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Hilbert space compression for free products and HNN-extensions

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Abstract

We investigate the behavior of equivariant and non-equivariant Hilbert space compression under group constructions. Given the (equivariant or non-equivariant) Hilbert space compression of two groups, we find bounds on the compression of their free product. We also investigate the case of HNN-extensions of a group relative to a subgroup which is finite or of finite index.

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1. Introduction

Gromov introduced in [7] the notion of uniform embeddability of a metric space into a Hilbert space. Since uniform embeddability is a quasi-isometric invariant and since the word length metric on a finitely generated group is unique up to quasi-isometry, we obtain the notion of uniformly embeddable finitely generated group. Gromov believed that such groups would satisfy the Novikov conjecture [8]. Six years later, Yu showed that such groups satisfy the coarse Baum–Connes conjecture [17], which is related to the Novikov conjecture. Together with Skandalis and Tu, the Novikov conjecture for uniformly embeddable groups was eventually proven in [15].

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Definition 1.1. A metric space (G, d) is *uniformly embeddable in a Hilbert space*, if there exist a Hilbert space $\mathcal{H}$, non-decreasing functions $\rho_-, \rho_+ : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{t\to\infty} \rho_-(t) = +\infty$, and a map $f : G \to \mathcal{H}$, such that

$$\rho_-\big(d(x, y)\big) \leqslant d\big(f(x), f(y)\big) \leqslant \rho_+\big(d(x, y)\big) \quad \forall x, y \in G.$$

The map f is called a uniform embedding of G in $\mathcal{H}$. It is called large-scale Lipschitz whenever ρ_+ can be taken of the form $\rho_+ : t \mapsto Ct + D$ for some $C > 0$, $D \geqslant 0$. It is Lipschitz if we can take $D = 0$.

Assume for a moment that (G, d) is a finitely generated group, equipped with the word length metric (always relative to some finite symmetric generating subset). There is then a standard result (see e.g. Proposition 2.9 in [9]), which shows that every uniform embedding $f : G \to \mathcal{H}$ is Lipschitz. On the other hand, one cannot always assume that ρ_- is of the form $\rho_- : t \mapsto (1/C)t - D$ for some $C > 0$, $D \geqslant 0$. Indeed, the free group on two generators is uniformly embeddable in a Hilbert space, but a theorem of Bourgain [2] implies that it cannot be quasi-isometrically embedded in a Hilbert space.

More generally, fixing a uniform embedding $f : G \to \mathcal{H}$, Guentner and Kaminker [9] introduced the notion of compression to measure *how close this embedding is to being quasi-isometric*. More precisely, the compression of a uniform embedding f, denoted $R(f)$, is defined as the supremum of all $\epsilon \in [0, 1]$ for which there exist $C > 0$, $D \geqslant 0$ with the property that

$$d\big(f(x), f(y)\big) \geqslant (1/C)d(x, y)^\epsilon - D \quad \forall x, y \in G.$$

Taking the supremum of $R(f)$ over all Hilbert spaces $\mathcal{H}$ and all uniform embeddings $f : G \to \mathcal{H}$, we obtain the Hilbert space compression, or shortly, the compression of G. We say that a uniform embedding $f : G \to \mathcal{H}$ is G-equivariant if there exists an affine isometric action $\mathcal{A}$ from G on $\mathcal{H}$ such that $f(xy) = x \cdot_{\mathcal{A}} f(y)$ $\forall x, y \in G$. We obtain the equivariant Hilbert space compression of G as the supremum of $R(f)$, taken over all Hilbert spaces $\mathcal{H}$ and all G-equivariant uniform embeddings $f : G \to \mathcal{H}$. We remark that the (equivariant) compression is invariant under the choice of finite symmetric generating subset.

The above definitions for (equivariant) compression generalize naturally from the case of finitely generated groups equipped with the word length metric to the case of discrete groups whose metric is induced by some length function (see Definition 2.2). In this case, it is no longer true that uniform embeddings are always Lipschitz and so we demand this explicitly in the definition for (equivariant) compression, i.e. we *only* take into account those (equivariant) uniform embeddings which are Lipschitz.

It is known that for every $\alpha \in [0, 1]$, there exists an infinite, finitely generated group with compression α (see [1]). Much less is known about the range of values of the equivariant compression. In fact, at the time of writing, the only known values for the equivariant compression are 0, 1/2 and $\frac{1}{2-2^{1-k}}$ with $k \geqslant 1$ (see also [12]).

In [9], Guentner and Kaminker prove that the compression of a direct product of two groups G_1 and G_2 equals the minimum of the compressions of G_1 and G_2. Moreover, their proof also holds in the equivariant case. In this article, we ask ourselves a similar question. Concretely, given the (equivariant) Hilbert space compressions of groups G_1 and G_2, we ask what can be said about the compression of the free product $G_1 * G_2$ and what can be said about the

compression of an HNN-extension of G_1. Although the proven results are more general, we decide to formulate them in this Introduction for the case of finitely generated groups equipped with the word length metric relative to a finite symmetric generating subset. The first result below is Corollary 2.5.

Theorem 1.2. *Assume that G_1 and G_2 are finitely generated groups equipped with the word length metric. Denoting their Hilbert space compressions by α_1 and α_2 respectively, the Hilbert space compression α of the free product $G = G_1 * G_2$, equipped with the word length metric, satisfies*

$$\min(\alpha_1, \alpha_2, 1/2) \leqslant \alpha \leqslant \min(\alpha_1, \alpha_2).$$

The result can be generalized for free products, amalgamated over finite groups, and for HNN-extensions $\mathrm{HNN}(H, F, \theta) = \langle H, t \mid t^{-1} f t = \theta(f)\ \forall f \in F\rangle$ where F is a *finite* subgroup of H. We refer the reader to the end of Section 2 for details.

In Section 3, we investigate the compression of an HNN-extension $\mathrm{HNN}(H, F, \theta)$ where both F and $\theta(F)$ are finite index subgroups of H. We obtain the following bounds (see Corollary 3.16).

Theorem 1.3. *Consider $G := \mathrm{HNN}(H, F, \theta)$ where both F and $\theta(F)$ are finite index subgroups of the finitely generated group H. Equip G with the word length distance d and H with the induced metric d_{in} from G. Then,*

$$\alpha_1/3 \leqslant \alpha \leqslant \alpha_1,$$

where α_1 and α denote the Hilbert space compressions of (H, d_{in}) and (G, d) respectively.

Regarding the equivariant Hilbert space compression, we obtain the following results in Sections 4 and 5 (see Theorem 4.9 and Theorem 5.3 respectively).

Theorem 1.4. *Let G_1 and G_2 be finitely generated groups equipped with the word length metric and denote their equivariant Hilbert space compressions by α_1 and α_2 respectively. Denote $G = G_1 *_F G_2$ an amalgamated free product where F is a finite subgroup of both G_1 and G_2. If α denotes the equivariant Hilbert space compression of G, when equipped with the word length metric, then*

1. $\alpha = 1$ *if F is of index 2 in both G_1 and G_2,*
2. $\alpha = \alpha_1$ *when $F = G_2$ and $\alpha = \alpha_2$ when $F = G_1$,*
3. $\alpha = \min(\alpha_1, \alpha_2, 1/2)$ *otherwise.*

Although condition 2 above is rather trivial, we have added it for completeness. For HNN-extensions over a finite group, we show

Theorem 1.5. *Let H be a finitely generated group with equivariant Hilbert space compression α_1 and let F be a finite subgroup of H. Denote $G = \mathrm{HNN}(H, F, \theta)$ and equip it with the word length metric. The equivariant Hilbert space compression α of G satisfies*

1. $\alpha = 1$ *whenever* $F = H$,
2. $\alpha = \min(\alpha_1, 1/2)$ *otherwise.*

Here, the first statement is rather trivial, but we have added it for completeness.

2. Hilbert space compression of a free product of groups

In the Introduction, we have only considered uniform embeddings of groups into Hilbert spaces, i.e. into L^2-spaces. By slightly modifying the definitions in the Introduction, one can choose any $p \geqslant 1$ and replace the class of L^2-spaces by the class of L^p-spaces. This way, one defines uniform embeddability into an L^p-space, equivariant and non-equivariant L^p-compression, etc.

In order to investigate the behavior of compression under free products, we start with the following general lemma.

Lemma 2.1. *Let (X, d) be a uniformly discrete metric space, i.e. $B := \inf\{d(x, y) \mid x, y \in X\} > 0$, which embeds uniformly into an L^p-space. If $f : X \to L^p$ is a map such that*

$$(1/C)d(x, y)^\epsilon - D \leqslant \left\| f(x) - f(y) \right\|_p \leqslant Cd(x, y) + D$$

for some $\epsilon > 0$, $C > 0$, $D \geqslant 0$ and $\forall x, y \in X$, then there exist $\tilde{f} : X \to L^p \oplus_p l^p(X)$ and a real number $\overline{C} > 0$ such that

$$(1/\overline{C})d(x, y)^\epsilon \leqslant \left\| f(x) - f(y) \right\|_p \leqslant \overline{C}d(x, y) \quad \forall x, y \in X. \tag{1}$$

Proof. Define $\tilde{f} : X \to L^p \oplus_p l^p(X)$, $x \mapsto f(x) \oplus \delta_x$ where δ_x is the Dirac function at x. Then for every two distinct elements x, y of X, we have that

$$\left\| \tilde{f}(x) - \tilde{f}(y) \right\|_p^p = \left\| f(x) - f(y) \right\|_p^p + 2.$$

Therefore

$$\left\| \tilde{f}(x) - \tilde{f}(y) \right\|_p \leqslant Cd(x, y) + D + 2^{1/p} \leqslant \left(C + \frac{D + 2^{1/p}}{B} \right) d(x, y),$$

and so we obtain an upper bound like the one in Eq. (1) by setting $\overline{C} = C + \frac{D+2^{1/p}}{B}$. With respect to the lower bound, we obtain that

$$\left\| \tilde{f}(x) - \tilde{f}(y) \right\|_p \geqslant \max\bigl((1/C)d(x, y)^\epsilon - D, 2^{1/p}\bigr).$$

When $d(x, y)^\epsilon \geqslant 2CD$, then

$$\left\| \tilde{f}(x) - \tilde{f}(y) \right\|_p \geqslant \left(\frac{1}{2C} d(x, y)^\epsilon + \frac{1}{2C} d(x, y)^\epsilon - D \right) \geqslant (1/\overline{C})d(x, y)^\epsilon,$$

if we take $\overline{C} \geqslant 2C$. When $d(x,y)^\epsilon \leqslant 2CD$, then

$$\left\| \tilde{f}(x) - \tilde{f}(y) \right\|_p \geqslant 2^{1/p} \geqslant 1 \geqslant \frac{1}{2CD} d(x,y)^\epsilon.$$

Finally, putting $\overline{C} := \max(2C, 2CD, C + \frac{D+2^{1/p}}{B})$, we obtain

$$(1/\overline{C}) d(x,y)^\epsilon \leqslant \left\| \tilde{f}(x) - \tilde{f}(y) \right\|_p \leqslant \overline{C} d(x,y) \quad \forall x, y \in X. \qquad \square$$

Definition 2.2. A length function l on a group G is a function $l : G \to \mathbb{R}^+$ satisfying

1. $\forall x \in G$: $l(x) = 0 \Leftrightarrow x = 1$,
2. $\forall x \in G$: $l(x) = l(x^{-1})$, and
3. $\forall x, y \in G$: $l(xy) \leqslant l(x) + l(y)$.

A length function on G induces a metric on G by setting $d(x,y) = l(x^{-1}y)$ $\forall x, y \in G$. If G is finitely generated and S is a finite symmetric generating subset for G, then we can define the length of an element $x \in G$ as the length of the shortest path between 1 and x on the Cayley graph of (G, S). Here, the length of every edge is counted as 1. The so obtained length function is called the word length function and the associated metric is called the word length distance on G relative to S. The (equivariant) L^p-compression is invariant under the choice of finite symmetric generating subset.

For groups, there is a standard definition for the free product. Analogously, given two metric spaces (X_1, d_1), (X_2, d_2) and points $\tilde{x}_1 \in X_1$, $\tilde{x}_2 \in X_2$, we will now define the free product of $(X_1, d_1, \tilde{x}_1)$ and $(X_2, d_2, \tilde{x}_2)$, denoted $X_1 * X_2$.

As a set, this space is equal to the collection of all words whose letters are alternately elements from $X_1 \setminus \{\tilde{x}_1\}$ and $X_2 \setminus \{\tilde{x}_2\}$. We also include the word $\tilde{x}_1$ which we identify with $\tilde{x}_2$ (the idea being that $\tilde{x}_1$ and $\tilde{x}_2$ play the role of "the identity elements" of X_1 and X_2). In order to specify that they are words, i.e. elements of X and not of X_1 or X_2, we will denote them by $\tilde{x}_i^w$. Two elements x, y of $(X_1 * X_2) \setminus \{\tilde{x}_1^w\}$ can always be written in reduced form as

$$\begin{aligned} x &= a_1 a_2 \dots a_m, \\ y &= b_1 b_2 \dots b_n, \end{aligned} \tag{2}$$

where m, n are natural numbers and where the a_i, b_j are elements of $X_1 \setminus \{\tilde{x}_1\} \sqcup X_2 \setminus \{\tilde{x}_2\}$ such that no two consecutive elements a_i, a_{i+1} or b_j, b_{j+1} both belong to X_1 or both belong to X_2. If i_0 is the highest index such that $a_1, a_2, \dots, a_{i_0-1}$ and a_{i_0} are equal to $b_1, b_2, \dots, b_{i_0-1}$ and b_{i_0} respectively, then $h := a_1 a_2 \dots a_{i_0}$ is called the common part of x and y. This way, we abbreviate

$$\begin{aligned} x &= h g_x x_1 x_2 \dots x_n, \\ y &= h g_y y_1 y_2 \dots y_m, \end{aligned} \tag{3}$$

where h is the common part of x and y (when $i_0 = 0$, then no h occurs in the above formulas). If $g_x \in X_1 \setminus \{\tilde{x}_1\}$, and $g_y \in X_2 \setminus \{\tilde{x}_2\}$ (which can happen only if there is no h), then redefine $y_{m+1} := y_m, y_m := y_{m-1}, \dots, y_2 := y_1, y_1 := g_y, g_y := \tilde{x}_1$, in order that g_x, g_y both belong

to X_1. Similar remarks hold when $g_x \in X_2 \setminus \{\tilde{x}_2\}$ and $g_y \in X_1 \setminus \{\tilde{x}_1\}$. With the convention that empty sums are zero, we define the distance between x and y by

$$\sum_{i=1}^{n} d_{1,2}(x_i, \tilde{x}_{1,2}) + d_{1,2}(g_x, g_y) + \sum_{j=1}^{m} d_{1,2}(y_j, \tilde{x}_{1,2}), \tag{4}$$

where $d_{1,2}$ stands for d_1 or d_2 and $\tilde{x}_{1,2}$ stands for $\tilde{x}_1$ or $\tilde{x}_2$ as appropriate.

The above definition is not random. Indeed, if X_1 and X_2 are finitely generated groups equipped with the word length metric relative to finite symmetric generating subsets S and T respectively, then the above definition gives the word length metric on $X_1 * X_2$ relative to $S \cup T$.

We prove the following

Theorem 2.3. *Assume that (X_1, d_1) and (X_2, d_2) are uniformly discrete metric spaces with L^p-compressions given by α_1 and α_2 respectively. Choose elements $\tilde{x}_1 \in X_1$ and $\tilde{x}_2 \in X_2$. The L^p-compression α of the free product $X := (X_1, d_1, \tilde{x}_1) * (X_2, d_2, \tilde{x}_2)$, equipped with the above defined metric, satisfies*

$$\min(\alpha_1, \alpha_2, 1/p) \leqslant \alpha \leqslant \min(\alpha_1, \alpha_2).$$

Remark 2.4. Part of the proof is based on Section 3 of [3], where given uniform embeddings of groups G_1 and G_2 in a Hilbert space, an explicit uniform embedding of the free product $G_1 * G_2$ in a Hilbert space is constructed.

Proof of Theorem 2.3. It is not hard to find the desired upper bound for α since X_1 and X_2 are metric subspaces of X and so $\alpha \leqslant \min(\alpha_1, \alpha_2)$.

Next, choose a number $0 \leqslant \epsilon < \min(\alpha_1, \alpha_2, 1/p)$ and let us take for each $i \in \{1, 2\}$ a map $f_i : X_i \to L_i^p$, where L_1^p and L_2^p are L^p-spaces, which maps the chosen element $\tilde{x}_i$ to 0 and such that

$$\exists C \geqslant 1,\ \forall x, y \in X_i\colon \quad (1/C)d(x, y)^\epsilon \leqslant \left\| f_i(x) - f_i(y) \right\|_p \leqslant C d(x, y). \tag{5}$$

If necessary, replace each L_i^p by $L_1^p \oplus_p L_2^p$ in order that we may assume that $L_1^p = L_2^p$. Write $L^p := L_1^p$ for short.

Denote by W_i $(i = 1, 2)$ the set of those elements of X whose expression as a reduced word begins with an element of X_i (see Eq. (2)). Notice that $W_1 \cap W_2$ is the singleton containing the word $\tilde{x}_1^w$. Define an L^p-space $\overline{L^p}$ by

$$\overline{L^p} = \left(\bigoplus_{W_1} L^p \right) \oplus \left(\bigoplus_{W_2} L^p \right),$$

where direct sums are l^p-direct sums. Consider a map $f : X \to \overline{L^p}$ defined as follows: set $f(\tilde{x}_1^w) = 0$. Next, choose any element $x \in X \setminus \{\tilde{x}_1^w\}$ and write it as a reduced word $x = x_1 x_2 \ldots x_n$. If $x_1 \in X_1$, then we define $f(x) = f(x)_1 \oplus_p f(x)_2 \in \overline{L^p}$ by setting $f(x)_2 = 0$ and

$$f(x)_1 = \bigoplus_{h \in W_1} \big(f(x)_1\big)_h$$

$$\text{where } \big(f(x)_1\big)_h = \begin{cases} f_1(x_{2k+1}), & \exists k \geqslant 0 \text{ such that } h = x_1 x_2 \dots x_{2k}, \\ f_2(x_{2k}), & \exists k \geqslant 1 \text{ such that } h = x_1 x_2 \dots x_{2k-1}, \\ 0 & \text{otherwise,} \end{cases}$$

using the convention that an empty product corresponds to $\tilde{x}_1^w$. In particular, $(f(x)_1)_{\tilde{x}_1^w} = f_1(x_1)$, $(f(x)_1)_{x_1} = f_2(x_2)$, $(f(x)_1)_{x_1 x_2} = f_1(x_3), \dots$. A similar formula is used when the reduced word expression of x begins with an element of X_2. Let us show that

$$(1/C)d(x,y)^\epsilon \leqslant \big\| f(x) - f(y) \big\|_p \leqslant C d(x,y) \quad \text{for all } x, y \in X.$$

Choose $x, y \in G$ and write $x = h g_x x_1 x_2 \dots x_n$ and $y = h g_y y_1 y_2 \dots y_m$ as in (3). With the convention that empty sums are 0, we obtain

$$\big\| f(x) - f(y) \big\|_p^p = \sum_{i=1}^n \big\| f_{1,2}(x_i) \big\|_p^p + \big\| f_{1,2}(g_x) - f_{1,2}(g_y) \big\|_p^p + \sum_{j=1}^m \big\| f_{1,2}(y_j) \big\|_p^p.$$

By Eq. (5), we obtain the upper bound

$$\big\| f(x) - f(y) \big\|_p^p \leqslant \sum_{i=1}^n C^p d_{1,2}(x_i, \tilde{x}_{1,2})^p + C^p d_{1,2}(g_x, g_y)^p + \sum_{j=1}^m C^p d_{1,2}(y_j, \tilde{x}_{1,2})^p,$$

and making use of Eq. (4) and the fact that $(a^p + b^p)^{1/p} \leqslant a + b$ for all $a, b \geqslant 0$, we obtain

$$\big\| f(x) - f(x) \big\|_p \leqslant C d(x,y).$$

On the other hand, we have that

$$\begin{aligned} \big\| f(x) - f(y) \big\|_p^p \geqslant \sum_{i=1}^n \big(1/C^p\big) d_{1,2}(x_i, \tilde{x}_{1,2})^{p\epsilon} + \big(1/C^p\big) d_{1,2}(g_x, g_y)^{p\epsilon} \\ + \sum_{j=1}^m \big(1/C^p\big) d_{1,2}(x_i, \tilde{x}_{1,2})^{p\epsilon} \end{aligned}$$

and since $a^{p\epsilon} + b^{p\epsilon} \geqslant (a+b)^{p\epsilon}$ for all $a, b \geqslant 0$, we get that

$$\big\| f(x) - f(y) \big\|_p \geqslant (1/C)\big(d(x,y)^{p\epsilon}\big)^{1/p} = (1/C) d(x,y)^\epsilon,$$

which concludes the proof of the theorem. □

It is particularly interesting to notice that for $p = 1$, we obtain that the L^1-compression of the free product $X_1 * X_2$ equals the minimum of the L^1-compressions of X_1 and X_2. Moreover, the same result holds when we replace the class of L^1-spaces, with a class $\mathcal{C}$ of Banach spaces which is stable under l^1-direct sum.

Applying the above result to finitely generated groups equipped with the word length metric relative to a finite symmetric generating subset, we obtain

Corollary 2.5. *Assume that G_1 and G_2 are finitely generated groups equipped with the word length metric. If we denote their L^p-compressions by α_1 and α_2 respectively, then the L^p-compression α of the free product $G = G_1 * G_2$, when equipped with the word length metric, satisfies*

$$\min(\alpha_1, \alpha_2, 1/p) \leqslant \alpha \leqslant \min(\alpha_1, \alpha_2).$$

We end this section by a remark regarding the L^p-compression ($p \geqslant 1$) of $G_1 *_F G_2$ and $\mathrm{HNN}(H, F)$ where F is a finite subgroup and where G_1, G_2 and H are finitely generated groups. Our claim is that the L^p-compressions of $G_1 *_F G_2$ and $\mathrm{HNN}(H, F)$ are equal to that of $G_1 * G_2$ and $H * \mathbb{Z}$ respectively. We distinguish two cases. First, assume that G_1, G_2 and H are finite (or more generally are hyperbolic). It follows from [11] that $G_1 *_F G_2$, $G_1 * G_2$, $H * \mathbb{Z}$ and $\mathrm{HNN}(H, F)$ are also hyperbolic and thus all have L^p-compression 1 [16]. Secondly, assume that at least one of G_1 and G_2 is infinite. Then our claim regarding amalgamated products follows from Theorem 0.2 in [13], where it is shown that $G_1 *_F G_2$ is quasi-isometric to $G_1 * G_2$. If H is infinite, then our claim regarding HNN-extensions follows from the same result, where it is proven that $\mathrm{HNN}(H, F)$ and $H * \mathbb{Z}$ are quasi-isometric for infinite H.

3. Hilbert space compression of an HNN-extension over a finite index subgroup

Before formulating the main result from this section, we will first elaborate on a different characterization of uniform embeddability, stated by Dadarlat and Guentner [5].

Proposition 3.1. *(See Proposition 2.1 from [5].) Let (X, d) be a metric space. Then X is uniformly embeddable in a Hilbert space if and only if for every $R > 0$ and $\epsilon > 0$ there exists a Hilbert space valued map $\xi : X \to \mathcal{H}$, $x \mapsto \xi_x$, such that $\|\xi_x\| = 1$ for all $x \in X$ and such that*

1. $\sup\{\|\xi_x - \xi_y\|\colon d(x, y) \leqslant R,\ x, y \in X\} \leqslant \epsilon$,
2. $\lim_{S\to\infty} \inf\{\|\xi_x - \xi_y\|\colon d(x, y) \geqslant S,\ x, y \in X\} = \sqrt{2}$.

One is allowed to replace the second condition by

$$\lim_{S\to\infty} \sup\bigl\{\bigl|\langle \xi_x, \xi_y\rangle\bigr|\colon d(x, y) \geqslant S,\ x, y \in X\bigr\} = 0.$$

Copying the first part of their proof word for word, one obtains more specifically the following

Lemma 3.2. *Let (X, d) be a metric space that uniformly embeds into a Hilbert space. Assume that $F : X \to \mathcal{H}$ satisfies*

$$\forall x, y \in X\colon \quad \rho_-\bigl(d(x, y)\bigr) \leqslant \bigl\|F(x) - F(y)\bigr\| \leqslant \rho_+\bigl(d(x, y)\bigr),$$

where $\lim_{t\to\infty}(\rho_-(t)) = +\infty$ and where ρ_+ is a monotone increasing function. Then for every $R, \epsilon > 0$, there exists a Hilbert space $\mathcal{H}^{\epsilon,R}$ and a family of unit vectors $(\xi_x)_{x\in X} \subset \mathcal{H}^{\epsilon,R}$ (explicitly constructed) such that

$$\left(1-\frac{\epsilon^2}{2}\right)^{\frac{\rho_+(d(x,y))^2}{\rho_+(R)^2}} \leqslant \langle \xi_x, \xi_y \rangle \leqslant \left(1-\frac{\epsilon^2}{2}\right)^{\frac{\rho_-(d(x,y))^2}{\rho_+(R)^2}},$$

for all $x, y \in X$.

The following lemma follows from quantifying the second part of the proof of Proposition 3.1.

Lemma 3.3. *Let* (X, d) *be a metric space, let* $p > 0$ *be a number and assume that there exists* $r > 0$ *such that for each* $m \in \mathbb{N}_0$ *large enough (i.e. greater or equal to some arbitrary number* δ*), we can find a collection of unit vectors* $(\xi_x^m)_{x\in X}$ *in a Hilbert space* $\mathcal{H}^m$ *satisfying*

1. $\|\eta_x^m - \eta_y^m\| \leqslant \frac{1}{m^{1/2+p}}$ *provided* $d(x, y) \leqslant R_m := \sqrt{m}$,
2. $|\langle \eta_x^m, \eta_y^m \rangle| \leqslant 1/2$ *whenever* $d(x, y) \geqslant S_m := m^r$.

Then the compression of (X, d) *is at least* $\frac{1}{2r}$.

If moreover, (X, d) *is quasi-geodesic as a metric space, then it suffices to find vectors* η_x^m *satisfying the above conditions for* $R_m = \ln(m)$ *and the same conclusion holds.*

Proof. Choose a base point $x_0 \in X$ and define $f : X \to \bigoplus_{n=\delta}^{\infty} \mathcal{H}^m$ by

$$f(x) = 1/2\big(\big(\eta_\delta(x) - \eta_\delta(x_0)\big) \oplus \big(\eta_{\delta+1}(x) - \eta_{\delta+1}(x_0)\big) \oplus \cdots\big).$$

We show that f is well defined and

$$\rho_-\big(d(x, y)\big) \leqslant \big\|f(x) - f(y)\big\| \leqslant d(x, y) + B \quad \text{for all } x, y \in X,$$

where $B \in \mathbb{R}^+$, $\rho_- = 1/2 \sum_{m=\delta}^{\infty} \sqrt{m-\delta+1}\,\chi_{[S_m, S_{m+1})}$, and the $\chi_{[S_m, S_{m+1})}$ are the characteristic functions of the sets $[S_m, S_{m+1})$.

Indeed, let $x, y \in X$. If m is such that $\sqrt{m-1} \leqslant d(x, y) < \sqrt{m}$, we have

$$\begin{aligned}\big\|f(x) - f(y)\big\|^2 &= \frac{1}{4} \sum_{\delta \leqslant i \leqslant m-1} \big\|\eta_i(x) - \eta_i(y)\big\|^2 + \frac{1}{4} \sum_{i \geqslant m} \big\|\eta_i(x) - \eta_i(y)\big\|^2 \\ &\leqslant (m-1) + (1/4) \sum_{i \geqslant m} \frac{1}{i^{1+2p}} \leqslant d(x, y)^2 + C\end{aligned}$$

where $C = (1/4) \sum_{i \geqslant 1} \frac{1}{i^{1+2p}} < \infty$.

If m is such that $S_m \leqslant d(x, y) < S_{m+1}$, we have

$$\big\|f(x) - f(y)\big\|^2 \geqslant \frac{1}{4} \sum_{\delta \leqslant i \leqslant m} \big\|\eta_i(x) - \eta_i(y)\big\|^2 \geqslant (m-\delta+1)/4 = \rho_-\big(d(x, y)\big)^2.$$

It is clear that the compression of the uniform embedding f exceeds β whenever $S_{m+1}^{\beta} \leqslant \sqrt{m-\delta+1}$ for all m sufficiently large. This is true whenever $\beta < \frac{1}{2r}$. Since the Hilbert space compression is defined as a supremum, we obtain that the compression of (X, d) is at least $\frac{1}{2r}$.

We needed $R_m = \sqrt{m}$ in order to show that f is large-scale Lipschitz. Assuming that X is quasi-geodesic, there is a standard result stating that every uniform embedding of X in a Hilbert space is large-scale Lipschitz (see e.g. Proposition 2.9 in [9]). Therefore, it suffices in this case to take $R_m = \ln(m)$. $\square$

Construction 3.4. Let H be a group, equipped with some length function l_1. Assume that $G := \mathrm{HNN}(H, F, \theta)$ where both F and $\theta(F)$ are of finite index in H. There is a natural way to equip G with a length function. Indeed, each element $x \in \mathrm{HNN}(H, F, \theta)$ can be represented by a collection of words $(a_1, t^{i_1}, a_2, t^{i_2}, \ldots, a_{n-1}, t^{i_{n-1}}, a_n)$ where n runs over the natural numbers, where $x = a_1 t^{i_1} a_2 t^{i_2} \cdot \ldots \cdot a_n$, where the a_i belong to H and where $i_1, i_2, \ldots, i_{n-1} \in \{1, -1\}$. We define the length of such a word as the sum $l_1(a_n) + \sum_{j=1}^{n-1}(l_1(a_i) + |i_j|)$. We define the length of x as the infimum of the lengths of all words representing x. It is easy to see that this defines a length function on G.

Remark 3.5. If H is finitely generated and l_1 is the word length metric relative to a finite symmetric generating subset S, then the above construction yields the word length metric on G, relative to $S \cup \{t, t^{-1}\}$.

Modifying an idea of Dadarlat and Guentner (see the proof of Theorem 5.3 in [5]), we obtain a lower bound on the Hilbert space compression of G.

Theorem 3.6. *Let H be a group, equipped with a length function l_1 and consider $G := \mathrm{HNN}(H, F, \theta)$ where both F and $\theta(F)$ are finite index subgroups of H. Equip G with the length function l, defined in the above construction. Equip H with the induced length l_{in} from G. Then,*

$$\alpha_1/6 \leqslant \alpha \leqslant \alpha_1,$$

where α_1 and α denote the Hilbert space compressions of (H, l_{in}) and (G, l) respectively.

To prove the result, let us introduce some notations and definitions as in Sections 5 and 6 of [5]. Recall that a *tree* consists of a set V of vertices, a set E of edges and two endpoint maps $E \to V$, associating to each edge its endpoints. Every two vertices in a tree are connected by a unique path without backtracking. Whenever two vertices $v, v' \in V$ are connected by an edge, then we denote this edge by $[v, v']$, or equivalently by $[v', v]$, i.e. edges do not carry an orientation.

A *tree of metric spaces* consists of families $(X_v)_{v \in V}$ and $(X_e)_{e \in E}$ of metric spaces and corresponding maps $\sigma_{e,v} : X_e \to X_v$ whenever v is an endpoint of e. The maps $\sigma_{e,v}$ are called *structural maps*, the spaces X_v are called *vertex spaces* and the sets X_e are called *edge spaces*. We do not require that the structural maps are isometric embeddings.

Given an HNN-extension $G := \mathrm{HNN}(H, F, \theta)$, we can use Bass–Serre theory to associate a tree T to it as follows. As the set V of vertices we take G/H, the collection of left cosets of H in G. As the set E of edges we take G/F, the left cosets of F in G. Given $x \in G$, the edge xF connects xH and xtH.

Notice that the vertices and edges of the above tree are actually subsets of G, so we can equip them as metric subspaces of G. Next, we can define structural maps $\sigma_{xF,xH} : xF \hookrightarrow xH$ by inclusion and $\sigma_{xF,xtH} : xF \to xtH$ by $xf \mapsto xft = xt\theta(f)$. This way, we obtain a tree of metric spaces which is called *the tree of metric spaces associated to the HNN-extension* $G = \mathrm{HNN}(H, F, \theta)$. Notice that the union of the vertex spaces equals G.

Remark 3.7. Choose $x, y \in G = \mathrm{HNN}(H, F, \theta)$. From Bass–Serre theory, it follows immediately that the distance $d(x, y)$ in G equals

$$d_T(xH, yH) + \inf\bigl\{d(x_0, x_1) + d(x_2, x_3) + \cdots + d(x_{p-1}, x_p)\bigr\},$$

where d_T is the distance on the underlying Bass–Serre tree T (i.e. the number of edges in the shortest path connecting xH and yH) and where the infimum is taken over all sequences $x_0, x_1, \ldots, x_p$, where $p = 2d_T(xH, yH) + 1$ and

- $x = x_0$, $y = x_p$,
- $x_{2k} = x_{2k-1}t$ or $x_{2k} = x_{2k-1}t^{-1}$ for $k = 1, \ldots, d_T(xH, yH)$,
- x_{2k}, x_{2k+1} lie in the same coset of H for $k = 0, 1, \ldots, d_T(xH, yH)$.

Notation 3.8. For a given vertex $v \in V$, we denote by $\alpha(v) \in V$ the unique vertex such that $[v, \alpha(v)]$ points towards the infinite geodesic $H, tH, t^2H, \ldots$. Here, just for this once, $[v, \alpha(v)]$ was considered as an *oriented* edge. Given vertices $v, v' \in V$, we denote by (k, l) the unique pair of natural numbers such that $\alpha^k(v) = \alpha^l(v')$ and $d_T(v, v') = k + l$. Write $Y_v = \sigma_{[v,\alpha(v)],v}(X_{[v,\alpha(v)]}) \subset X_v$ and remark that it is a left coset of F or $\theta(F)$. Set $f_v = \sigma_{[v,\alpha(v)],\alpha(v)} \circ \sigma^{-1}_{[v,\alpha(v)],v} : Y_v \to X_{\alpha(v)}$. Finally, let $Z > 0$ be a real number such that every right coset of F and $\theta(F)$ in H contains a representative whose length is strictly smaller than Z.

Definition 3.9. Given $x_0 \in G$, an *s-chain starting in* x_0 is a sequence $\mathbf{x} = (x_0, x_1, \ldots, x_{s-1})$ with $x_i \in X_{\alpha^i(v)}$ such that for each $0 \leqslant i \leqslant s-2$ there exists $\overline{x_i} \in Y_{\alpha^i}(v)$ such that $d(x_i, \overline{x_i}) < d(x_i, Y_{\alpha^i(v)}) + 1$ and $x_{i+1} = f_{\alpha^i(v)}(\overline{x_i})$.

If F were not of finite index in H and if (x_0, x_1) and (x_0', x_1') were 2-chains with $x_0, x_0' \in X_v$ for some $v \in V$ and $d(x_0, x_0') = n$, then it is, apart from special cases, impossible to give estimates on $d(x_1, x_1')$. Indeed, $d(x_0, Y_v)$ could be very large but also very small, which makes it hard to use the triangle inequality. In the same spirit, it is possible that x_0 and x_0' are close to each other but $\overline{x_0}$ and $\overline{x_0}'$ are very far away from each other. In our case, good estimates *can* be made, since F and $\theta(F)$ are of finite index in H (see the proofs of Lemmas 3.10 and 3.12 below). Dadarlat and Guentner found bounds in a different way, namely by assuming that the structural maps preserve distances. We will not work under this assumption, but still the upcoming lemmas can be recognized as specific versions of Lemmas 6.1, 6.5, 6.6, 6.7 in [5].

Lemma 3.10. *Assume that R is a strictly positive real number, let $x_0 \in X_v$ and $x_0' \in X_{v'}$ with $d(x_0, x_0') < R$ and let k, l and Z be as in Notation* 3.8. *Then, any chains $(x_0, x_1, \ldots, x_k)$, $(x_0', x_1', \ldots, x_l')$ are such that*

$$\max\Bigl\{\Bigl(\sup_{0\leqslant i\leqslant k-1} d(x_i, x_{i+1})\Bigr), \Bigl(\sup_{0\leqslant j\leqslant l-1} d\bigl(x_j', x_{j+1}'\bigr)\Bigr), d\bigl(x_k, x_l'\bigr)\Bigr\} < (Z+3)R.$$

Proof. If $R < 1$, then the statement is trivial since $v = v'$ and $k = l = 0$.

If $R \geqslant 1$, then fix $i \in \{0, 1, \ldots, k-1\}$. Write $v_i = \alpha^i(v)$, denote $e = [v_i, \alpha(v_i)]$ and take $a \in G$ such that $X_e = aF$. This implies either that $X_{v_i} = aH$ and $X_{\alpha(v_i)} = atH$ or that $X_{v_i} = atH$ and $X_{\alpha(v_i)} = aH$. We only prove the second case, leaving the first case as an exercise to the reader.

The elements of Y_{v_i} are of the form $aft = at\theta(f)$ where $f \in F$. Writing $x_i = ath$ for some $h \in H$, take b a representative of $\theta(F)h$ whose length is smaller than Z. Then clearly $d(x_i, x_{i+1}) < d(x_i, Y_{v_i}) + 2 \leqslant d(x_i, x_i b^{-1}) + 2 < Z + 2 < (Z+3)R$.

Analogously, one proves that $d(x'_j, x'_{j+1}) < Z + 2 < (Z+3)R$ for $j \in \{0, 1, \ldots, l-1\}$. For the case $i = k$, we use the triangle inequality to get that

$$d\big(x_k, x'_l\big) < (Z+2)(k+l) + R \leqslant (Z+3)R. \qquad \square$$

Notation 3.11. Given $R > 0$ and $\epsilon > 0$, choose and fix $s, n \in \mathbb{N}_0$ such that

$$\sqrt{2/s} \leqslant \frac{\epsilon}{2(R+1)}, \qquad n \geqslant (Z+3)R. \tag{6}$$

Next, using Lemma 3.2, find a Hilbert space $\overline{\mathcal{H}}$ and unit vectors $\{\tilde{\xi}_x \mid x \in H\} \subset \overline{\mathcal{H}}$ satisfying the conditions

$$\sup\big\{\|\tilde{\xi}_y - \tilde{\xi}_{y'}\|\colon d\big(y, y'\big) \leqslant n + 2s(Z+2)\big\} \leqslant \frac{\epsilon}{2(R+1)}, \tag{7}$$

$$\lim_{S\to\infty} \sup\big\{\big|\langle \tilde{\xi}_y, \tilde{\xi}_{y'}\rangle\big|\colon d\big(y, y'\big) \geqslant S\big\} = 0. \tag{8}$$

For each $v \in V$, denote $\mathcal{H}_v := \overline{\mathcal{H}}$. Since G is the disjoint union of the vertex spaces X_v, we can take unit vectors $\{\xi_x \mid x \in G\} \subset \mathcal{H} := \bigoplus_{v\in V} \mathcal{H}_v$ such that $\xi_x \in \mathcal{H}_v$ whenever $x \in X_v$ and such that

$$\sup\big\{\|\xi_y - \xi_{y'}\|\colon d\big(y, y'\big) \leqslant n + 2s(Z+2),\ y, y' \in X_v,\ v \in V\big\} \leqslant \frac{\epsilon}{2(R+1)}, \tag{9}$$

$$\lim_{S\to\infty} \sup\big\{\big|\langle \xi_y, \xi_{y'}\rangle\big|\colon d\big(y, y'\big) \geqslant S,\ y, y' \in X_v,\ v \in V\big\} = 0. \tag{10}$$

Finally, for every s-chain $\mathbf{x} = (x_0, x_1, \ldots, x_{s-1})$, define the unit vector $\eta^{\mathbf{x}} \in \mathcal{H}$ by

$$\eta^{\mathbf{x}} = \sqrt{1/s} \sum_{i=0}^{s-1} \xi_{x_i}. \tag{11}$$

Similarly, Dadarlat and Guentner [5] define unit vectors $\eta^{\mathbf{x}}$ (see their formula (30)), where instead of simply summing the unit vectors ξ_{x_i}, they define numbers c_i and sum the weighted vectors $c_i \xi_{x_i}$. The numbers c_i are introduced to take the distances between the consecutive elements x_i, x_{i+1} into account. Since we assume that F and $\theta(F)$ are of finite index in H, we can discard them.

Our initial goal is to prove Proposition 3.15, namely that the vectors $\eta^{\mathbf{x}}$ satisfy properties similar to those of Proposition 3.1.

Lemma 3.12. *Let $\mathbf{x} = (x_0, x_1, \ldots, x_{s-1})$ and $\mathbf{x}' = (x'_0, x'_1, \ldots, x'_{s-1})$ be s-chains starting in X_v. If $d(x_0, x'_0) \leqslant n$, then $\|\eta^{\mathbf{x}} - \eta^{\mathbf{x}'}\| \leqslant \frac{\epsilon}{2(R+1)}$.*

Proof. We write

$$\|\eta^{\mathbf{x}} - \eta^{\mathbf{x}'}\| = \left\| \sqrt{1/s} \sum_{i=0}^{s-1} (\xi_{x_i} - \xi_{x'_i}) \right\|. \tag{12}$$

Since by the triangle inequality $d(x_i, x'_i) \leqslant n + 2i(Z+2) \leqslant n + 2s(Z+2)$, we can bound (12) by

$$\sup_{0 \leqslant i \leqslant s-1} \|\xi_{x_i} - \xi_{x'_i}\| \leqslant \sup_{\Delta} \|\xi_y - \xi_{y'}\| \leqslant \frac{\epsilon}{2(R+1)},$$

where $\Delta = \{(y, y') \mid d(y, y') \leqslant n + 2s(Z+2),\ y, y' \in X_v,\ v \in V\}$. $\square$

Lemma 3.13. *Let* $\mathbf{x} = (x_1, \ldots, x_s)$ *and* $\mathbf{x}' = (x_0, x'_1, x'_2, \ldots, x'_{s-1})$ *be* s*-chains with* $x_0 \in X_v$ *and* $x_1 \in X_{\alpha(v)}$. *If* (x_0, x_1) *is a* 2*-chain and* $d(x_0, x_1) \leqslant n$, *then* $\|\eta^{\mathbf{x}} - \eta^{\mathbf{x}'}\| \leqslant \frac{\epsilon}{R+1}$.

Proof. Denote $\bar{x}_0 = f_v^{-1}(x_1)$ and set $\bar{\mathbf{x}} = (\bar{x}_0, x_1, \ldots, x_{s-1})$. Then

$$\eta^{\mathbf{x}} = \eta^{(x_1, x_2, \ldots, x_s)} = \sqrt{1/s} \left(\sum_{i=1}^{s-1} \xi_{x_i} + \xi_{x_s} \right),$$

$$\eta^{\bar{\mathbf{x}}} = \eta^{(\bar{x}_0, x_1, \ldots, x_{s-1})} = \sqrt{1/s} \left(\xi_{\bar{x}_0} + \sum_{i=1}^{s-1} \xi_{x_i} \right).$$

Therefore,

$$\|\eta^{\bar{\mathbf{x}}} - \eta^{\mathbf{x}}\| = \sqrt{1/s} \|\xi_{\bar{x}_0} - \xi_{x_s}\| = \sqrt{2/s} \leqslant \frac{\epsilon}{2(R+1)},$$

where the final inequality comes from the choice of s in Eq. (6). Since $d(x_0, \bar{x}_0) = d(x_0, x_1) - 1 \leqslant n$, we can apply Lemma 3.12 to the chains $\mathbf{x}'$ and $\bar{\mathbf{x}}$ to conclude that

$$\|\eta^{\mathbf{x}} - \eta^{\mathbf{x}'}\| \leqslant \|\eta^{\mathbf{x}} - \eta^{\bar{\mathbf{x}}}\| + \|\eta^{\bar{\mathbf{x}}} - \eta^{\mathbf{x}'}\| \leqslant \frac{\epsilon}{2(R+1)} + \frac{\epsilon}{2(R+1)} = \frac{\epsilon}{R+1}. \qquad \square$$

Lemma 3.14. *For any* 2 s*-chains* $\mathbf{x} = (x_0, x_1, \ldots, x_{s-1})$ *and* $\mathbf{x}' = (x'_0, x'_1, \ldots, x'_{s-1})$,

$$\left|\langle \eta^{\mathbf{x}}, \eta^{\mathbf{x}'} \rangle\right| \leqslant \sup\left\{ \left|\langle \xi_y, \xi_{y'} \rangle\right| :\ y, y' \in X_v,\ v \in T,\ d(y, y') \geqslant d(x_0, x'_0) - 2s(Z+2) \right\}.$$

Proof. Assume that $x_0 \in X_{\tilde{v}}$ and $x'_0 \in X_{v'}$. As before, denote (k, l) the unique pair of natural numbers such that $d_T(\tilde{v}, v') = k + l$ and $\alpha^k(\tilde{v}) = \alpha^l(v')$. By symmetry, we will assume that $k \geqslant l$. Further, we will assume that $k < s$, because $k \geqslant s$ implies that $\langle \eta^{\mathbf{x}}, \eta^{\mathbf{x}'} \rangle = 0$. We obtain by definition that

$$\langle \eta^{\mathbf{x}}, \eta^{\mathbf{x}'} \rangle = (1/s) \sum_{i=0}^{s-k-1} \langle \xi_{x_{k+i}}, \xi_{x'_{l+i}} \rangle.$$

Notice that $d(x_{k+i}, x'_{l+i}) \geqslant d(x_0, x'_0) - (k+i)(Z+2) - (l+i)(Z+2) \geqslant d(x_0, x'_0) - 2s(Z+2)$, so

$$\left|\langle \eta^{\mathbf{x}}, \eta^{\mathbf{x}'} \rangle\right| \leqslant \sup_{\Omega} \left|\langle \xi_y, \xi_{y'} \rangle\right|,$$

where $\Omega = \{(y, y')\colon\ y, y' \in X_v,\ v \in T,\ d(y, y') \geqslant d(x_0, x'_0) - 2s(Z+2)\}$. $\square$

Proposition 3.15. *Given $R > 0$ and $\epsilon > 0$, let s and $(\xi_x)_{x \in G}$ be constructed as in Notation* 3.11. *For each $x_0 \in G$, choose and fix an s-chain $\mathbf{x} = (x_0, x_1, \ldots, x_{s-1})$ and consider the corresponding vector $\eta^{\mathbf{x}} = \eta^{(x_0, x_1, \ldots, x_{s-1})}$. Then*

$$\sup\left\{\left\|\eta^{\mathbf{x}} - \eta^{\mathbf{x}'}\right\|\colon\ d\left(x_0, x'_0\right) < R\right\} \leqslant \epsilon, \tag{13}$$

and

$$\left|\langle \eta^{\mathbf{x}}, \eta^{\mathbf{x}'} \rangle\right| \leqslant \sup\left\{\left|\langle \xi_y, \xi_{y'} \rangle\right|\colon\ y, y' \in X_v,\ v \in V,\ d\left(y, y'\right) \geqslant d\left(x_0, x'_0\right) - 2s(Z+2)\right\}. \tag{14}$$

Proof. Condition (14) was proven in Lemma 3.14. To prove (13), let us choose $x_0, x'_0 \in G$ such that $d(x_0, x'_0) < R$. Choose any two s-chains $\mathbf{x}$ and $\mathbf{x}'$ starting at x_0 and x'_0, respectively. We want to prove that $\|\eta^{\mathbf{x}} - \eta^{\mathbf{x}'}\| \leqslant \epsilon$.

Therefore, let k and l be as before. Take chains $(x_0, x_1, \ldots, x_k)$ and $(x'_0, x'_1, \ldots, x'_l)$. By Lemma 3.10, we have that

$$\max\left\{\left(\sup_{0 \leqslant i \leqslant k-1} d(x_i, x_{i+1})\right), \left(\sup_{0 \leqslant j \leqslant l-1} d\left(x'_j, x'_{j+1}\right)\right), d\left(x_k, x'_l\right)\right\} < (Z+3)R \leqslant n,$$

where the last inequality follows from Eq. (6). Consequently, we can apply Lemma 3.12 and Lemma 3.13 repeatedly to s-chains $\mathbf{x}(i)$ and $\mathbf{x}'(j)$ whose initial elements are x_i, $i = 0 \ldots k$ and x'_j, $j = 0 \ldots l$, respectively. We obtain

$$\begin{aligned}\left\|\eta^{\mathbf{x}} - \eta^{\mathbf{x}'}\right\| &\leqslant \sum_{i=0}^{k-1} \left\|\eta^{\mathbf{x}(i)} - \eta^{\mathbf{x}(i+1)}\right\| + \left\|\eta^{\mathbf{x}(k)} - \eta^{\mathbf{x}'(l)}\right\| + \sum_{j=0}^{l-1} \left\|\eta^{\mathbf{x}'(j)} - \eta^{\mathbf{x}'(j+1)}\right\| \\ &\leqslant \frac{(k+l+1)\epsilon}{R+1} \leqslant \epsilon. \qquad \square\end{aligned}$$

We are now ready to conclude the proof of Theorem 3.6.

Proof of Theorem 3.6. Clearly $\alpha \leqslant \alpha_1$ since (H, d_{in}) embeds isometrically in (G, d).

Conversely, assume that $\alpha_1 > 0$ and fix any real number $0 < p < \alpha_1$. Next, choose $C > 0$ and $D \geqslant 0$ such that there exists a uniform embedding f of (H, d_{in}) in a Hilbert space satisfying

$$\begin{aligned}\rho_-\left(d_{in}(x, y)\right) &:= (1/C) d_{in}(x, y)^{\alpha_1 - p} - D \leqslant d\left(f(x), f(y)\right) \leqslant C d_{in}(x, y) + D \\ &:= \rho_+\left(d_{in}(x, y)\right),\end{aligned}$$

for every $x, y \in H$. For each $m \in \mathbb{N}_0$, define $\epsilon_m = \frac{1}{m^{1/2+p}}$, $R_m = \sqrt{m}$ and define $n_m = m^{1/2+p}$ and $s_m = m^{2+4p}$. Clearly then

$$n_m \geqslant (Z+3)R_m, \qquad \sqrt{2/s_m} \leqslant \frac{\epsilon_m}{2(R_m+1)}, \tag{15}$$

whenever m is larger than some natural number r_p. Next, use Lemma 3.2 to find unit vectors $\{\xi_y \mid y \in H\}$ in some Hilbert space $\overline{\mathcal{H}}$ such that

$$\|\xi_y - \xi_{y'}\| \leqslant \frac{\epsilon_m}{2(R_m+1)} = \frac{1}{2m^{1/2+p}(\sqrt{m}+1)} =: \bar{\epsilon}_m$$
$$\text{when } d\big(y, y'\big) \leqslant n_m + 2s_m(Z+2) := \overline{R}_m.$$

We should denote these vectors by ξ_y^m, but we drop the upper index to lighten notation. The so obtained vectors also satisfy

$$\big|\langle \xi_y, \xi_{y'}\rangle\big| \leqslant \left(1 - \frac{1}{2}\bar{\epsilon}_m^2\right)^{\frac{\rho_-(d(y,y'))^2}{(C\overline{R}_m+D)^2}} = \left(1 - \frac{1}{2}\bar{\epsilon}_m^2\right)^{\frac{((1/C)d(y,y')^{\alpha_1-p}-D)^2}{(C\overline{R}_m+D)^2}}.$$

Therefore, $|\langle \xi_y, \xi_{y'}\rangle| \leqslant 1/2$ whenever

$$\left(1 - \frac{1}{8m^{1+2p}(\sqrt{m}+1)^2}\right)^{\frac{((1/C)d(y,y')^{\alpha_1-p}-D)^2}{(C\overline{R}_m+D)^2}} \leqslant 1/2,$$

and so whenever

$$\frac{((1/C)d(y,y')^{\alpha_1-p} - D)^2}{(C\overline{R}_m + D)^2} \geqslant \frac{-\ln(2)}{\ln(1 - \frac{1}{8m^{1+2p}(\sqrt{m}+1)^2})}.$$

Using the fact that $\lim_{m\to\infty} \frac{-1}{\ln(1-\frac{1}{8m^{1+2p}(\sqrt{m}+1)^2})} \frac{1}{8m^{2+3p}} = 0$, we see that for m larger than some natural number $\tilde{r}_p$, the above inequality is true if

$$d\big(y, y'\big)^{\alpha_1-p} \geqslant \sqrt{\ln(2)8m^{2+3p}}(C\overline{R}_m + D)C + CD.$$

We conclude that for m larger than $\delta(p) := \max(r_p, \tilde{r}_p)$, Eq. (15) holds and

$$\big|\langle \xi_y, \xi_{y'}\rangle\big| \leqslant 1/2 \quad \text{whenever} \quad d\big(y, y'\big) \geqslant m^{\frac{3+6p}{\alpha_1-p}}.$$

Denote $S_m = m^{\frac{3+6p}{\alpha_1-p}}$. For every $x_0, x_0' \in G$ and for every $m \in \mathbb{N}_0$ larger than $\delta(p)$, Proposition 3.15 gives vectors $(\eta_m^{\mathbf{x}}), (\eta_m^{\mathbf{x}'}) \in \mathcal{H} := \bigoplus_{v\in V} \overline{\mathcal{H}}$ where $\mathbf{x}$ and $\mathbf{x}'$ are s_m-chains starting in x_0 and x_0' respectively. Moreover,

$$\sup\big\{\big\|\eta_m^{\mathbf{x}} - \eta_m^{\mathbf{x}'}\big\|\colon d\big(x_0, x_0'\big) < R_m\big\} \leqslant \epsilon_m$$

and

$$\left|\left\langle \eta_m^{\mathbf{x}}, \eta_m^{\mathbf{x}'}\right\rangle\right| \leqslant 1/2 \quad \text{whenever} \quad d\left(x_0, x_0'\right) \geqslant S_m + 2s_m(Z+2).$$

Denote $S_m' = S_m + 2s_m(Z+2)$, which is smaller than $m^{\frac{3+7p}{\alpha_1 - p}}$ when m is sufficiently large. By Lemma 3.3, we see that the compression of G is at least $\alpha_1/6$. □

Corollary 3.16. *Let H be a group, equipped with a length function l_1 and let G be an HNN-extension* $\mathrm{HNN}(H, F, \theta)$ *where both F and $\theta(F)$ are finite index subgroups of H. Equip G with the length function l from Construction* 3.4 *and assume that (G, l) is quasi-geodesic as a metric space (e.g. if H is finitely generated and l_1 is the word length metric relative to a finite symmetric generating subset). Equip H with the induced metric d_{in} from G. Then,*

$$\alpha_1/3 \leqslant \alpha \leqslant \alpha_1,$$

where α_1 and α denote the Hilbert space compressions of (H, d_{in}) and (G, d) respectively.

Proof. First, all of the previous lemmas stay valid. Then, instead of taking $R_m = \sqrt{m}$ in the proof above, we now take $R_m = \ln(m)$. This way, we get $n_m = m^p$ and $s_m = m^{1+4p}$. Consequently, $\bar{\epsilon}_m$ becomes equal to $\frac{1}{2m^{1/2+p}(\ln(m)+1)}$. Going through the previous proof, exactly as before, we obtain finally that $S_m = m^{\frac{3/2+6p}{\alpha_1 - p}}$ and by Lemma 3.3 for quasi-geodesic spaces, we obtain that the compression of G is at least $\alpha_1/3$. □

4. Equivariant Hilbert space compression for a free product of groups

We now investigate the behavior of the equivariant Hilbert space compression under an amalgamated free product $G := G_1 *_F G_2$, taken over a finite group F. Throughout this section, we only deal with discrete groups G, and all of the length functions l under consideration will be uniformly discrete, i.e. $\inf\{l(x) \mid x \in G \setminus \{1\}\} > 0$.

In order to relate the equivariant compression of the amalgamated free product $G = G_1 *_F G_2$ to the equivariant compressions of the factors G_1 and G_2, one needs a length function on G which is somehow related to the length functions on G_1 and G_2. Similarly as in Section 2, we define the length of a word in $G_1 * G_2$, but because the product now is amalgamated, there is more than 1 way to represent an element by a word in $G_1 * G_2$. We define the length of an element of G as the infimum of the lengths of all words representing the element (cf. Construction 3.4). Specifically, we obtain

Construction 4.1. Assume that G_1 and G_2 are countable groups equipped with (always uniformly discrete) length functions l_1 and l_2 respectively. Each element $x \in G$ can be represented by a collection of words $(a_1, a_2, a_3, \ldots, a_n)$ where n runs over the natural numbers, where $x = \prod_{i=1}^n a_i$ and where the a_i belong alternately to G_1 and G_2. We define the length of a word as the sum $l_1(a_1) + l_2(a_2) + l_1(1_3) + \cdots + l_{1,2}(a_n)$ where $l_{1,2}$ is l_1 or l_2 as appropriate. We define the length of x as the minimum of the lengths of all words representing x.

Remark 4.2. If G_1 and G_2 are finitely generated groups equipped with the word length relative to finite symmetric generating subsets S and T respectively, then the above construction equips $G_1 *_F G_2$ with the word length metric relative to $S \cup T$.

A major advantage of this construction is the following

Proposition 4.3. *Let G_1 and G_2 be discrete groups, equipped with uniformly discrete length functions, and equip $G_1 *_F G_2$ with the length function from Construction* 4.1. *If F is finite, then G_1 and G_2 embed bi-Lipschitzly into the amalgamated free product $G_1 *_F G_2$.*

Proof. Let us show that G_1 embeds bi-Lipschitzly into G; the case of G_2 is analogous. Denote the length functions on G_1 and G_2 by l_1 and l_2 respectively and assume w.l.o.g. that $\inf(l_1(x) \mid x \in G_1) \geqslant 1$ and $\inf(l_2(x) \mid x \in G_2) \geqslant 1$. Let l be the length function from Construction 4.1 and also define the length $\tilde{l}$ of words in $G_1 * G_2$ as in Construction 4.1.

Clearly, $l_{|G_1} \leqslant l_1$. Conversely, fix $x \in G_1$ and choose a reduced word $w = (s_1, s_2, \ldots, s_n) \neq (x)$ in $G_1 * G_2$ representing it. By a well-known analogue of Britton's lemma for amalgamated free products (see Theorem 11, 5.2 in [14]), there is at least one s_i which belongs to F. Take s_i and merge it with its neighbor(s) to get a new word w_1 containing strictly less letters. It is clear that $\tilde{l}(w_1) \leqslant \tilde{l}(w) + M$ where $M = \max(l(f) \mid f \in F)$. We can now continue the same reasoning over and over, each time obtaining a word with strictly less letters than the previous one. This can be done at most $\tilde{l}(w)$ times. One finally obtains the word (x) whose length is $l_1(x)$. We conclude $l_1(x) \leqslant \tilde{l}(w) + \tilde{l}(w)M$ and so $l_{|G_1} \leqslant l_1 \leqslant (M+1)l_{|G_1}$. □

We proceed with some basic facts and some observations related to isometric actions of discrete groups on Hilbert spaces.

4.1. On isometric actions

Any affine isometric action χ of a discrete group H on a Hilbert space $\mathcal{H}$ can be written as

$$\chi(x)v = \pi(x)v + b(x) \quad \forall v \in \mathcal{H},\ \forall x \in H,$$

where $\pi : H \to \mathcal{O}(\mathcal{H})$ is a group homomorphism from H to the orthogonal group of $\mathcal{H}$ and where $b : H \to \mathcal{H}$ is a map satisfying the 1-cocycle identity, i.e.

$$b(xy) = \pi(x)\big(b(y)\big) + b(x) \quad \forall x, y \in H.$$

We summarize some standard properties of 1-cocyles in the lemma below.

Lemma 4.4. *Let H be a discrete group. Assume as above that χ is an affine isometric action with associated* 1*-cocycle $b : H \to \mathcal{H}$. Then,*

1. $\|b(x) - b(y)\| = \|b(y^{-1}x)\|\ \forall x, y \in H$,
2. *if $f : H \to \mathcal{H}$ is an H-equivariant map relative to χ and the left multiplication action by H on itself, then the compression of b equals the compression of f, i.e. for a given $r > 0$, there exist $C, D > 0$ such that $\frac{1}{C} l_H(x)^r - D \leqslant \|f(x)\|$, for all $x \in H$ if and only if there exist $C', D' > 0$ such that $\frac{1}{C'} l_H(x)^r - D' \leqslant \|b(x)\|$ for all $x \in H$.*

Proof. We leave the proof as an exercise to the reader. □

Definition 4.5. A function $\psi : H \to \mathbb{R}^+$ is *conditionally negative definite* if $\psi(x) = \psi(x^{-1})$ for all $x \in H$, $\psi(1) = 0$, and

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \psi\left(x_i^{-1} x_j\right) \leqslant 0,$$

for all $n \in \mathbb{N}_0$, $x_1, x_2, \ldots, x_n \in H$ and all $a_1, a_2, \ldots, a_n \in \mathbb{R}$ with $\sum_{i=1}^{n} a_i = 0$.

The proofs of the following results can be found in [10, p. 63] and [4, Lemma 6.2.1] respectively.

Theorem 4.6. *Assume that $\psi : H \to \mathbb{R}^+$ is a conditionally negative definite function on a discrete group H. There exists a real Hilbert space $\mathcal{H}_\psi$, an orthogonal representation $\pi_\psi : H \to \mathcal{O}(\mathcal{H})$ and a 1-cocycle b_ψ associated to π_ψ such that*

$$\psi(x) = \left\|b_\psi(x)\right\|^2,$$

for every $x \in H$.

Conversely, if b is a 1-cocycle relative to some affine isometric action of H on a Hilbert space, then the map $\psi : x \mapsto \|b(x)\|^2$ is conditionally negative definite.

Lemma 4.7. *Let F be a finite subgroup of the discrete group H. If ψ is a conditionally negative definite function on H, then there exists a conditionally negative definite function ψ' on H such that*

1. *ψ' is F-bi-invariant (that is,*

$$\psi'\left(fxf'\right) = \psi'(x) \quad \forall x \in H,\ \forall f, f' \in F);$$

2. *$\psi'(f) = 0\ \forall f \in F$, and $\psi'(x) \geqslant 1$ for all $x \in H \setminus F$;*
3. *$\psi - \psi'$ is bounded.*

Notice that from 2 in Lemma 4.4, Theorem 4.6 and 2, 3 of Lemma 4.7 (take $F = \{1\}$), it follows that the equivariant compression of a group H is the supremum of all $\epsilon \in [0, 1]$ such that there exist $C > 0$ and a 1-cocycle b, satisfying

$$(1/C)l(x)^\epsilon \leqslant \left\|b(x)\right\| \leqslant Cl(x) \quad \forall x \in H. \tag{16}$$

In fact, together with Proposition 4.3, this is the only time that we will use the uniform discreteness of the length functions. We need one more

Lemma 4.8. *Denote H any discrete group equipped with some length function l and let F be a finite normal subgroup of H. If we define the length of an element $\bar{x}$ of H/F as the minimum of $l(y)$ where $y \in xF$, then the equivariant Hilbert space compressions of H and H/F are equal.*

Proof. Given a 1-cocycle $b : H \to \mathcal{H}$ which is Lipschitz, we get a conditionally negative definite map $\psi : x \mapsto \|b(x)\|^2$. By Lemma 4.7, there exists a conditionally negative definite function, ψ', at bounded distance from ψ such that

1. ψ' is F-bi-invariant;
2. $\psi'(f) = 0\ \forall f \in F$, and $\psi'(x) \geqslant 1$ for all $x \in H \setminus F$.

This implies that the map ψ' is in fact a conditionally negative definite function on H/F. The 1-cocycle b' associated to ψ' by Theorem 4.6 has the same compression as b and is again a Lipschitz map.

Conversely, starting with a Lipschitz 1-cocycle $b' : H/F \to \mathcal{H}$, we obtain a conditionally negative definite function $\psi'(\bar{x}) = \|b'(\bar{x})\|^2$. Define $\psi : H \to \mathbb{R}^+$ by setting $\psi(x) = \psi'(\bar{x})$. This map is clearly conditionally negative definite and so Theorem 4.6 associates a 1-cocycle b to it. It is clear that the compressions of b and b' are again equal and that b is Lipschitz. □

4.2. Main result

Theorem 4.9. *Let G_1 and G_2 be discrete groups equipped with uniformly discrete length functions l_1 and l_2 respectively. Denote the equivariant Hilbert space compressions of G_1 and G_2 by α_1 and α_2 respectively. Denote $G = G_1 *_F G_2$ an amalgamated free product where F is a finite subgroup of both G_1 and G_2 and equip G with a uniformly discrete length function as in Construction* 4.1. *If α denotes the equivariant Hilbert space compression of G, then*

1. $\alpha = 1$ *if F is of index* 2 *in both G_1 and G_2,*
2. $\alpha = \alpha_1$ *if $F = G_2$ and $\alpha = \alpha_2$ if $F = G_1$,*
3. $\alpha = \min(\alpha_1, \alpha_2, 1/2)$ *otherwise.*

Proof. Let us first focus our attention to the easy case where F is of index 2 in both G_1 and G_2. These assumptions imply that F is a normal subgroup of both G_1 and G_2 and so it is a normal subgroup of G with quotient $G_1/F * G_2/F = \mathbb{Z}_2 * \mathbb{Z}_2$. Lemma 4.8 implies that $\alpha = 1$.

Case 2 follows from Proposition 4.3.

To prove 3, note that F is of index at least 3 in G_1 or in G_2 and that $G_1 \neq F \neq G_2$. Assume that F is of index at least 3 in G_2 and take $x \in G_1 \setminus F$ and $y_1, y_2 \in G_2 \setminus F$ with $y_1 F \neq y_2 F$. A well-known analogue of Britton's lemma for amalgamated free products (see Theorem 11, 5.1 in [14]), shows immediately that the group generated by $S_{F_2} = \{(xy_1)^2, (xy_1)^{-2}, (xy_2)^2, (xy_2)^{-2}\}$ generates the free group F_2. It also shows that if you multiply an element of $\langle S_{F_2} \rangle$ ending on $s \in S_{F_2}$ by $s_1 \in S_{F_2} \setminus \{s^{-1}\}$, that you go further away from the vertex G_1 in the Bass–Serre tree. By uniform discreteness of l_1 and l_2, this implies that the subspace length on $\langle S_{F_2} \rangle < G$ is greater than a fixed multiple of the word length relative to S_{F_2}. Clearly, we have that the subspace metric on $\langle S_{F_2} \rangle < G$ is smaller than the word length metric on $\langle S_{F_2} \rangle$ times $\max(l(s) \mid s \in S_{F_2})$. We conclude that the equivariant compression of G is bounded by that of $\langle S_{F_2} \rangle \cong F_2$, namely $1/2$. Using Proposition 4.3, we conclude finally that $\alpha \leqslant \min(\alpha_1, \alpha_2, 1/2)$.

Conversely, assume $0 \leqslant \epsilon < \min(\alpha_1, \alpha_2, 1/2)$. There exist $C > 0$ and conditionally negative definite functions $\psi_i : G_i \to \mathbb{R}^+$ for $i \in \{1, 2\}$, such that

1. ψ_i is F-bi-invariant;
2. $\psi_i(f) = 0\ \forall f \in F$ and $\psi_i(x) \geqslant 1$ for all $x \in G_i \setminus F$;
3. the 1-cocycle b_i associated to ψ_i satisfies $(1/C) l_i(g)^\epsilon \leqslant \|b_i(g)\| \leqslant C l_i(g)\ \forall g \in G_i \setminus \{F\}$.

Let R and S be sets of representatives for the left cosets of F in G_1 and G_2 respectively. Assume $1_{G_1} \in R$, $1_{G_2} \in S$ and denote elements of R, S and F by α_i, β_j and f respectively.

According to [14], every element $x \in G$ has a unique normal form:

$$x = \alpha_1\beta_1\alpha_2\beta_2\ldots\alpha_k\beta_k f, \tag{17}$$

where k is as small as possible and might be 0. In [4] (see the proof of Proposition 6.2.3), it is shown that the map $\psi : G \to \mathbb{R}^+$, defined by

$$\psi(x) = \sum_{i=1}^{k} \psi_1(\alpha_i) + \sum_{j=1}^{k} \psi_2(\beta_j),$$

is a conditionally negative definite function on G. Application of Theorem 4.6 gives an affine isometric action of G on a Hilbert space $\mathcal{H}$ with 1-cocycle b satisfying $\|b(x)\|^2 = \psi(x)$. Choose $x \in G$, and write $x = \alpha_1\beta_1\alpha_2\beta_2\ldots\alpha_l\beta_l f$ in normal form as above. We obtain that

$$\begin{aligned}\|b(x)\|^2 &= \psi(x)\\ &= \sum_{i=1}^{k} \psi_1(\alpha_i) + \sum_{j=1}^{k} \psi_2(\beta_j)\\ &= \sum_{i=1}^{k} \|b_1(\alpha_i)\|^2 + \sum_{j=1}^{k} \|b_2(\beta_j)\|^2.\end{aligned}$$

Consequently,

$$\|b(x)\| \leqslant C\left[\sum_{i=1}^{k} l_1(\alpha_i) + \sum_{j=1}^{k} l_2(\beta_j)\right]. \tag{18}$$

Abusing the word "length", we now introduce $l^{SB}(x)$, the *shortest blocklength* of an element $x \in G \setminus F$. By definition, it denotes the minimum of the lengths of all words representing x which are of the form $(\gamma_1, \delta_1, \gamma_2, \ldots, \gamma_k, \delta_k)$ where k is as in Eq. (17) and where the γ_i and δ_i belong to G_1 and G_2 respectively. Take such a word, say $(\gamma_1, \delta_1, \gamma_2, \ldots, \gamma_k, \delta_k)$, representing x. It follows from Bass–Serre theory that $\gamma_i \in F\alpha_i F$ and $\delta_i \in F\beta_i F$ for all $i \in \{1, 2, \ldots, k\}$. Using Eq. (18), we obtain

$$\|b(x)\| \leqslant C\big[l^{SB}(x) + 4Mk\big] \leqslant C\left(\frac{4M}{M'} + 1\right) l^{SB}(x),$$

where $M = \max\{l_i(c) \mid c \in C,\ i = 1, 2\}$ and $M' = \min(1, \min\{l_1(\gamma) \mid \gamma \in G_1 \setminus \{1\}\}, \min\{l_2(\delta) \mid \delta \in G_2 \setminus \{1\}\})$.

If $B \geqslant 1$ is such that $\forall g \in G_1$: $l_1(g) \leqslant Bl(g)$ and $\forall g \in G_2$: $l_2(g) \leqslant Bl(g)$, then $l^{SB} \leqslant Bl$. Indeed, take a word $(s_1, s_2, \ldots, s_m)$ representing x where the s_i belong alternately to G_1 and G_2. This word gives a path in the Bass–Serre tree T, namely

$$G_1 \to s_1G_2 \to s_1s_2G_1 \to \cdots.$$

Assume that the last time that this path passes through G_1 is at the ith vertex, i.e. $G_1 = s_1 s_2 s_3 \dots s_{i-1} G_1$. Then the sum

$$\begin{aligned} l_1(s_1) + l_2(s_2) + \dots + l_2(s_{i-1}) + l_1(s_i) &\geqslant l(s_1 s_2 \dots s_i) \\ &\geqslant \frac{1}{B} l_1(s_1 s_2 \dots s_{i-1} s_i), \end{aligned}$$

since $s_1 s_2 \dots s_i \in G_1$. The next vertex is $s_1 s_2 \dots s_i G_2$, and again you can look at the last time that this path passes through this vertex. Following the same reasoning, we obtain finally that

$$l_1(s_1) + l_2(s_2) + \dots + l_{1,2}(s_n) \geqslant \frac{1}{B} l^{SB}(x),$$

so that $l^{SB} \leqslant Bl$. We conclude that

$$\|b(x)\| \leqslant \left(\frac{4M}{M'} + 1\right) BCl(x) \quad \forall x \in G,$$

and so b is Lipschitz.

Conversely, we have

$$\|b(x)\|^2 \geqslant \sum_{i=1}^{l} (1/C^2) l_1(\alpha_i)^{2\epsilon} + \sum_{j=1}^{l} (1/C^2) l_2(\beta_j)^{2\epsilon}.$$

Since $\forall a, b \in \mathbb{R}^+$: $a^{2\epsilon} + b^{2\epsilon} \geqslant (a+b)^{2\epsilon}$, we obtain

$$\|b(x)\|^2 \geqslant (1/C^2) \left(\sum_{i=1}^{l} l_1(\alpha_i) + \sum_{j=1}^{l} l_2(\beta_j) \right)^{2\epsilon}.$$

Setting $\overline{M} = \sup_{f \in F}\{l(f)\}$, we obtain

$$\|b(x)\| \geqslant (1/C)\big(l(x) - \min\big(l(x), \overline{M}\big)\big)^{\epsilon} \geqslant (1/C') l(x)^{\epsilon} - D', \tag{19}$$

for some $C' > 0$, $D' \geqslant 0$. It follows that the equivariant compression of G is also greater or equal than $\min(\alpha_1, \alpha_2, 1/2)$. □

5. Equivariant Hilbert space compression for HNN-extensions over finite groups

We continue by proving a similar result for HNN-extensions $\mathrm{HNN}(H, F, \theta)$ where F is finite. Throughout this section, all groups will be discrete. We put no assumption on the length function l_H of H, but the length function on G must be that of Construction 3.4. We now obtain a proposition similar to Proposition 4.3.

Proposition 5.1. *Let $G := \mathrm{HNN}(H, F, \theta)$ be an HNN-extension where F is finite. Then H embeds quasi-isometrically into G if G is equipped with the length function from Construction* 3.4.

Proof. Denote l_H the length function on H, l the length function on G defined in Construction 3.4 and $\tilde{l}$ the length of an element of $H * \mathbb{Z}$ as defined in Construction 3.4. Clearly $l_{|H} \leqslant l_H$. Conversely, choose an element $x \in H$ and represent it by a reduced word w in $H * \mathbb{Z}$. If w contains more than one letter, then Britton's lemma implies that it contains some relation of the form $t^{-1}yt$ with $y \in F$ or tyt^{-1} with $y \in \theta(F)$. Replacing this relation by the corresponding element of $\theta(F)$ or F respectively and then reducing, we obtain a word $\tilde{w}$ in $H * \mathbb{Z}$ such that $\tilde{l}(\tilde{w}) \leqslant \tilde{l}(w) + M$ where M equals $\max(l(z) \mid z \in F \cup \theta(F))$. Continuing in this fashion, one removes all the powers of t in less than $l(w)$ steps and so $l_H \leqslant l(M+1)$. This implies that $l_{|H} \leqslant l_H \leqslant (M+1)l_{|H}$. $\square$

The above lemma implies that the equivariant compression of H is an upper bound for the equivariant compression of G. The proof of the following lemma is very basic and we leave it as an exercise.

Lemma 5.2. *Let v, w be vectors in a Hilbert space $\mathcal{H}$. Then*

$$\|v+w\|^2 \geqslant \frac{1}{2}\|v\|^2 - \|w\|^2.$$

Theorem 5.3. *Let H be a discrete group equipped with a length function and denote its equivariant Hilbert space compression by α_1. Assume that F is a finite subgroup of H. Equip $G := \mathrm{HNN}(H, F, \theta)$ with the length function from Construction* 3.4. *The equivariant Hilbert space compression α of $G := \mathrm{HNN}(H, F, \theta)$ satisfies*

1. $\alpha = 1$ *whenever* $F = H$,
2. $\alpha = \min(\alpha_1, 1/2)$ *otherwise.*

The first claim follows trivially from Lemma 4.8, but we have added it for completeness.

Remark 5.4. Originally, we proved the above result under the additional assumption that $F \cup \theta(F)$ generates a finite group, which is for example true if F is a finite *normal* subgroup of H. We are thankful to Stefaan Vaes for explaining us how this assumption can be removed. Instead of using conditionally negative definite maps, the main idea is to start from a 1-cocycle on H and explicitly construct a 1-cocycle on G of which we estimate the compression.

Proof of Theorem 5.3. Since $F \neq H \neq \theta(F)$, we can choose $b \in H \setminus F$ and $c \in H \setminus \theta(F)$. Denote $S_{F_2} = \{t^{-1}b, (t^{-1}b)^{-1}, tc, (tc)^{-1}\}$. Using simple Bass–Serre theory, one shows that the subgroup $\langle S_{F_2} \rangle$ of G is free of order 2. Indeed, if you multiply an element $s \in S_{F_2}$ with an element from $S_{F_2} \setminus \{s^{-1}\}$, you obtain a point outside H. So, if one multiplies a product of elements of S_{F_2} ending on $s \in S_{F_2}$ by $s_1 \in S_{F_2} \setminus \{s^{-1}\}$, then the step you take in the Bass–Serre tree brings you further away from the vertex H. This shows that the subgroup $\langle S_{F_2} \rangle$ of G has no non-trivial relations. Denoting the word length metric on F_2 relative to S_{F_2} by l_{F_2}, one observes that for all $z \in \langle S_{F_2} \rangle < G$, the word length $l_{F_2}(z)$ is equal to the length of the geodesic path from H to zH in the Bass–Serre tree (where edges have length 1). This implies that $l_{F_2} \leqslant l$ and it is easy to see that l is smaller than $\max(l(t^{-1}b), l(tc))$ times l_{F_2}. This shows that F_2 embeds bi-Lipschitzly into G and so $\alpha^*(G) \leqslant \alpha^*(F_2) = 1/2$. Together with the fact that H embeds quasi-isometrically into G, we obtain that $\alpha^*(G) \leqslant \min(\alpha^*(H), 1/2)$. We now proceed to show that this upper bound is also a lower bound.

Denote the length function on H by l_H and that on G by l. Fix $0 < \epsilon < \min(\alpha_1, 1/2)$ and let $b : H \to \mathcal{H}$ be a 1-cocycle for H with respect to an orthogonal representation $\pi : H \to \mathcal{O}(\mathcal{H})$ satisfying

$$\frac{1}{C} l_H(h)^\epsilon - D \leqslant \|b(h)\| \leqslant C l_H(h) + D,$$

for all $h \in H$, where $C > 1$ and $D \geqslant 0$ are real numbers. We will construct a large-scale Lipschitz 1-cocycle b_{HNN} and $D' > 0$ such that

$$\|b_{\mathrm{HNN}}(x)\| \geqslant \frac{1}{4C} l(x)^\epsilon - D',$$

for all $x \in G$.

Denote by $t^{\mathbb{Z}} \subset G$ the cyclic subgroup generated by the stable letter t. The action of $t^{\mathbb{Z}} \times H$ on G by left–right (inverse) multiplication is free. So, we can choose a map $r : G \to H$ such that $r(1) = 1$ and $r(t^n x h) = r(x)h$ for all $n \in \mathbb{Z}$, $h \in H$, $x \in G$. Define the corresponding 1-cocycle $\omega : G \times G/H \to H$ by the formula $\omega(x, kH) = r(xk)r(k)^{-1}$ for all $x, k \in G$. Induce the representation π to a representation of G, concretely given as

$$\tilde{\pi} : G \to \mathcal{O}\big(l^2(G/H) \otimes \mathcal{H}\big): \quad \tilde{\pi}(x)(\delta_{kH} \otimes \xi) = \delta_{xkH} \otimes \pi\big(\omega(x, kH)\big)\xi,$$

for all $x, k \in G$, $\xi \in \mathcal{H}$.

One checks that for all $\xi \in \mathcal{H}$, we have

$$\begin{aligned}
\tilde{\pi}(h)(\delta_H \otimes \xi) &= \delta_H \otimes \pi(h)\xi \quad \text{for all } h \in H,\\
\tilde{\pi}(f)(\delta_{tH} \otimes \xi) &= \delta_{tH} \otimes \pi\big(\theta(f)\big)\xi \quad \text{for all } f \in F,\\
\tilde{\pi}(t)(\delta_{kH} \otimes \xi) &= \delta_{tkH} \otimes \xi \quad \text{for all } k \in G.
\end{aligned}$$

Using Lemma 4.7 and Theorem 4.6, we may assume that $b(f) = 0$ for all $f \in H$. Moreover, if we restrict the affine action $\pi + b$ to $\theta(F)$, then a vector is fixed since $\theta(F)$ is finite (namely $\frac{1}{|\theta(F)|} \sum_{f \in F} b(\theta(f)) := -\xi_0$ is fixed). We can thus find $\xi_0 \in \mathcal{H}$ such that $b(\theta(f)) = \pi(\theta(f))\xi_0 - \xi_0$ for all $f \in F$. Define a 1-cocycle $\tilde{b} : G \to l^2(G/H) \otimes \mathcal{H}$ relative to $\tilde{\pi}$ by

$$\tilde{b}(h) := \delta_H \otimes b(h) \quad \text{for all } h \in H \text{ and } \tilde{b}(t) := \delta_{tH} \otimes \xi_0.$$

In order to check that $\tilde{b}$ is well defined, it suffices to check that the defining relation $ft = t\theta(f)$ is respected. Now,

$$\tilde{b}(f) + \tilde{\pi}(f)\tilde{b}(t) = 0 + \tilde{\pi}(f)(\delta_{tH} \otimes \xi_0) = \delta_{ftH} \otimes \pi\big(\theta(f)\big)\xi_0 = \delta_{tH} \otimes \pi\big(\theta(f)\big)\xi_0.$$

On the other hand,

$$\tilde{b}(t) + \tilde{\pi}(t)\tilde{b}\big(\theta(f)\big) = \delta_{th} \otimes \xi_0 + \tilde{\pi}(t)\big(\delta_H \otimes \big(\pi\big(\theta(f)\big)\xi_0 - \xi_0\big)\big) = \delta_{tH} \otimes \pi\big(\theta(f)\big)\xi_0.$$

This implies that $\tilde{b}$ extends to a well-defined 1-cocycle.

Next, consider the Bass–Serre tree T associated to the HNN-extension G. The vertex set is G/H. Let $d_T(g_1H, g_2H)$ be the tree distance between vertices g_1H, g_2H. It is known that, on a tree, the distance is a conditionally negative definite kernel [10, Proposition 2 in §6.a] and the corresponding 1-cocycle satisfies $\|b_{BS}(x)\| = \sqrt{d_T(H, xH)}$ for all $x \in G$. We define b_{HNN} as the direct sum of $\tilde{b}$ and $\sqrt{E+1}b_{BS}$ where $E = 4\|\xi_0\|^2 + D^2$. In order to show that b_{HNN} is Lipschitz and has compression at least ϵ, we first investigate $\tilde{b}$.

Let $x = h_0t^{n_1}h_1t^{n_2}\dots h_k$ be a *reduced expression*. By this we mean that $n_1, \dots, n_k \in \mathbb{Z}_0$, $h_1, h_2, \dots, h_{k-1} \in H \setminus \{e\}$ and finally the following two conditions hold: if $n_i > 0$ and $n_{i+1} < 0$, then $h_i \notin \theta(F)$; if $n_i < 0$ and $n_{i+1} > 0$, then $h_i \notin F$. The 1-cocycle equality for $\tilde{b}$ implies that

$$\begin{aligned}\tilde{b}(x) = \tilde{b}(h_0) + \tilde{\pi}(h_0)\big(\tilde{b}\big(t^{n_1}\big)\big) + \tilde{\pi}\big(h_0t^{n_1}\big)\big(\tilde{b}(h_1)\big) + \tilde{\pi}\big(h_0t^{n_1}h_1\big)\big(\tilde{b}\big(t^{n_2}\big)\big) + \cdots \\ + \tilde{\pi}\big(h_0t^{n_1}h_1\dots t^{n_k}\big)\big(\tilde{b}(h_k)\big),\end{aligned} \tag{20}$$

$$\tilde{b}\big(t^n\big) = \sum_{i=1}^{n}(\delta_{t^iH} \otimes \xi_0) \quad \text{for } n > 0, \tag{21}$$

and

$$\tilde{b}\big(t^n\big) = -\sum_{i=n+1}^{0}(\delta_{t^iH} \otimes \xi_0) \quad \text{for } n < 0. \tag{22}$$

View $\tilde{b}(x)$ as a map from G/H to $\mathcal{H}$. Then the value of $\tilde{b}(x)$ in H equals $b(h_0)$ if $n_1 > 0$. If however, $n_1 < 0$, then Eq. (22) implies that the value at H is $b(h_0) - \pi(h_0)\xi_0$. For $i \leqslant k$, denote $x_i = h_0t^{n_1}h_1\dots t^{n_i}$. Similarly as above, one verifies that the value of $\tilde{b}(x)$ at x_iH equals $\pi(\omega(x_i, H))(b(h_i) + \eta_i)$, where η_i is defined as follows

$$\eta_i := \begin{cases} \xi_0 & \text{if } n_i > 0 \text{ and } n_{i+1} > 0, \\ \xi_0 - \pi(h_i)\xi_0 & \text{if } n_i > 0 \text{ and } n_{i+1} < 0, \\ -\pi(h_i)\xi_0 & \text{if } n_i < 0 \text{ and } n_{i+1} < 0, \\ 0 & \text{if } n_i < 0 \text{ and } n_{i+1} > 0. \end{cases}$$

Further, the value at $x_ih_it^jH$ with $j \in \{1, \dots, n_i - 1\}$ if $n_{i+1} > 0$ (or $j \in \{-1, -2, \dots, n_i + 1\}$ if $n_{i+1} < 0$) is equal to $\pi(\omega(x_ih_it^{j-1}, tH)(\xi_0))$ (or $\pi(\omega(x_ih_it^{j+1}, t^{-1}H))(\xi_0)$).

Lemma 5.5. *The* 1*-cocycle* $\tilde{b}_{\mathrm{HNN}}$ *is Lipschitz.*

Proof. Since $\tilde{b}_{BS}$ is Lipschitz, it suffices to show that $\tilde{b}$ is Lipschitz.

Let y_i be the subword of x which contains only the first i letters, where letters are elements of $H \cup \{t, t^{-1}\}$ (x has $k + 2 + |n_1| + |n_2| + \dots + |n_k|$ such subwords). Since $\|\eta_i\| \leqslant 2\|\xi_0\|$, we see that the value of $\tilde{b}(x)$ at x_iH can be bounded by $\|b(h_i)\| + 2\|\xi_0\|$. Moreover, if y_iH differs from $x_0H, x_1H, x_2H, \dots, x_kH$, then the value at y_iH can be bounded by $\|\xi_0\|$. Using $(a+b)^2 \leqslant 2a^2 + 2b^2$, we get that

$$\big\|\tilde{b}(x)\big\|^2 \leqslant \big(|n_1| + |n_2| + \dots + |n_k|\big)\|\xi_0\|^2 + \sum_{i=0}^{k}\big(2\big\|b(h_i)\big\|^2 + 8\|\xi_0\|^2\big).$$

Since b is Lipschitz, we can without loss of generality assume that $k \neq 0$. Then the above sum contains less than

$$8(k+1) + |n_1| + |n_2| + \cdots + |n_k| \leqslant 9\big(1 + |n_1| + |n_2| + \cdots + |n_k|\big) \leqslant 18 l(x)$$

times $\|\xi_0\|^2$ and so

$$\begin{aligned}\big\|\tilde{b}(x)\big\|^2 &\leqslant 18\|\xi_0\|^2 l(x) + 2\sum_{i=0}^{k}\big(2C^2 l_H(h_i)^2 + 2D^2\big)\\ &\leqslant \big(18\|\xi_0\|^2 + 8D^2\big)l(x) + 4C^2\sum_{i=0}^{k} l_H(h_i)^2.\end{aligned}$$

This implies

$$\big\|\tilde{b}(x)\big\| \leqslant \sqrt{18\|\xi_0\|^2 + 8D^2}\, l(x) + 2C\sum_{i=0}^{k} l_H(h_i).$$

Previously as before, we can introduce the shortest blocklength $l^{SB}(x)$ of x as the length of a shortest reduced expression of x. Here, we look at a reduced expression as a word and we emphasize that l^{SB} is not a length function in the strict sense of Definition 2.2. Since the above inequality is valid for every reduced expression, one immediately sees that

$$\big\|\tilde{b}(x)\big\| \leqslant \big(\sqrt{18\|\xi_0\|^2 + 8D^2} + 2C\big)l^{SB}(x).$$

Using the fact that (H, l_H) embeds bi-Lipschitzly into (G, l) (Lemma 6.2.5 in [6]), then as in the proof of Theorem 4.9, one can conclude that $\tilde{b}$ is Lipschitz. □

Lemma 5.6. *There exists $D' > 0$ such that*

$$\big\|b_{\mathrm{HNN}}(x)\big\| \geqslant \frac{1}{2\sqrt{4C^2+1}} l(g)^r - D',$$

for every $x \in G$.

Proof. We start by estimating $\|\tilde{b}(x)\|$ from below.

$$\begin{aligned}\big\|\tilde{b}(x)\big\|^2 &\geqslant \sum_{i=0}^{k}\big\|b(h_i) + \eta_i\big\|^2\\ &\geqslant \sum_{i=0}^{k}\big(1/2\big\|b(h_i)\big\|^2 - \|\eta_i\|^2\big)\end{aligned}$$

$$\geqslant \left(1/2\sum_{i=0}^{k}\left\|b(h_i)\right\|^2\right) - 4(k+1)\|\xi_0\|^2$$
$$\geqslant \left(\frac{1}{4C^2}\sum_{i=0}^{k} l_H(h_i)^{2\epsilon}\right) - (k+1)\big(4\|\xi_0\|^2 + D^2\big),$$

where the second and final inequality follow from Lemma 5.2. Recalling that b_{HNN} is the direct sum of $\tilde{b}$ and $\sqrt{E+1}b_{BS}$ where $E = 4\|\xi_0\|^2 + D^2$, we get that

$$\begin{aligned}\left\|b_{\mathrm{HNN}}(x)\right\|^2 &\geqslant -E + \frac{1}{4C^2}\sum_{i=0}^{k} l_H(h_i)^{2\epsilon} + \sum_{i=1}^{k}|n_i| \\ &\geqslant -E + \frac{1}{4C^2}\left(\sum_{i=0}^{k} l_H(h_i) + \sum_{i=1}^{k}|n_i|\right)^{2\epsilon} \\ &\geqslant -E + \frac{1}{4C^2} l(x)^{2\epsilon}.\end{aligned}$$

It follows that $\|b_{\mathrm{HNN}}(x)\| \geqslant \frac{1}{4C} l(x)^{\epsilon} - D'$, for some positive real number D'. □

The two lemmas above conclude the proof of Theorem 5.3. □

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