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Soft Set Theory—First Results

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Abstract—The soft set theory offers a general mathematical tool for dealing with uncertain, fuzzy, not clearly defined objects. The main purpose of this paper is to introduce the basic notions of the theory of soft sets, to present the first results of the theory, and to discuss some problems of the future.

Keywords—Soft set, Soft function, Soft game, Soft equilibrium, Soft integral, Internal regularization.

1. INTRODUCTION

To solve complicated problems in economics, engineering, and environment, we cannot successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties.

Theory of probabilities can deal only with stochastically stable phenomena. Without going into mathematical details, we can say, e.g., that for a stochastically stable phenomenon there should exist a limit of the sample mean μ_n in a long series of trials. The sample mean μ_n is defined by

$$\mu_n = \frac{1}{n}\sum_{i=1}^{n} x_i,$$

where x_i is equal to 1 if the phenomenon occurs in the trial, and x_i is equal to 0 if the phenomenon does not occur. To test the existence of the limit, we must perform a large number of trials. We can do it in engineering, but we cannot do it in many economic, environmental, or social problems.

Interval mathematics have arisen as a method of taking into account the errors of calculations by constructing an interval estimate for the exact solution of a problem. This is useful in many cases, but the methods of interval mathematics are not sufficiently adaptable for problems with different uncertainties. They cannot appropriately describe a smooth changing of information, unreliable, not adequate, and defective information, partially contradicting aims, and so on.

The most appropriate theory, for dealing with uncertainties is the theory of fuzzy sets developed by Zadeh [1]. We recall the definition of the notion of *fuzzy* set.

For every set $A \subset X$, define its indicator function μ_A

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

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This correspondence between a set and its indicator function is obviously one-to-one correspondence.

A fuzzy set F is described by its membership function μ_F. To every point $x \in X$, this function associates a real number $\mu_F(x)$ in the interval $[0,1]$. The number $\mu_F(x)$ is interpreted for the point as a degree of belonging x to the fuzzy set F.

The theory of fuzzy sets offers natural, from the first glance, operations for fuzzy sets. Let F and G be fuzzy sets, and μ_F, μ_G be their membership functions. Then, the complement CF is defined by its membership function

$$\mu_{CF}(x) = 1 - \mu_F(x).$$

The intersection $F \cap G$ can be defined by one of the following membership functions

$$\mu_{F\cap G}(x) = \min\{\mu_F(x), \mu_G(x)\},$$
$$\mu_{F\cap G}(x) = \mu_F(x) \cdot \mu_G(x),$$
$$\mu_{F\cap G}(x) = \max\{0, \mu_F(x) + \mu_G(x) - 1\}.$$

There are three possibilities of membership functions for the union $F \cup G$

$$\mu_{F\cup G}(x) = \max\{\mu_F(x), \mu_G(x)\},$$
$$\mu_{F\cup G}(x) = \mu_F(x) + \mu_G(x) - \mu_F(x) \cdot \mu_G(x),$$
$$\mu_{F\cup G}(x) = \min\{1, \mu_F(x), \mu_G(x)\}.$$

At the present time, the theory of fuzzy sets is progressing rapidly. But there exists a difficulty: how to set the membership function in each particular case.

We should not impose only one way to set the membership function. The nature of the membership function is extremely individual. Everyone may understand the notation $\mu_F(x) = 0.7$ in his own manner. So, the fuzzy set operations based on the arithmetic operations with membership functions do not look natural. It may occur that these operations are similar to the addition of weights and lengths.

The reason for these difficulties is, possibly, the inadequacy of the parametrization tool of the theory. In the next section, we propose a mathematical tool for dealing with uncertainties which is free of the difficulties mentioned above.

2. MAIN NOTIONS OF SOFT SET THEORY

2.1. Definition of the Soft Set

To avoid difficulties, one must use an adequate parametrization. Let U be an initial universe set and let E be a set of parameters.

DEFINITION 2.1. *A pair (F, E) is called a soft set (over U) if and only if F is a mapping of E into the set of all subsets of the set U.*

In other words, the soft set is a parametrized family of subsets of the set U. Every set $F(\varepsilon)$, $\varepsilon \in E$, from this family may be considered as the set of ε-elements of the soft set (F, E), or as the set of ε-approximate elements of the soft set.

As an illustration, let us consider the following examples.

(1) A soft set (F, E) describes the attractiveness of the houses which *Mr. X* is going to buy.
  U – is the set of houses under consideration.
  E – is the set of parameters. Each parameter is a word or a sentence.
  $E = \{$*expensive; beautiful; wooden; cheap; in the green surroundings; modern; in good repair; in bad repair*$\}$.

In this case, to define a soft set means to point out *expensive* houses, *beautiful* houses, and so on.

It is worth noting that the sets $F(\varepsilon)$ may be arbitrary. Some of them may be empty, some may have nonempty intersection.

(2) Zadeh's fuzzy set may be considered as a special case of the soft set. Let A be a fuzzy set, and μ_A be the membership function of the fuzzy set A, that is μ_A is a mapping of U into $[0, 1]$.

Let us consider the family of α-level sets for function μ_A

$$F(\alpha) = \{x \in U \mid \mu_A(x) \geq \alpha\}, \qquad \alpha \in [0, 1].$$

If we know the family F, we can find the functions $\mu_A(x)$ by means of the following formulae:

$$\mu_A(x) = \sup_{\substack{\alpha \in [0,1] \\ x \in F(\alpha)}} \alpha.$$

Thus, every Zadeh's fuzzy set A may be considered as the soft set $(F, [0, 1])$.

(3) Let (X, τ) – be a topological space, that is, X is a set and τ is a topology, in other words, τ is a family of subsets of X, called the open sets of X.

Then, the family of open neighborhoods $T(x)$ of point x, where $T(x) = \{V \in \tau \mid x \in V\}$, may be considered as the soft set $(T(x), \tau)$.

The way of setting (or describing) any object in the soft set theory principally differs from the way in which we use classical mathematics.

In classical mathematics, we construct a mathematical model of an object and define the notion of the exact solution of this model. Usually the mathematical model is too complicated and we cannot find the exact solution. So, in the second step, we introduce the notion of approximate solution and calculate that solution.

In the soft set theory, we have the opposite approach to this problem. The initial description of the object has an approximate nature, and we do not need to introduce the notion of exact solution.

The absence of any restrictions on the approximate description in soft set theory makes this theory very convenient and easily applicable in practice. We can use any parametrization we prefer: with the help of words and sentences, real numbers, functions, mappings, and so on.

It means that the problem of setting the membership function or any similar problem does not arise in the soft set theory.

2.2. Operations with Soft Sets

Assume that we have a binary operation, denoted by $*$, for subsets of the set U. Let (F, A) and (G, B) be soft sets over U. Then, the operation $*$ for soft sets is defined in the following way:

$$(F, A) * (G, B) = (H, A \times B),$$

where $H(\alpha, \beta) = F(\alpha) * G(\beta)$, $\alpha \in A$, $\beta \in B$, and $A \times B$ is the Cartesian product of the sets A and B.

This definition takes into account the individual nature of any soft set.

If we produce a lot of operations with soft sets, the result will be a soft set with a very wide set of parameters. Sometimes such expansion of the set of parameters may be useful. So, the intersection of the soft set from Example 1 with itself gives the soft set with more detailed description. The resulting soft set points out the houses which are *expensive* and *beautiful*, *modern* and *cheap*, and so on.

In cases when such expansion of the set of parameters is not convenient, we may use a lot of cutting operations. Of course, the acceptance of these cutting operations depends on the special case and on the problem under consideration.

If we want to construct a general mathematical tool, we do not introduce a universal cutting operation for the set of parameters. If we look at the operations with fuzzy sets from the point of view of the soft set theory, we will figure out that all binary operations with fuzzy sets include the universal cutting operation. Let us consider, for example, the first version of the intersection of two fuzzy sets A and B,

$$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}.$$

To three fuzzy sets A, B, and $A \cap B$, three soft sets correspond to $(F_A, [0,1])$, $(F_B, [0,1])$, $(F_{A\cap B}, [0,1])$, where

$$\begin{aligned} F_A(\alpha) &= \{x \in U \mid \mu_A(x) \geq \alpha\}, && \alpha \in [0,1], \\ F_B(\alpha) &= \{x \in U \mid \mu_B(x) \geq \alpha\}, && \alpha \in [0,1], \\ F_{A\cap B}(\alpha) &= \{x \in U \mid \mu_A(x) \geq \alpha,\ \mu_B(x) \geq \alpha\}, && \alpha \in [0,1]. \end{aligned}$$

The intersection of soft sets $(F_A, [0,1])$ and $(F_B, [0,1])$ is denoted by $(H, [0,1]) \times [0,1]$. Then, we have

$$H(\alpha,\beta) = F_A(\alpha) \cap F_B(\beta) = \{x \in U \mid \mu_A(x) \geq \alpha,\ \mu_B(x) \geq \beta\}.$$

Comparing $H(\alpha,\beta)$ and $F_{A\cap B}(\alpha)$, we can see that, in this case, the cutting operation means the changing of the Cartesian product $[0,1] \times [0,1]$ to its diagonal.

The individual nature of a fuzzy set contradicts to the universal cutting operation. It causes many difficulties in application areas of the theory.

3. SOME APPLICATIONS OF THE SOFT SET THEORY

3.1. Stability and Regularization

Let (M, ρ) be a metric space, where ρ is a metric. We shall call space (M, ρ) a model space. Let U be a set and for every $m \in M$, we have a soft set $(F(m), E)$ over U. Such a pair (F, E) is said to be an s-function (soft function) and we shall use the notation $(F, E) : M \to U$.

Now, we need to introduce the notion of "smoothness" for s-functions which is similar to "continuity" in the classical case. We understand smoothness as the proximity of two soft function values under the condition that their models are close, too.

To give a formal definition, we have to make more exact the notions of proximity for models m, n and for s-function values $F(m,\alpha)$ and $F(n,\beta)$. To measure model proximity, we shall use the metric ρ, and for s-function values we shall define the closeness as inclusion for sets $F(m,\alpha)$ and $F(n,\beta)$.

DEFINITION 3.1.1. *The s-function (F, E) is said to be internally smooth from above on the pair (m,α) if and only if there exist such parameter $\beta \in E$ and positive number δ that for every model $n \in M$, for which $\rho(m,n) \leq \delta$, the following inclusion holds:*

$$\emptyset \neq F(n,\beta) \subset F(m,\alpha).$$

DEFINITION 3.1.2. *The s-function (F, E) is said to be internally smooth from below on the pair (m,α) if and only if there exist a parameter $\beta \in E$ and a positive number δ such that for every model $n \in M$, for which $\rho(m,n) \leq \delta$, the following inclusion holds:*

$$\emptyset \neq F(m,\beta) \subset F(n,\alpha).$$

We say that s-function (F, E) is internally smooth from above (from below) on the subset K of the set $M \times E$ if for every pair $(m, \alpha) \in K$, the s-function (F, E) is internally smooth from above (from below) on the pair (m, α).

The words "from above" and "from below" in the Definitions 3.1.1 and 3.1.2 express the connection with the classical notions of continuity from above and continuity from below for point-set mapping. To prove it, let us consider the s-function (G, P),

$$(G, P) : M \to T,$$

where (M, ρ), (T, τ) are metric spaces, P is the set of all positive numbers, and G has the following form:

$$G(m, \alpha) = \{t \in T \mid r(t, g(m)) \leq \alpha\},$$

and g is a point-set mapping from M into the family of all subsets of the set T, $\alpha \in P$.

Let (G, P) be an internally smooth from above s-function on the set $m \times P$. What does it mean for the point-set mapping g?

It is clear from Definition 3.1.1, for every positive number α, there exists a positive number δ, such that for every model $n \in M$, such that $r(m, n) < \delta$, the following inclusion holds:

$$g(n) \subset \{t \in T \mid r(t, g(m)) \leq \alpha\}.$$

It simply means that mapping g is semicontinuous from above on the model m.

Now, let (G, P) be an s-function internally smooth from below on the set $m \times P$. From Definition 3.1.2, it follows that for every positive number α, there exists a positive number δ such that for every model $n \in M$, such that $r(m, n) < \delta$, the following inclusion,

$$g(m) \subset \{t \in T \mid r(t, g(n)) \leq \alpha\},$$

is valid. It simply means that mapping g is semicontinuous from below on the model m.

Both types of smoothness for s-functions can be applied in practice. At first, let us consider the possible application of smoothness from above.

Assume that the s-function (F, E) is internally smooth from above on the pair (m, α) and we want to find a point from the set $F(m, \alpha)$.

Often, to solve this problem, we use an approximation of the model m. It may appear in discretization, computer calculation errors, and so on. So, instead of the model m, our computer will deal with a close model n and we can control only the level of proximity δ, $\delta \geq \rho(m, n)$, but not the model n.

But, due to the smoothness of the s-function (F, E), we can solve the initial problem, in spite of perturbations of the model m. To do it, we have to change the parameter α in the initial setting of the problem to the appropriate parameter β.

Then, the computer will find a point x from the set $F(n, \alpha)$. If a model n is sufficiently close to the model m, then the smoothness from above guarantees that x is a solution of the initial problem, i.e., $x \in F(m, \alpha)$.

Now, assume that the s-function (F, E) is internally smooth from below on the pair (m, α). The initial problem is to find a point from the set $F(n, \alpha)$ under the condition that n is an uncertain factor. We know only the model m and the fact that $\rho(n, m) \leq \delta$.

This is a typical case from real life where almost all measurements have approximate character. Let us consider the auxiliary problem: find a point from the set $F(m, \beta)$, where β and δ satisfy Definition 3.1.2. This is a problem without any uncertain factors. It is easy to show that every solution of the auxiliary problem is a solution of the initial one. So, the notion of internal smoothness from below serves in solving problems with uncertain factors.

We gave new definitions of smoothness (an analogy of continuity) but it is natural to ask: why introduce these new definitions? Why can't we apply the classical notions of continuity to s-functions?

Avoiding mathematical details, we may say that the family of sets which appears when we define the soft set plays the role of topology in the smoothness notions. From this point of view, the essential difference between soft set theory and classical theories consists of the fact that in the classical theory, every notion of continuity has only one topology for all functions, but in the soft set theory every function has its own topology.

A topology from the point of view of soft set theory defines the structure of the soft set and if we use only one topology, we must consider soft sets with only one structure.

We do not need such limitation in our theory. So, the classical notion of continuity is not appropriate for this theory.

The new notions of smoothness are very effective tools in the "struggle" with the instability of ill-posed problems. Often, when we deal with an unstable problem, we can choose a natural s-function for setting the problem and it makes the problem a stable one.

Let us consider one example. Let W be a closed bounded subset of the n-dimensional Euclidean space $\mathbb{E}^n$.

The Pareto set $\Pi(W)$ of the set W is the set of all points from W which cannot be dominated by any point from W. The domination is understood in the following way: a vector $u = (u_1, \dots, u_n)$ dominates vector $v = (v_1, \dots, v_n)$, if and only if $u_i \geq v_i$ for every i, and there exists an index j such that $u_j > v_j$

It is well known that arbitrarily small variations of the set W may cause big changes in the Pareto set. There are simple examples of a set W such that there exists a neighborhood $O(\Pi(W))$ of the Pareto set $\Pi(W)$ such that for every neighborhood $U(W)$ of the set W, there exists a set $W' \subset U(W)$, such that $\Pi(W') \setminus O(\Pi(W)) \neq \emptyset$. It means that the mapping $\Pi(W)$ is not upper semicontinuous. Often, in practice, we can find only a set W' which approximates the set W, but $\Pi(W')$ is not close to $\Pi(W)$.

Let us introduce a new notion of domination which can be considered as a variation of the initial one. Let α and β be positive numbers. We say that vector $u = (u_1, \dots, u_n)$ (α, β)-dominates vector $v = (v_1, \dots, v_n)$, if and only if $u_i \geq v_i - \alpha$ for every i, and there exists an index j such that $u_j > v_j + \beta$.

The (α, β)-Pareto set $\Pi(W, \alpha, \beta)$ of the set W is the set of all points from W which cannot be (α, β)-dominated by any point from W. The s-function (Π, P^2), where P is a set of positive numbers, will be called soft Pareto function. In [2], it was shown that soft Pareto function is internally smooth from below and from above under very weak assumptions.

In [2], natural s-functions were introduced describing the notion of solution for linear and nonlinear optimization, for multi max-min problems, and for multicriteria optimization. All these s-functions are also internally smooth from below and from above under very weak assumptions. In addition, these s-functions approximate classical solutions for problems mentioned above. There were also introduced new notions of smoothness for soft functions. Some of them can be considered as analogues of the classical notions of continuity.

When the problem under consideration is not continuous (or not smooth) and any modification of the solution notion is not allowed, there arises the question: how do we solve these problems with the help of approximate numerical methods?

The notions of regularization and approximation for s-function are very useful in this case. There are formal definitions for interval approximation and regularization. Let (F, A) and (G, B) be soft functions,

$$(F, A), (G, B) : M \to U,$$

and H be a subset of the Cartesian product of the sets M, B.

DEFINITION 3.1.3. *The s-function (F, A) internally approximates the s-function (G, B) on the set H if and only if for every pair $(m, \beta) \in H$ there exists parameter $\alpha \in A$ such that the inclusion*

$$\emptyset \neq F(m, \alpha) \subset G(m, \beta),$$

is valid.

DEFINITION 3.1.4. *The s-function (F, E) is said to be internal regularization from above for the s-function (G, B) on the set H if and only if for every pair $(m, \beta) \in H$, there exist parameter $\alpha \in E$ and positive number δ for which for every model $n \in M$ such that $\rho(m, n) \leq \delta$, the inclusion*

$$\emptyset \neq F(n, \alpha) \subset G(m, \beta)$$

is valid.

DEFINITION 3.1.5. *The s-function (F, E) is said to be internal regularization from below for the s-function (G, B) on the set H if and only if for every pair $(m, \beta) \in H$, there exist parameter $\alpha \in E$ and positive number δ such that for every model $n \in M$ for which $\rho(m, n) \leq \delta$, the inclusion*

$$\emptyset \neq F(m, \alpha) \subset G(n, \beta)$$

is valid.

The notions of external approximation and regularization have similar structure but they are based on the inverse inclusion

$$F(m, \alpha) \supset G(m, \beta) \neq \emptyset.$$

If we know the s-function (F, A) which is internal regularization from above for the s-function (G, B) on the pair (m, β), it helps to find a point of the set $G(m, \beta)$, in spite of calculation errors.

In the theory of ill-posed problems [3], we do not have any universal method of constructing a regularization. There are a lot of methods which present a regularization for particular classes of problems.

In the soft set theory, we have a simple universal method for constructing a regularization of s-function (F, A). To construct the internal regularization of any type, we have to find a soft function which is the internal approximation of the s-function $(\operatorname{int} F, A \times \mathbb{E}_+)$. To construct the external regularization of any type for s-function (F, A), we have to find a soft function which is the external approximation of the s-function $(\operatorname{ext} F, A \times \mathbb{E}_+)$.

The definitions of the s-function $(\operatorname{int} F, A \times \mathbb{E}_+)$ and s-function $(\operatorname{ext} F, A \times \mathbb{E}_+)$ are very clear,

$$\operatorname{int} F(m, \alpha, \delta) = \bigcap_{\substack{n \in M \\ \rho(m,n) < \delta}} F(n, \alpha),$$

$$\operatorname{ext} F(m, \alpha, \delta) = \bigcup_{\substack{n \in M \\ \rho(m,n) < \delta}} F(n, \alpha),$$

here $\mathbb{E}_+$ is a set of positive real numbers.

It is significant that we cannot improve the regularizing s-functions $(\operatorname{int} F, A \times \mathbb{E}_+)$ and $(\operatorname{ext} F, A \times \mathbb{E}_+)$. It means that the general problem of regularization in the soft set theory is closed. Furthermore, the regularizing s-functions are constructed for particular classes of problems: linear and nonlinear optimization, multi max-min problems, multicriteria optimization, and the computation of the derivative and integral.

In particular, the soft Pareto function which was mentioned above is the internal and external regularization for the classical Pareto set.

3.2. Game Theory and Operations Research

At present, the generally accepted model of human behaviour in game theory and operations research is described as aspiration for maximization of the pay function. Usually, the construction of the pay function in practice is a difficult problem. It is much easier to describe a human behaviour directly showing the set of strategies which a person may choose in a particular situation. A mapping which associates a set of such strategies with a given situation is called choice function. At first glance, the notion of choice function seems promising and useful for applications in cases with many players and with various uncertain parameters. In fact, it is not so. The choice function cannot give appropriate description of various compromises and concessions which are typical for such problems. The soft set theory gives an opportunity to construct a new mathematical tool which keeps all good sides of choice function and eliminates its drawbacks.

So, we will describe the person's behaviour with the help of the s-function which for any set of strategies indicates the set of ε-optimal choices. Now, the principal question is: which s-function has to be chosen for describing the person's behaviour under uncertainty?

We will present some approaches to this problem for different types of uncertainties. At first, we consider a formal description of the game with s-function modelling the person's behaviour.

Let us introduce some notations:

n	is the number of players,
S_i	is a set of strategies of player i,
E_i	is a set of parameters of player i,
$S = S_1 \times \cdots \times S_n$	is a set of situations,
$\mathcal{M}(S)$	is a set of all subsets of the set S,
$(F_i, E_i) : \mathcal{M}(S) \to S, (F_i, E_i)$	is a soft choice function of player i.

If $P \subset S$, that is, P is a subset of admissible strategies, and ε is a parameter, $\varepsilon \in E_i$, then $F_i(P, \varepsilon)$ is a set of ε-optimal situations for player i.

We will call such a game a soft game in a normal form with the following notation:

$$\langle (F_i, E_i), S_i,\ i = 1, \dots, n \rangle.$$

For a soft game given in normal form, the analogue of the Nash equilibrium is the following construction.

DEFINITION 3.2.1. *Situation $s \in S$ is called a situation of soft ε-equilibrium, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$, $\varepsilon_i \in E_i$, if and only if*

$$s \in F_i(s_1 \times \cdots \times s_{i-1} \times S_i \times s_{i+1} \times \cdots \times s_n, \varepsilon_i),$$

for every $i = 1, \dots, n$.

We denote the set of all situations of soft ε-equilibrium by $N(\varepsilon)$. Then, it is natural to call the soft set

$$(N, E_1 \times \cdots \times E_n),$$

a soft equilibrium.

Now, we will suggest the soft guarantee concept.

Suppose, we have only one player and

X	is the set of strategies of the player,
E	is the set of parameters of the player,
S	is the set of situations,
$(F, E) : \mathcal{M}(S) \to S$	is a soft choice function of the player.

Let π be a point-set mapping

$$\pi : X \to \mathcal{M}(S).$$

The player knows that if he chooses the strategy $x \in X$, then he will get one of the situations $s,\ s \in \pi(x)$.

We suggest to consider the soft set $(\mathrm{Gar}\, F, E)$, where

$$\mathrm{Gar}\, F(X,\varepsilon) = \{x \in X \mid \pi(x) \subset F(\pi(x),\varepsilon)\}, \qquad \pi(X) = \bigcup_{x\in X} \pi(x),$$

as a soft guarantee concept.

Let us apply the soft sets to the notion of Stackelberg solution. Suppose we have a soft game of two players in normal form

$$\langle (F_i, E_i), S_i,\ i = 1,2\rangle.$$

Player 1 makes the first move. He chooses a strategy $s_1 \in S_1$ and informs Player 2 about his choice. Since Player 2 knows the strategy s_1, he chooses his strategy $s_2 \in S_2$ so that $(s_1, s_2) \in F_2(s_1 \times S_2, \varepsilon_2)$. We assume that Player 1 knows the value of the parameter $\varepsilon_2 \in E_2$.

Player 1 considers uncertain the possible choices of Player 2. Applying the guarantee concept, described above, we come to the soft Stackelberg set $(St_1, E_1 \times E_2)$ for Player 1, where

$$St_1(S_1 \times S_2, \varepsilon_1, \varepsilon_2) = \{s_1 \in S_1 \mid F_2(s_1 \times S_2, \varepsilon_2) \subset F_1(S_1 \times S_2, \varepsilon_1)\}.$$

In [4,5], more details can be found and a number of examples for other games.

3.3. Soft Analysis

As it was mentioned above, a soft set is the collection of approximate descriptions of an object. The exact description is not necessary. If we want to keep this spirit of approximate descriptions, we should not base the soft set analysis on the classical notion of the limit.

We suggest the notion of "soft limit" for real function, it is based on the following treatment: a number A is a soft limit of the function f at a point a, if from the fact that x is close to a, it follows that $f(x)$ is close to A. To give a formal definition, we have to define exactly the notion of proximity. We will assume that for every point $x \in \mathbb{E}$, we have the set $\tau(x) \subset \mathbb{E}$ which is defined as a set of τ-close points to the point x. Let $\alpha, \beta, \varepsilon$ be positive numbers, too.

DEFINITION 3.3.1. *The upper (ε, τ)-softlimit of function f at a point x is the following set:*

$$\overline{\mathrm{Softlimit}}\, [f, \varepsilon, \tau](x) = \{v \in \mathbb{E} \mid f(y) \le v + \varepsilon,\ \forall\, y \in \tau(x)\}.$$

The lower (ε, τ)-softlimit of function f at a point x is the following set:

$$\underline{\mathrm{Softlimit}}\, [f, \varepsilon, \tau](x) = \{v \in \mathbb{E} \mid f(y) \ge v - \varepsilon,\ \forall\, y \in \tau(x)\}.$$

The set

$$\mathrm{Softlimit}\, [f, \alpha, \beta, \tau](x) = \{v \in \mathbb{E} \mid v - \alpha \le f(y) \le v + \beta,\ \forall\, y \in \tau(x)\},$$

is called (α, β, τ)-softlimit of the function f at the point x.

The collection of all these softlimits forms the notions of upper softlimit, lower softlimit, and softlimit of the function f, respectively.

The reader can find more details on softlimits in [2]. Now, we will construct the notion of "soft approximator" which is the analogue of the classical differential. Let α and β be positive numbers.

DEFINITION 3.3.2. *The set*

$$\bar{D}[f,\alpha,\beta,\tau](x) = \{v \in \mathbb{E} \mid f(y) \le f(x) + (v + \alpha(x))(y - x) + \beta(x),\ \forall\, y \in \tau(x)\}$$

is called an upper (α,β,τ)-approximator of function f at the point x.

The set

$$\underline{D}[f,\alpha,\beta,\tau](x) = \{v \in \mathbb{E} \mid f(y) \ge f(x) + (v - \alpha(x))(y - x) - \beta(x),\ \forall\, y \in \tau(x)\},$$

is called a lower (α,β,τ)-approximator of function f at the point x.

The collection of upper and lower (α,β,τ)-approximators forms upper and lower soft approximators. Under the soft approximator D we mean the intersection of upper and lower soft approximators

$$D[f,\alpha,\beta,\gamma,\delta,\tau](x) = \bar{D}[f,\alpha,\beta,\tau](x) \cap \underline{D}[f,\gamma,\delta,\tau](x).$$

The constructions used in Definition 3.3.2 are well known in convex analysis [6].

Here, we present some of the simplest properties of the soft approximators

$$\bar{D}[-f,\alpha,\beta,\tau](x) = -\underline{D}[f,\alpha,\beta,\tau](x),$$

$$\bar{D}[kf,\alpha,\beta,\tau](x) = k\bar{D}[f,\alpha,\beta,\tau](x), \qquad k > 0,$$

$$\bar{D}[f,\alpha,\beta,\tau](x) + \bar{D}[g,\gamma,\delta,\tau](x) \subset \bar{D}[f+g,\alpha+\gamma,\beta+\delta,\tau](x),$$

$$D\,[f,\alpha,\beta,\gamma,\delta,\tau](x) + D[g,\alpha',\beta',\gamma',\delta',\tau]\,(x)$$

$$\subset D\,[f+g,\alpha+\alpha',\beta+\beta',\gamma+\gamma',\delta+\delta',\tau]\,(x),$$

$$D\,[f+g,\alpha+\alpha',\beta+\beta',\gamma+\gamma',\delta+\delta',\tau]\,(x)$$

$$\subset D[f,\alpha,\beta,\gamma,\delta,\tau](x) + D[g,\alpha',\beta',\gamma',\delta',\tau](x) + [-\chi,\chi],$$

where

$$\chi = \alpha(x) + \gamma(x) + \alpha'(x) + \gamma'(x) + (\beta(x) + \delta(x) + \beta'(x) + \delta'(x)) \left(\sup_{y \in \tau(x)} |y - x| \right)^{-1}.$$

Of course, these properties are not as simple as those for classical differential, but in contrast to the classical differential, the soft approximators are smooth soft functionals. Moreover, the soft approximator is the regularization from above and from below for the classical differential [2]. Because of these properties of smoothness and regularization, the soft approximators are more convenient to deal with uncertain information and approximate calculation methods than the classical differentials.

We have introduced the soft analogue of differential, and it naturally raised up the question on the soft analogue of the integral. Two approaches can be used to construct the soft integral. The first one, called Riemann approach, is based on the integral sums.

Consider the interval $[a,b] \subset \mathbb{E}$. For simplicity we will assume that $\tau(x) \subset x + \mathbb{E}_+$, that is, the τ-close points of the point x lay on the right side of x. We will consider the sequences of points where every pair of neighbor points are close.

Let us denote $\mathrm{Pro}[x,\tau]$ the set of points which can be reached from the point x going only to the τ-close points.

Here the formal definition is given,

$$\mathrm{Pro}[x,\tau] = \bigcup_{k=0}^{\infty} \tau^k(x), \qquad \tau^0 = \{x\}, \qquad \tau^k(x) = \tau\left(\tau^{k-1}(x)\right).$$

For $b \in \mathrm{Pro}[x,\tau]$, we introduce the set of admissible divisions for the interval $[a,b]$,

$$\mathrm{dis}[a,b,\tau] = \{\bar{x}(x_0,\dots,x_n) \mid x_0 = a,\ x_n = b,\ x_{i+1} \in \tau(x_i)\}.$$

Note, that number n depends on the division. Let ε be a nonnegative real function.

DEFINITION 3.3.3. *The value*

$$\bar{I}R_a^b[f,\varepsilon,\tau] = \sup_{\bar{x}\in\mathrm{dis}[a,b,\tau]} \sum_{i=0}^{n(\bar{x})-1} \{f(x_i)(x_{i+1}-x_i) - \varepsilon(x_i)\},$$

is called the upper Riemann (ε,τ)-integral of the function f between limits a and b. The value

$$\underline{I}R_a^b[f,\varepsilon,\tau] = \inf_{\bar{x}\in\mathrm{dis}[a,b,\tau]} \sum_{i=0}^{n(\bar{x})-1} \{f(x_i)(x_{i+1}-x_i) + \varepsilon(x_i)\}$$

is called the lower Riemann (ε,τ)-integral of the function f between limits a and b.

The second approach of constructing a soft integral is based on the ideas of Perron. Denote $\tau_b(x) = \tau(x) \cap (-\infty, b]$.

We say that function F is (ε,τ)-subfunction of the function f between limits a and b, if and only if:

(1) F is defined on the set $\mathrm{Pro}[a,b,\tau]$,
(2) $F(a) = 0$,
(3) $f(x) \in \bar{D}[F,0,\varepsilon,\tau_b](x)$ for every $x \in \mathrm{Pro}[a,b,\tau] \setminus \{b\}$.

We say that function F is (ε,τ)-superfunction of the function f between limits a and b, if and only if:

(1) F is defined on the set $\mathrm{Pro}[a,b,\tau]$,
(2) $F(a) = 0$,
(3) $f(x) \in \underline{D}[F,0,\varepsilon,\tau_b](x)$ for every $x \in \mathrm{Pro}[a,b,\tau] \setminus \{b\}$.

DEFINITION 3.3.4. *We call the upper Perron (ε,τ)-integral of the function f between limits a and b the following value:*

$$\bar{I}P_a^b[f,\varepsilon,\tau] = \inf F(b),$$

where the infimum is considered with respect to all (ε,τ)-superfunctions F for function f between limits a and b.

We call the lower Perron (ε,τ)-integral of the function f between limits a and b the following value:

$$\underline{I}P_a^b[f,\varepsilon,\tau] = \sup F(b),$$

where the supremum is considered with respect to all (ε,τ)-subfunctions F for function f between limits a and b.

It is well known, that in the classical case, Riemann's approach and Perron's approach give us two different notions of the integral. For our case, these two approaches give identical results, that is

$$\bar{I}R_a^b[f,\varepsilon,\tau] = \bar{I}P_a^b[f,\varepsilon,\tau], \qquad \underline{I}R_a^b[f,\varepsilon,\tau] = \underline{I}P_a^b[f,\varepsilon,\tau].$$

Further, for upper and lower integrals, we will use the following notations:

$$\bar{I}_a^b[f,\varepsilon,\tau], \qquad \underline{I}_a^b[f,\varepsilon,\tau].$$

Let us introduce the notion of soft integral. Let u and v be the nonnegative real functions, γ and δ be real numbers. The role of parameter for soft integral is played by the vector $(u, v, \gamma, \delta, \tau)$.

We will call the soft $(u, v, \gamma, \delta, \tau)$-integral of function f between limits a and b the following interval:

$$I[f,u,v,\gamma,\delta,\tau] = \left[\bar{I}_a^b[f,u,\tau] - \gamma,\ \underline{I}_a^b[f,v,\tau] + \delta\right].$$

It is easy to see that the soft integral has the properties which are very close to the properties of convex positively homogeneous functions.

Now, the study of the soft integral is at the beginning but some interesting results have been obtained. We have obtained the analogue of the Newton-Leibniz formula for soft integrals, sufficient conditions for the existence of soft integral, the correlation of the soft integral and the Riemann and Lebesgue integrals. We also have proved the stability of the soft integral [2]. The interval base of the soft integral was generalized and the general theory of abstract interval integrals was constructed in [7,8].

On this basis, a new theory of interval probability was proposed in [9–11]. In this theory, the probability of an event is not a number but a real subinterval of the interval [0,1]. This simple fact is very important in practice. It allows us to apply the interval probability theory to any series of events, whereas the classical probability theory may be applied only to the stochastically stable events.

Although the soft set theory was a stimulator for constructing and developing the interval probability theory, the mathematical tool of the interval probability theory is more close to the convex analysis than to the theory of soft sets. Interested readers can find more details in [9–11].

4. SOME PROSPECTS OF THE SOFT SET THEORY

It is obvious, that the soft set theory can be applied to a wide range of problems in economics, engineering, physics, and so on. Here, we want to show only some directions which are most interesting at the present time. It may be interesting to construct a theory of measurements based on the soft sets. We propose to consider the following model of measurement.

For simplicity, we consider a set Y of real numbers which are the results of measurement,

$$Y = \{y_1, \dots, y_n\}, \qquad y_i \in \mathbb{E}.$$

With the set Y, we associate the soft set $(SY, [0,1])$ over the set of closed intervals $\mathbb{E} = \{[a,b] \mid a, b \in \mathbb{E},\ a \le b\}$ as follows.

Let α be a real number, $\alpha \in [0,1]$. The set $SY(\alpha)$ includes those and only those intervals $[a,b]$, for which the number of points belonging to the set Y and to the interval $[a,b]$ is greater or equal to αn. The real number α may be interpreted as a degree of trust of the interval $[a,b]$ because at least αn points of the measurement belong to the interval $[a,b]$. Note, that the notion "degree of trust" has nothing common with classical probability, so this construction may be a basis of a new theory of probability.

For example, let W be a space of elementary events and f be a bounded real valued function on the set W (an analogue of random variable).

Denote $w_1, \dots, w_n$, where $w_i \in W$, n realizations of the function f. Let T be a subset of the set of natural numbers $\{1, \dots, n\}$, and $\#T$ be the number of elements of the set T.

The following real number,

$$\langle f, T\rangle = \left(\sum_{i\in T} f(w_i)\right)(\#T)^{-1},$$

is called sample T-mean for function f.

We can choose some subsets T (for example, $\#T \geq m$) and consider the set $\{\langle f,T\rangle\}$ of all these sample T-means for function f. The set $\{\langle f,T\rangle\}$ may be considered as the result of measurement, and we can apply to it the soft set construction mentioned above. We shall obtain the soft set $(S\{\langle f,T\rangle\},[0,1])$ which may be considered as an analogue of the expectation of f.

If the function f is an indicator function of the set A, that is,

$$f(w) = \begin{cases} 1, & \text{if } w \in A, \\ 0, & \text{if } w \notin A, \end{cases}$$

then the soft set $(S\{\langle f,T\rangle\},[0,1])$ can be considered as the soft probability of the set A.

It is important to emphasize some features of this approach to the soft probability. In contrast with the axiomatic approach in the classical probability theory, the soft probability is directly based on measurements. The soft probability may be applied to any event, in contrast to the classical probability which may be applied only to stochastically stable events.

If the event is stochastically stable, then the soft probability will be a sufficiently narrow interval with a large trust degree if n is large enough. If the event is not stochastically stable, then the soft probability may be a large interval. The proposed approach can also be applied to construct a theory of soft clusters.

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