-1}} \mathbb{E}\left[|\langle W_v, w\rangle|^\alpha\right] \sigma_{\Gamma_1}(dw) = \int_{\mathbb{S}^{d-1}} \mathbb{E}\left[|\langle W_v, \Phi(0,w)\rangle|^\alpha\right] \sigma_{\Lambda_b}(dw)\\ &\quad = \mathbb{E}\left[|W_v|^\alpha \int_{\mathbb{S}^{d-1}} |\langle W_v/|W_v|, \Phi(0,w)\rangle|^\alpha \, \sigma_{\Lambda_b}(dw)\right] \geq C_{\Lambda_b} \mathbb{E}\left[|W_v|^\alpha\right]\end{aligned} \tag{3.24}$$

for $C_{\Lambda_b} = \min_{u\in\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} |\langle u, \Phi(0,w)\rangle|^\alpha \, \sigma_{\Lambda_b}(dw)$ which is strictly positive. Indeed, if for some $u_0 \in \mathbb{S}^{d-1}$, $\int_{\mathbb{S}^{d-1}} |\langle u_0, \Phi(0,w)\rangle|^\alpha \, \sigma_{\Lambda_b}(dw)$ were equal to 0, then the set $\Phi[\{0\} \times \operatorname{supp} \sigma_{\Lambda_b}]$ would be contained in the hyperplane $u_0^\perp$, which contradicts our assumptions. This completes the proof of Theorem 1.16. □

## Acknowledgments

The authors are grateful to the referees for a very careful reading of the manuscript and useful remarks that led to improvement of the presentation.

D. Buraczewski and E. Damek were partially supported by MNiSW N N201 393937. M. Mirek was partially supported by MNiSW grant N N201 392337. D. Buraczewski was also supported by the European Commission via IEF Project (contract number PIEF-GA-2009-252318 - SCHREC).

## References

[1] G. Alsmeyer, S. Mentemeier, Tail behavior of stationary solutions of random difference equations: the case of regular matrices, preprint, http://arxiv.org/abs/1009.1728.

[2] K. Bartkiewicz, A. Jakubowski, T. Mikosch, O. Wintenberger, Stable limits for sums of dependent infinite variance random variables, Probab. Theory Related Fields 150 (3–4) (2011) 337–372.

[3] B. Basrak, R.A. Davis, T. Mikosch, A characterization of multivariate regular variation, Ann. Appl. Probab. 12 (3) (2002) 908–920.

[4] M. Benda, A central limit theorem for contractive stochastic dynamical systems, J. Appl. Probab. 35 (1998) 200–205.

[5] J. Boman, F. Lindskog, Support theorems for the Radon transform and Cramer–Wold theorems, J. Theoret. Probab. 22 (3) (2009) 683–710.

[6] L. Breiman, On some limit theorems similar to the arc-sin law, Teor. Verojatnost. i Primenen. 10 (1965) 351–360.

[7] D. Buraczewski, E. Damek, Y. Guivarc'h, Convergence to stable laws for a class of multidimensional stochastic recursions, Probab. Theory Related Fields 148 (2010) 333–402.

[8] D. Buraczewski, E. Damek, Y. Guivarc'h, A. Hulanicki, R. Urban, On tail properties of stochastic recursions connected with generalized rigid motions, Probab. Theory Related Fields 145 (2009) 385–420.

[9] N.H. Bingham, C.M. Goldie, J.L. Teugels, Regular Variation, in: Encyclopedia of Mathematics and its Applications, vol. 27, Cambridge University Press, Cambridge, 1987, p. xx+491.

[10] R.A. Davis, S.I. Resnick, Limit theory for bilinear processes with heavy-tailed noise, Ann. Appl. Probab. 6 (4) (1996) 1191–1210.

[11] P. Diaconis, D. Freedman, Iterated random functions, SIAM Rev. 41 (1) (1999) 45–76. electronic.

[12] R. Durrett, Probability: Theory and Examples, third ed., in: Duxbury Advanced Series, Thomson Brooks/Cole, 1968.

[13] H. Furstenberg, Noncommuting random products, Amer. Math. Soc. 108 (1963) 377–428.

[14] H. Furstenberg, H. Kesten, Products of random matrices, Ann. Math. Statist. 31 (1960) 457–469.

[15] C.M. Goldie, Implicit renewal theory and tails of solutions of random equations, Ann. Appl. Probab. 1 (1) (1991) 126–166.

[16] D.R. Grey, Regular variation in the tail behaviour of solutions of random difference equations, Ann. Appl. Probab. 4 (1) (1994) 169–183.

[17] A.K. Grincevicius, On limit distribution for a random walk on the line, Lith. Math. J. 15 (1975) 580–589. English translation.

[18] Y. Guivarc'h, Heavy tail properties of multidimensional stochastic recursions, in: IMS Lecture Notes—Monograph Series, Dyn. & Stoch. 48 (2006) 85–99.

[19] Y. Guivarc'h, E. Le Page, On spectral properties of a family of transfer operators and convergence to stable laws for affine random walks, Ergodic Theory Dynam. Systems 28 (2) (2008) 423–446.

[20] H. Hennion, L. Hervé, Limit theorems for Markov chains and stochastic properties of dynamical systems by quasi compactness, in: Lecture Notes in Math., vol. 1766, Springer, Berlin, 2001.

[21] D. Hay, R. Rastegar, A. Roitershtein, Multivariate linear recursions with Markov-dependent coefficients, J. Multivariate Anal. 102 (2011) 521–527.

[22] H. Kesten, Random difference equations and renewal theory for products of random matrices, Acta Math. 131 (1973) 207–248.

[23] G. Keller, C. Liverani, Stability of the spectrum for transfer operators, Ann. Sc. Norm. Super. Pisa Cl. Sci. 28 (4) (1999) 141–152.

[24] C. Klüppelberg, S. Pergamenchtchikov, The tail of the stationary distribution of a random coefficient AR$(q)$ model, Ann. Appl. Probab. 14 (2) (2004) 971–1005.

[25] É. Le Page, Théoremes de renouvellement pour les produits de matrices aléatoires. Équations aux differences aleátoires, in: Séminaires de Probabilités Rennes 1983, Univ. Rennes I, Rennes, 1983, p. 116.

[26] C. Liverani, 2004, Invariant measures and their properties, a functional analytic point of view, Dynamical Systems, Part II: Topological Geometrical and Ergodic Properties of Dynamics, Pubblicazioni della Classe di Scienze, Scuola Normale Superiore, Pisa, Centro di Ricerca Matematica Ennio De Giorgi: Proceedings, Published by the Scuola Normale Superiore in Pisa.

[27] M. Mirek, Heavy tail phenomenon and convergence to stable laws for iterated Lipschitz maps, Probab. Theory Related Fields (2011), doi:10.1007/s00440-010-0312-9.

[28] R. Rastegar, A. Roitershtein, V. Roytershteyn, J. Suh, Discrete-time Langevin motion in a regularly varying potential induced by a Gibbs' state, Preprint.

[29] S.I. Resnick, E. Willekens, Moving averages with random coefficients and random coefficient autoregressive models, Comm. Statist. Stoch. Models 7 (4) (1991) 511–525.

[30] M. Woodroofe, W.B. Wu, A central limit theorem for iterated random functions, J. Appl. Probab. 37 (2000) 748–755.