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1-Factors and Polynomials

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In this paper we give a fuller exposition of a property of 1-factors discussed in [1]. The 1-factors of cubic graphs are found to be enumerated by a graph-function closley related to the chromatic and flow polynomials. The first part of the paper is a short account, with some minor improvements, of the theory of V-functions and ϕ-functions first set out in [1].

1. V-Functions

Given a link A in a graph G we can define two derived graphs G'_A and G''_A. The first is the spanning subgraph of G obtained by deleting A. The second is derived from G by contracting A, with its two ends, into a single new vertex.

We consider *graph-functions*. Such a function assigns to each abstract graph a unique value. Here we suppose this value to be an integer, or an element of some ring of polynomials over the integers.

There are graph-functions $V(G)$ that satisfy the following identity:

$$V(G) = V(G'_A) + V(G''_A). \tag{1}$$

The *complexity*, or number of spanning trees is an example.

In other interesting cases we usually have

$$V(G) = V(H) \cdot V(K) \tag{2}$$

whenever G is the union of two disjoint subgraphs H and K. We refer to a graph-function satisfying (1) and (2) as a *V-function*. One example is the number of spanning subgraphs of G whose intersection with each component of G is a tree. Examples related to the chromatic polynomial and what is now known as the flow polynomial are given in [1, 2]. It follows from (2) that a V-function must take the value 1 for the null graph, except in the trivial case in which it is zero for all graphs.

Let us write X_n, $(n = 0, 1, 2, 3, \ldots)$, for the graph consisting of a single vertex and n loops. We observe that the cyclomatic number of X_n is n.

Theorem 1.1. *Let G be a graph of cyclomatic number n. Let V be any V-function. Then $V(G)$ can be written as a polynomial with integral coefficients, and with a form independent of V, in those expressions $V(X_k)$ for which $k \leq n$. Hence V is uniquely determined when its value for each X_k is known.*

In proving this we bear in mind that contraction of a link does not change the cyclomatic number, that deletion of a link does not increase the cyclomatic number, and that the cyclomatic number of a graph is the sum of the cyclomatic numbers of its components.

The proof is by induction over $m(G)$, the number of edges and vertices of G. At each step in the induction we use Equation (1) if G has a link A, and Equation (2) if G is disconnected. In the remaining case there is nothing to prove.

Theorem 1.1 tells us that a V-function is completely determined when it is known for each X_n (and for the null graph). There is a converse theorem saying that there exists a

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V-function having arbitrary assigned values for the graphs X_n, but it is a little more difficult to prove.

To prove the converse theorem we first define a graph-function $Z(G)$ as follows:

$$Z(G)=\sum_S \prod_{j=0}^{\infty} (z_j)^{\sigma(S,j)}. \tag{3}$$

Here S denotes any subset of the set E of edges of G. The z_j are independent indeterminates over the integers. S determines a spanning subgraph $G:S$ of G, and we write $\sigma(S, j)$ for the number of components of $G:S$ with cyclomatic number j.

By a simple exercise in graph theory we find that $Z(G)$ satisfies (1) and (2), and is therefore a V-function. (See [1] and [2].)

THEOREM 1.2. *There exists a V-function V taking an arbitrarily assigned value (within the appropriate ring) for each of the graphs X_n.*

PROOF. Suppose we wish $V(X_n)$ to be v_n. Then we write

$$z'_n = \sum_{j=0}^{n} (-1)^j \binom{n}{j} v_n, \tag{4}$$

for each n. Substituting z'_j for z_j on the right of (3) we obtain a V-function which we denote by V.

By inversion of (4) we have

$$v_n = \sum_{j=0}^{n} \binom{n}{j} z'_j. \tag{5}$$

But this is $V(X_n)$, by (3).

Theorems 1.1 and 1.2 tell us that the graph-function $Z(G)$ defined by (3) is the most general V-function taking the value 1 for the null graph. We have indeed noted an exceptional V-function which is zero for all graphs. But let us ignore that triviality from now on.

2. TOPOLOGICALLY INVARIANT V-FUNCTIONS

Special interest attaches to those V-functions V for which

$$V(X_0) = -1. \tag{6}$$

We call them the *topologically invariant* V-functions.

THEOREM 2.1. *Let G be a graph having a monovalent vertex x. Let V be a topologically invariant V-function. Then*

$$V(G) = 0.$$

PROOF. Let A be the link of G incident with x. Then G'_A is the union of two disjoint subgraphs, of which one is X_0 and the other is isomorphic with G''_A. (See Figure 1.) So by (1) and (2) we have

$$V(G) = V(G''_A)\{(-1)+1\} = 0.$$

Consider next a graph G having a divalent vertex x incident with two links A and B. Let the other ends of A and B be a and b respectively. The vertices a and b may coincide. To

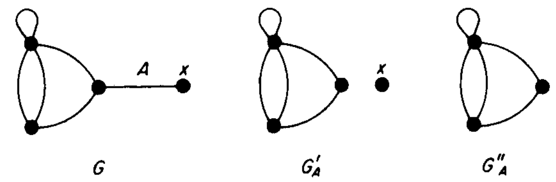

FIGURE 1

suppress x in G is to replace A, B and x by a single new edge D with ends a and b. The reverse operation is called *subdividing* D by means of a new divalent vertex c.

THEOREM 2.2. *Let G be a graph having a divalent vertex x incident with two links A and B. Let G_x be the graph derived from G by suppressing x. Let V be any topologically invariant V-function.* Then

$$V(G_x) = V(G).$$

PROOF. The vertex x is monovalent in G'_A. Moreover G''_A is isomorphic with G_x. (See Figure 2.) Hence $V(G) = V(G''_A) = V(G_x)$, by Theorem 2.1.

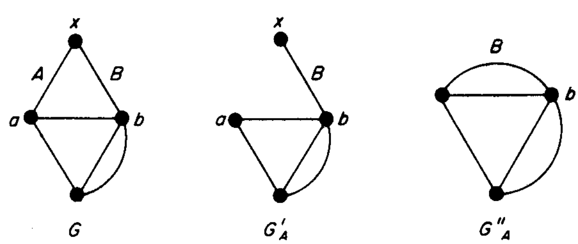

FIGURE 2

Theorem 2.2 justifies the term "topologically invariant V-function".

Let us consider also the case of a graph G having a divalent vertex x incident with a loop A. It is sometimes convenient to be able to speak of suppressing x even in this case. We take it that the operation removes x and turns A into a *loose edge*, incident with no vertex at all.

By allowing loose edges we are of course extending the usual definition of a graph. A generalized graph G will be an ordinary graph H to which $m \geqslant 0$ loose edges have been adjoined. Each loose edge is considered to constitute by itself a component of G that is also a circuit. So the components of G are those of H and m others defined by the loose edges. The cyclomatic number of G is defined as that of H, plus m. It is still the least number of edges whose deletion destroys all the circuits. But the rule that the cyclomatic number is the number of edges minus the number of vertices plus the number of components must be modified. Instead of "number of edges" we should say "number of attached edges".

Replacing a loose edge A in a generalized graph by a loop on a single new divalent vertex x is called *subdividing* A by means of x.

Let V be any topologically invariant V-function. With G and H as above we extend V to the generalized graph G by the following rule.

$$V(G) = V(H)\{V(X_1)\}^m. \tag{7}$$

With this extension V still remains invariant under subdivision, and it still satisfies (1) and (2), even in the realm of generalized graphs.

In diagrams we shall represent a loose edge by a small simple closed curve on which no vertices are marked.

THEOREM 2.3. *Let G be a connected non-null graph having no monovalent vertex. Let its cyclomatic number be n. Let V be any topologically invariant V-function. Then*

$$V(G) = V(X_n) + P, \tag{8}$$

where P is a polynomial whose form does not depend on the choice of V, in those expressions $V(X_k)$ *for which* $1 \leq k < n$.

PROOF. We proceed by induction over the number e of edges of G. If $e = 0$ then $G = X_0$ and $V(G) = -1$, so that the theorem holds. Assume it true whenever e is less than some positive integer q, and consider the case $e = q$.

Suppose first that G has no link. Then $G = X_n$, unless $n = 1$ and G consists of a single loose edge. But in any case $V(G) = V(X_n)$, and so the theorem holds.

We may now suppose G to have a link A. Then

$$V(G) = V(G'_A) + V(G''_A), \tag{9}$$

by (1). We note that G''_A is connected, that it has cyclomatic number n, and that it satisfies the theorem by the inductive hypothesis. For G'_A there are several possibilities. However, in no case can its cyclomatic number exceed n.

First, G'_A may have a monovalent vertex. Then $V(G'_A) = 0$, by 2.1. It now follows from (9) that the theorem holds for G.

We may now assume that G'_A has no monovalent vertex. Suppose it connected. Then its cyclomatic number is $n - 1$, since G has a circuit through A. By the inductive hypothesis the theorem holds for G'_A, with n replaced by $n - 1$. Hence, by (9), the theorem holds for G.

We may now make the further assumption that G'_A is disconnected. It must be the union of two disjoint connected subgraphs H and K, these being joined by A in G. Each of H and K has a circuit, since otherwise G would have a monovalent vertex. Hence each has a non-zero cyclomatic number. But the sum of their cyclomatic numbers is that of G'_A, which cannot exceed n. Hence each of H and K has a cyclomatic number less than n. So by (2) and the inductive hypothesis $V(G'_A)$ can be expressed as a polynomial, with a form independent of V, in those expressions $V(X_k)$ for which $1 < k < n$. Hence G satisfies the theorem, by (9).

This completes the proof that the theorem holds whenever $e = q$. It follows in general by induction.

THEOREM 2.4. *For each positive integer k let there be given a connected graph* H_k *of cyclomatic number k and with no monovalent vertex. Then there exists a topologically invariant V-function V taking an arbitrarily assigned value for each* H_k. *Moreover V is uniquely determined.*

PROOF. We use Equation (8) with G replaced successively by H_1, H_2, H_3, and so on. Solving the resulting equations, in order of increasing k we find values for the $V(X_n)$, $n \geq 1$, which force the desired values on the $V(H_k)$. Moreover these values of the $V(X_n)$ are uniquely determined. The theorem now follows from 1.1 and 1.2.

3. CUBIC GRAPHS AND ϕ-FUNCTIONS

A graph is called "cubic" if the valency of each of its vertices is 3. We consider that this definition allows a cubic graph to have any number of loose edges. Indeed any graph without vertices but with one or more loose edges is counted as cubic.

We discuss a cubic graph G. For convenience in describing constructions we bear in mind also the graph Q obtained from G by subdividing each attached edge by a single new divalent vertex. We refer to the attached edges of Q as *half-edges* of G.

Let A be a link of G having ends x and y. Let x be incident with the half-edges α and β not in A. These may unite to form a loop of G on x, or they may be halves meeting x of two distinct links of G. Similarly let y be incident with the half-edges γ and δ. In the graph G'_A the vertices x and y are divalent, but we can get a new cubic graph G_A by suppressing them. G_A may have more loose edges than G (see Figure 4).

Consider now the graph G''_A. It has a tetravalent vertex w formed by the collapse of A, x and y, and the half-edges incident with w are α, β, γ and δ. To recover G we have to arrange these four half-edges in two pairs, introduce x as a vertex incident with the members of one pair and y as one incident with those of the other pair, and then bring in A as an edge joining x and y. Since there are three possible pairings of the half-edges we can in general get three cubic graphs G, H and K in this way. (See Figure 3.) As abstract graphs they are not necessarily all distinct. Each of G, H and K is said to be derived from each of the others by *twisting* A. We note that

$$G''_A = H''_A = K''_A. \tag{10}$$

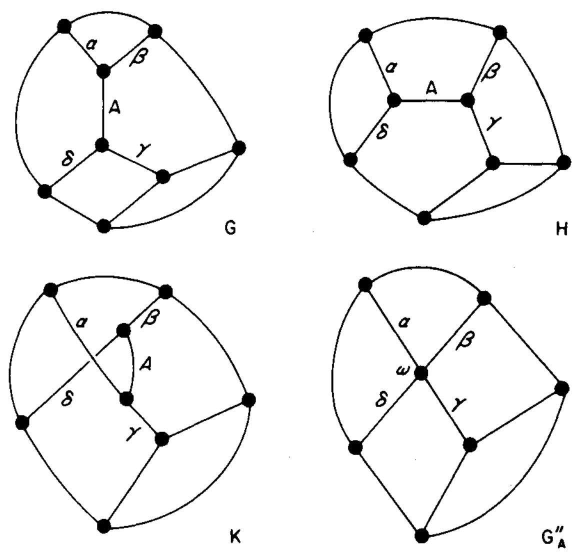

FIGURE 3

In [1] we define a "ϕ-function" of an abstract cubic graph. It is a function ϕ which is defined for all cubic graphs and which satisfies the two following rules.

THEOREM 3.1. *Let G and H be cubic graphs such that H is derived from G by twisting a link A. Then*

$$\phi(G) - \phi(G_A) = \phi(H) - \phi(H_A). \tag{11}$$

THEOREM 3.2. *Let the cubic graph G be the union of two disjoint subgraphs H and K. Then*

$$\phi(G) = \phi(H) \cdot \phi(K). \tag{12}$$

To these rules it is convenient to add that ϕ must take the value 1 for the null graph.

Figure 4 shows the graphs involved in Equation (11) in one very simple case

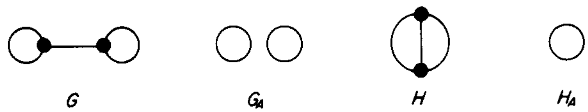

FIGURE 4

It is now convenient to describe a family of cubic graphs of specially simple structure. So for each non-negative integer m we define a cubic graph Y_m as follows. Y_0 consists of a single loose edge. For a positive m Y_m has exactly $2m$ vertices, conveniently enumerated as $v_1, v_2, v_3, \ldots, v_{2m}$. When the suffix j is odd v_j and v_{j+1} are joined by a single edge. When j is even and less than $2m$ the vertices v_j and v_{j+1} are joined by exactly two edges. Finally there is a loop on v_1 and another on v_{2m}. We observe that the cyclomatic number of Y_m is $m+1$. Some of the graphs Y_m are shown in Figure 5.

If the loop on v_1 is deleted from Y_m we obtain a *frond* of *order* m, with *root* v_1.

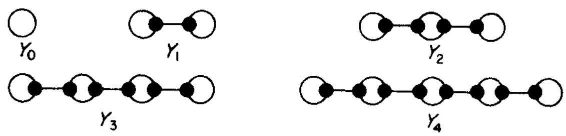

FIGURE 5

THEOREM 3.3. *Let G be a connected cubic graph of $2n$ vertices. Then by twisting links of G we can transform it into Y_n.*

PROOF. If $n=0$ then G is necessarily Y_0, and there is nothing to prove. In the remaining case G has a circuit C. If C is not a monogon we can transform it into one by twisting all but one of its edges. We thus transform G into a graph G_1 containing a frond of order 1.

Suppose that, by some sequence of twistings, we have transformed G into a graph H having a frond. Let F be a frond in G of highest order k. Let its root be r. Then H is the union of F with another subgraph F_1, these having only the vertex r in common. If F_1 consists of a single loop, then $H = Y_n$ and the theorem is verified. If F_1 contains a circuit through r, other than a monogon, we can twist edges of this circuit so as to transform H into a graph K with a frond of order $k+1$.

In the remaining case F_1 has no circuit through r. But it must have some circuit C, and we can find a shortest arc L in F_1 joining C to r. By twisting the edges of L we can transform to the case in which F_1 has a circuit through r, and we can go on to find K as before.

We repeat the process with K replacing H, and so on until it terminates with Y_n.

THEOREM 3.4. *A ϕ-function ϕ is uniquely determined when its value is known for each Y_n.*

PROOF. Suppose ϕ is known for each Y_n. We now have to show that it is determined for each cubic graph G. We proceed by induction over the number $2n(G)$ of vertices of G.

If $n(G)=0$ then $\phi(G)$ is determined as a power of $\phi(Y_0)$, by 3.2. Assume (G) determined whenever $n(G)$ is less than some positive integer q, and consider the case $n(G)=q$.

If G is connected then

$$\phi(G) = \phi(Y_q) + L,$$

where L is a linear form in expressions $\phi(H)$ such that $n(H) < q$, by 3.3 and the definition of a ϕ-function. So $\phi(G)$ is determined, by the inductive hypothesis.

If G is not connected then $\phi(G)$ is determined, by 3.2 and the inductive hypothesis, together with the result just proved if one component of G contains all the vertices. The step to $n(G) = q$ is now complete, and the theorem follows.

The next two theorems relate ϕ-functions to topologically invariant V-functions.

THEOREM 3.5. *Let V be any topologically invariant V-function. Then its restriction to cubic graphs is a ϕ-function.*

PROOF. Let G and H be any cubic graphs such that H is derived from G by twisting a link A. Then, by (1) and (10),

$$V(G) - V(G'_A) = V(G''_A) = V(H''_A) = V(H) - V(H'_A).$$

Hence, by topological invariance,

$$V(G) - V(G_A) = V(H) - V(H_A).$$

The restriction ϕ of V to cubic graphs thus satisfies (11). But it satisfies (12) by (2). It is therefore a ϕ-function.

THEOREM 3.6. *Let ϕ be any ϕ-function. Then there exists a topologically invariant V-function V such that the restriction of V to cubic graphs is ϕ. Moreover V is uniquely determined.*

PROOF. By 2.4 there exists a unique V-function V such that $V(Y_m) = \phi(Y_m)$ for each Y_m. Its restriction to cubic graphs is a ϕ-function, by 3.5. This ϕ-function is identical with ϕ, by 3.4.

4. A ϕ-FUNCTION ASSOCIATED WITH l-FACTORS

We recall that a 1-*factor* of a graph G is a spanning subgraph of G in which each vertex is monovalent. We denote the number of 1-factors in a cubic graph G by $f(G)$.

Evidently the null graph has just one 1-factor. Moreover any 1-factor of a graph G remains a 1-factor when we adjoin to or delete from it any loose edge of G. For the cubic graph consisting of k loose edges and no vertex we therefore write

$$f(G) = 2^k. \tag{13}$$

We note also that

$$f(Y_m) = 1 \quad \text{if } m \geq 1. \tag{14}$$

THEOREM 4.1. *Let G and H be cubic graphs such that H is derived from G by twisting a link A. Then*

$$f(G) + f(G_A) = f(H) + f(H_A). \tag{15}$$

For an example of this identity we take the graphs of Figure 4. Evidently $f(G) = 1$ and $f(H) = 3$, whereas $f(G_A) = 4$ and $f(H_A) = 2$, by (13).

PROOF OF 4.1. We return to the notation of Figure 3. Figure 6 shows how the four half-edges α, β, γ and δ relate to one another in G, G_A, H and H_A.

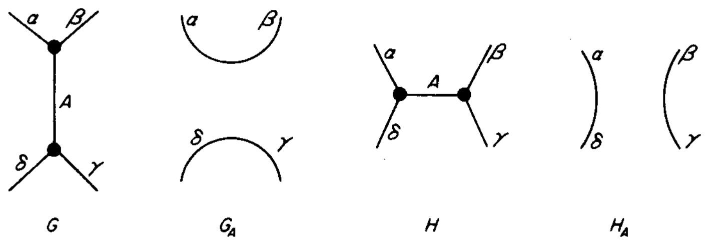

FIGURE 6

Let us now write S for the set of all edges of G, other than A, which do not contain α, β, γ or δ. Let T be a subset of S, and U a subset of $\{\alpha, \beta, \gamma, \delta\}$. We study those 1-factors of G, G_A, H and H_A which meet S in T and $\{\alpha, \beta, \gamma, \delta\}$ in U. We say that these 1-factors are of Type (T, U). We readily verify that for given T and U each of the four graphs has at most one 1-factor of Type (T, U).

Consider first the case in which U is null. Then if any of the four graphs has a 1-factor of Type (T, U) so does each of the others. This 1-factor would contain A in the cases of G and H. Hence the 1-factors of Type (T, U), with U null, balance on the two sides of (15).

We have to verify such a balance for each of the subsets U. If U has 1 or 3 members this is trivial. No corresponding 1-factor exists in any of the four graphs.

Suppose next that $U = \{\alpha, \beta\}$. No corresponding 1-factor is possible in G or H_A. But if one of G_A and H has a 1-factor of Type (T, U) then so does the other. Again we have balance in (15). Balance in the cases $U = \{\beta, \gamma\}$, $U = \{\gamma, \delta\}$, $U = \{\delta, \alpha\}$, follows by symmetry.

Suppose next that $U = \{\alpha, \gamma\}$. There can be no corresponding 1-factor in G_A or H_A. But if one of G and H has a 1-factor of Type (T, U) then so does the other. Balance is again achieved. Balance in the case $U = \{\beta, \delta\}$ follows by symmetry.

It remains only to consider the case $U = \{\alpha, \beta, \gamma, \delta\}$. Then 1-factors of Type (T, U) can exist only for G_A and H_A, and if one of these two graphs has such a 1-factor, then so does the other. This completes the verification of Equation (15).

THEOREM 4.2. *Let a cubic graph G be the union of two disjoint subgraphs H and K. Then $f(G) = f(H) \cdot f(K)$.*

This result is obvious. It is also clear that we can combine 4.1 and 4.2 into the following statement.

THEOREM 4.3. *Let the number of vertices of the cubic graph G be denoted by $2n(G)$. Then*

$$(-1)^{n(G)} f(G)$$

is a ϕ-function.

A more general ϕ-function is defined in [1]. It is written in the form

$$(-1)^{n(G)} D(G, x),$$

where $D(G, x)$ is a polynomial in a variable x, and is defined by the following equation:

$$D(G, x) = \sum_k \pi_k(G) x^k. \tag{16}$$

Here $\pi_k(G)$ denotes the number of spanning subgraphs of G which have no vertex of zero valency and exactly $2k$ vertices of odd valency. Equivalently we can use complementary spanning subgraphs, saying that $\pi_k(G)$ is the number of spanning subgraphs of G with no vertex of valency 3 and with exactly $2k$ vertices of even valency. We note that $\pi_0(G)$ is $f(G)$. It must be left to the reader to verify that $\pi_k(G)$ satisfies an analogue of (15), by an argument similar to the proof of 4.1.

5. The Extension of f

The ϕ-function of 4.3 extends as a topologically invariant V-function V to all finite graphs, by 3.6. We can extend f to all these graphs by writing

$$f(G)=(-1)^{e+v}V(G),$$

where G has e edges and v vertices. The function f, thus extended has the property (2), but in place of (1) it satisfies the identity

$$f(G''_A)=f(G)+f(G'_A) \tag{17}$$

for each link A of G. It also inherits from V the property of topological invariance.

Theorem 5.1. *$f(G)$ is a non-negative integer for each graph G.*

Proof. By repeated use of (17), with G''_A initially replaced by G, we can write $f(G)$ as a sum of expressions $f(H)$, where H has no vertex whose valency exceeds 3. By 2.1 and 2.2 we can arrange that each H is cubic. But then each $f(H)$ is a non-zero integer, being the number of 1-factors of H. The theorem follows.

Analogously we can show that $D(G, x)$ extends to all graphs G, and that for each G it is a polynomial in x with no negative coefficients.

We proceed to evaluate $f(G)$ for a few simple non-cubic graphs. Let $A(k, m)$ be the graph obtained from a frond of order m by adjoining k loops on the root r. Thus $A(1, m)=Y_m$. Let $B(k, m)$ be the graph obtained from $A(k, m)$ by contracting the link incident with r. Thus

$$B(k, 1)=X_{k+1}. \tag{18}$$

Other examples are shown in Figure 7.

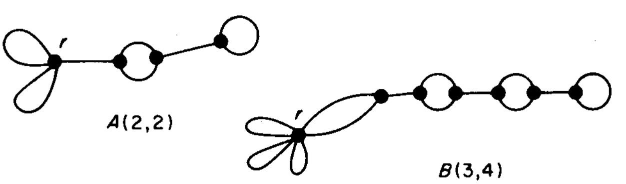

Figure 7

By applying (17) to a link A incident with r, first in $A(k, m)$ and then in $B(k, m)$, we obtain the recursion formulae (19) and (20). In these and some following equations we adopt the convention of writing G for $f(G)$.

$$B(k, m)=A(k, m)+X_k\cdot Y_{m-1}\quad (k\geqslant 0, m\geqslant 1). \tag{19}$$

$$A(k+1, m-1)=B(k, m)+A(k, m-1),\quad (k\geqslant 0, m\geqslant 2). \tag{20}$$

Now we know that $f(X_0)=-1$, $f(A(0,m))=0$ by topological invariance and $f(B(0,m))=f(Y_{m-1})$. Moreover $f(Y_0)=2$ and $f(Y_m)=1$ if $m>0$, by counting 1-factors in the cubic graphs concerned. But if f is known for X_k, $A(k,m)$ and $B(k,m)$, for a given k and all positive m, then it can be found for X_{k+1}, $A(k+1,m)$ and $B(k+1,m)$ by the use of Equations (18), (19) and (20). We can indeed establish the following three formulae by a simple induction:

$$f(A(k,m))=\tfrac{1}{2}(3^k-1), \tag{21}$$

$$f(B(k,m))=3^k \tag{22}$$

and

$$f(X_k)=\tfrac{1}{2}(3^k+1). \tag{23}$$

Let us now define $C(k,m)$ as consisting fo two vertices x and y, k loops on x, and $m\geqslant 1$ links joining x and y. Then

$$C(k,1)=0 \tag{24}$$

by 2.1. For $m>1$ we can use (17) to obtain the following recursion formula.

$$X_{k+m-1}=C(k,m)+C(k,m-1). \tag{25}$$

This enables us to prove inductively that

$$f(C(k,m))=\frac{3^{k+1}(3^{m-1}+(-1)^m)}{8}+\tfrac{1}{4}(1+(-1)^m). \tag{26}$$

In particular

$$f(C(k,3))=3^{k+1}. \tag{27}$$

Now let $D(k,m)$, where $m\geqslant 1$, be the graph derived from an arc L of length m, and one extra vertex x, by the following construction. We join x to each end of L by exactly two links, and to each internal vertex of L by exactly one link. We then attach k loops to x. For examples, see Figure 8.

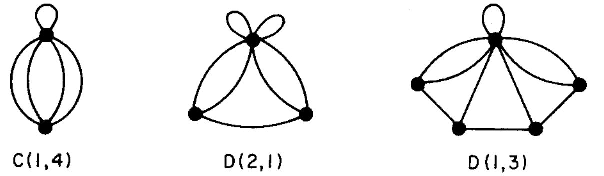

FIGURE 8

From (17), taking A to be one of the links joining x to an end-vertex of L, we find that

$$D(k,1)=C(k+1,3)-C(k,3). \tag{28}$$

Hence, by (27),

$$f(D(k,1))=2\cdot 3^{k+1}. \tag{29}$$

For $m>1$ we find similarly that

$$D(k,m)=D(k+1,m-1)-D(k,m-1). \tag{30}$$

Hence, by induction,

$$f(D(k,m))=2^m\cdot 3^{k+1}. \tag{31}$$

The *wheel* W_m, $(m>1)$, is obtained from a circuit C_m of length m by adjoining a new vertex x called the *hub*, and then joining x to each vertex of C_m by a single new link called a *spoke*. (See Figure 9.)

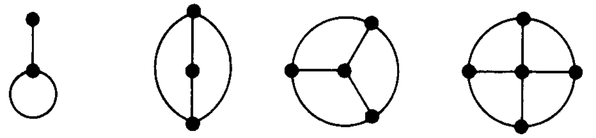

FIGURE 9

Let $W(k, m)$ be the graph derived from W_m by adjoining k loops on the hub. We observe that

$$W(k, 1) = A(k, 1) \quad \text{and} \quad W(k, 2) = C(k, 3) - W(k, 1).$$

Hence $f(W(k, 1)) = \frac{1}{2}(3^k - 1)$ and $f(W(k, 2)) = \frac{1}{2}(5 \,.\, 3^k + 1)$, by (21) and (27). For $m > 2$ we find, using (17) and taking A to be a spoke, that

$$W(k, m) = D(k, m-2) - W(k, m-1). \tag{32}$$

Hence, by induction,

$$f(W(k, m)) = \tfrac{1}{2}\{(2^m + (-1)^m)3^k + (-1)^m\}, \tag{33}$$

$$f(W_m) = 2^{m-1} + (-1)^m. \tag{34}$$

From these and other examples we may hope to induce a graph-theoretical interpretation of $f(G)$ in the general case.

6. A POSTSCRIPT

After studying the examples I did indeed find the graph-theoretical interpretation of $f(G)$ called for above. It runs as follows.

THEOREM 6.1. *Let G be any graph. Let W be the set of a subgraphs S of G such that* (*i*) *each vertex of S has non-zero even valency in S and* (*ii*) *each vertex with odd valency in G is a vertex of S. Let $a(S)$ denote the number of attached edges of S, and $b(S)$ the number of vertices of S. Then*

$$f(G) = \sum_{S \in W} 2^{a(S) - b(S)}. \tag{35}$$

PROOF. We easily verify that when G is cubic the sum on the right is the number of 2-factors, that is the number of 1-factors, of G. It remains only to show that $f(G)$, as defined by (35), satisfies (2) and (17). The first requirement is trivial; the second is a simple exercise in graph theory best left to the reader.

The question of the graph-theoretical interpretation of the extension of $D(G, x)$ is still open.

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