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Triangle-free distance-regular graphs with an eigenvalue multiplicity equal to their valency and diameter 3

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Abstract

In this paper, triangle-free distance-regular graphs with diameter 3 and an eigenvalue θ with multiplicity equal to their valency are studied. Let Γ be such a graph. We first show that $\theta = -1$ if and only if Γ is antipodal. Then we assume that the graph Γ is primitive. We show that it is formally self-dual (and hence Q-polynomial and 1-homogeneous), all its eigenvalues are integral, and the eigenvalue with multiplicity equal to the valency is either second largest or the smallest.

Let $x, y \in V\Gamma$ be two adjacent vertices, and $z \in \Gamma_2(x) \cap \Gamma_2(y)$. Then the intersection number $\tau_2 := |\Gamma(z) \cap \Gamma_3(x) \cap \Gamma_3(y)|$ is independent of the choice of vertices x, y and z. In the case of the coset graph of the doubly truncated binary Golay code, we have $b_2 = \tau_2$. We classify all the graphs with $b_2 = \tau_2$ and establish that the just mentioned graph is the only example. In particular, we rule out an infinite family of otherwise feasible intersection arrays.

1. Introduction

Let Γ be a triangle-free distance-regular graph with $a_2 \neq 0$ (i.e., Γ has induced pentagons) and an eigenvalue multiplicity m_t equal to its valency k, i.e., $m_t = k$ (note that the graphs under assumption $a_2 = 0$ instead of $a_2 \neq 0$ have already been classified in [7]). Then the graph Γ is an example of a triangle-free distance-regular graph for which the inequality

$$m_t \geq k$$

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(that follows from the Terwilliger tree bound [10] whenever $t \neq \pm k$) is satisfied with equality. There are many interesting triangle-free distance-regular graphs for which there is an eigenvalue with multiplicity k. An important class of such examples comes from triangle-free distance-regular graphs whose association scheme, as determined by their distance matrices, is formally self-dual (for the definition see Section 2). The family of Hermitean form graphs over $\mathrm{GF}(2^2)$ (see Brouwer, Cohen and Neumaier [2, 9.5C]) belongs to this class [2, Theorem 8.4.3].

In [8], it was shown that the set of projections into the eigenspace corresponding to t of the neighbours of any vertex of the graph Γ forms a basis for this eigenspace, see also Lemma 2.2. Furthermore, many such graphs were shown to have the 1-homogeneous property in the sense of Nomura, including the graphs with diameter three.

Let Γ be a triangle-free distance-regular graph with diameter $d = 3$, and assume that Γ has an eigenvalue t with multiplicity k. We first show that $t = -1$ if and only if Γ is antipodal. Then we assume Γ is primitive. The study of this case is our main focus. The only known example of such a graph is the coset graph of the doubly truncated binary Golay code with intersection array $\{21, 20, 16; 1, 2, 12\}$. This graph belongs to the abovementioned family of Hermitean form graphs over $\mathrm{GF}(2^2)$. In Section 3, we show our main result, namely that Γ is always formally self-dual, that all its eigenvalues are integral, and finally that the eigenvalue t is either the second largest or the least eigenvalue of Γ. We use the properties obtained to calculate all feasible intersection arrays with valency up to 4000, see Appendix.

The formal self-duality of Γ implies the Q-polynomial property and also the 1-homogeneous property. In Section 4, we study the intersection number $\tau_2 := |\Gamma(z) \cap \Gamma_3(x) \cap \Gamma_3(y)|$, which is independent of the choice of adjacent vertices x, y and $z \in \Gamma_2(x) \cap \Gamma_2(y)$ by the 1-homogeneous property. The condition $b_2 = \tau_2$ gives rise to a 1-parameter infinite family of feasible intersection arrays. Its first member corresponds to the abovementioned coset graph of doubly truncated binary Golay code. Ivanov and Shpectorov [6], cf. [2, Theorem 11.3.6], showed the uniqueness of this graph by showing that the second subconstituent consists of the disjoint union of 21 Petersen graphs. We use similar combinatorial arguments to rule out all the members of the above family except the first one.

2. Preliminaries

In this section we review some definitions and basic concepts. See Brouwer, Cohen and Neumaier [2] and Godsil [4] for more background information.

Throughout this paper, Γ will denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set $V\Gamma$, edge set $E\Gamma$, shortest path-length distance function ∂, and diameter $d := \max\{\partial(x, y) | x, y \in V\Gamma\}$. For $x \in V\Gamma$ and for an integer i, define $\Gamma_i(x)$ to be the set of vertices of Γ at distance i from x. We abbreviate $\Gamma(x) := \Gamma_1(x)$. The graph Γ is said to be *distance-regular* whenever for all integers h, i, j $(0 \le h, i, j \le d)$, and all $x, y \in V\Gamma$ with $\partial(x, y) = h$, the number $p^h_{ij} := |\Gamma_i(x) \cap \Gamma_j(y)|$ is independent of the vertices x and y. The constants p^h_{ij} $(0 \le h, i, j \le d)$ are known as the *intersection numbers* of Γ. For notational convenience, define $c_i := p^i_{1,i-1}$ $(1 \le i \le d)$, $a_i := p^i_{1i}$ $(0 \le i \le d)$, $b_i := p^i_{1,i+1}$ $(0 \le i \le d-1)$, $k_i := p^0_{ii}$ $(0 \le i \le d)$, and set $c_0 = 0 = b_d$ and $n := |V\Gamma|$. We observe that $a_0 = 0$ and $c_1 = 1$. Moreover, $a_i + b_i + c_i = k$ $(0 \le i \le d)$, where $k := k_1$.

It is well known that a distance-regular graph Γ of diameter d has precisely $d + 1$ eigenvalues, say $\theta_0 > \theta_1 > \cdots > \theta_d$. Also, $\theta_0 = k$, its multiplicity, is equal 1, and $\theta_d = -k$ if and only if Γ is bipartite. Let U_i be a matrix with its columns forming an orthonormal basis for the eigenspace corresponding to θ_i. Then $E_i := U_i U_i^*$ is called *the principal idempotent* of A corresponding to θ_i. For each integer i $(0 \le i \le d)$, we denote the multiplicity of θ_i by m_{θ_i}.

Let Γ be a graph of diameter d. We can define distance matrices as follows. For each i $(0 \le i \le d)$ let A_i be the matrix with rows and columns indexed by $V\Gamma$, and the x, y entry of A_i equal to 1 if $\partial(x, y) = i$, and 0 otherwise. We call A_i the ith *distance matrix* of Γ, and $A := A_1$ is the adjacency matrix of Γ. Suppose now that the graph Γ is distance-regular. Then the distance matrices $A_0, A_1, \ldots, A_d$ form a basis for a commutative semi-simple $\mathbb{R}$-algebra $\mathcal{M}$, known as the *Bose–Mesner algebra* of the graph Γ. The set of principal idempotents $\{E_0, E_1, \ldots, E_d\}$ is another basis of the algebra $\mathcal{M}$. Therefore, we can define two $(d+1)\times(d+1)$ matrices P and Q by:

$$A_i = \sum_{j=0}^{d} P_{ji} E_j \quad \text{and} \quad E_i = \frac{1}{n} \sum_{j=0}^{d} Q_{ji} A_j \quad (0 \le i \le d). \tag{1}$$

When $P = Q$, the graph Γ is called *formally self-dual*. The following relations between P and Q, which will be used in the next section to calculate these two matrices, can be found in [4, p. 226]

$$PQ = nI \quad \text{and} \quad P^T \Delta_m = \Delta_k Q, \tag{2}$$

where Δ_m and Δ_k are the $(d+1)\times(d+1)$ diagonal matrices with $(\Delta_m)_{ii} = m_{\theta_i}$ and $(\Delta_k)_{ii} = k_i$.

Let θ be an eigenvalue of the graph Γ, and let E be the associated principal idempotent. Let $w_0, w_1, \ldots, w_d$ be the numbers satisfying

$$E = \frac{m_\theta}{n} \sum_{j=0}^{d} w_j A_j, \tag{3}$$

where m_θ denotes the multiplicity of θ. We refer to w_i as the ith *cosine* of Γ with respect to θ (or E), and call $w_0, w_1, \ldots, w_d$ the *cosine sequence* of Γ associated with θ (or E). Note that, by the right relation of (1), (3) and (2), the cosines are related to the entries of matrices P and Q, i.e., for $\theta = \theta_i$ and $w_j = w_j(\theta_i)$

$$Q_{ji} = m_{\theta_i}\, w_j(\theta_i) \quad \text{and} \quad P_{ij} = k_j\, w_j(\theta_i). \tag{4}$$

For notational convenience, we identify $V\Gamma$ with the standard orthonormal basis in the Euclidean space $(V, \langle\,,\,\rangle)$, where $V = \mathbb{R}^n$ (column vectors), and where $\langle\,,\,\rangle$ is the dot product $\langle u, v\rangle = u^t v$ for $u, v \in V$. The following basic result can be found, for example, in Brouwer, Cohen and Neumaier [2, p. 128] ((i) follows immediately from (3)).

Lemma 2.1. *Let Γ be a distance-regular graph with diameter d. Let θ be an eigenvalue of Γ with multiplicity m_θ, associated principal idempotent E, and associated cosine sequence $w_0, w_1, \ldots, w_d$. Then the following two conditions hold.*

(i) *For all $x, y \in V\Gamma$ with $\partial(x, y) = i$ we have $\langle Ex, Ey\rangle = w_i\, m_\theta/n$.*
(ii) *The cosine sequence satisfies $w_0 = 1$ and $c_i w_{i-1} + a_i w_i + b_i w_{i+1} = \theta w_i$ $(0 \le i \le d)$, where w_{-1} and w_{d+1} are set arbitrarily.* ■

In particular, we have $w_1 = \theta/k$, and for $d \ge 2$ also:

$$w_2 = (\theta^2 - a_1\theta - k)/(kb_1), \quad \text{and} \quad kb_1(1 - w_2) = (k - \theta)(\theta + k - a_1). \tag{5}$$

If we assume furthermore that $d = 3$ and $a_1 = 0$, then we have also:

$$w_3 = \frac{w_2(\theta - a_2) - c_2 w_1}{b_2} = \frac{\theta^3 - a_2\theta^2 - \theta(k + kc_2 - c_2) + ka_2}{k(k-1)b_2}$$
$$= \frac{-\left(\theta^2 + c_2\theta - (k - c_2)\right) c_3}{k(k-1)b_2}. \tag{6}$$

The cubic term of θ in w_3 was reduced using the relations (2) and (4) (or alternatively, (8)).

Lemma 2.2 (*[8, Lemma 3.3]*). *Let Γ be a non-bipartite triangle-free distance-regular graph with diameter $d \geq 2$ and valency $k \geq 3$. Let θ be an eigenvalue of Γ with multiplicity equal to k. Let E be the associated principal idempotent and $w_0, \ldots, w_d$ the associated cosine sequence. Let $x \in V\Gamma$ and $y \in \Gamma_i(x)$, where $1 \leq i \leq d$. Then $w_1(w_0 - w_2) \neq 0$, and*

$$Ey = C_i \left(\sum_{z \in \Gamma(x) \cap \Gamma_{i-1}(y)} Ez \right) + A_i \left(\sum_{z \in \Gamma(x) \cap \Gamma_i(y)} Ez \right) + B_i \left(\sum_{z \in \Gamma(x) \cap \Gamma_{i+1}(y)} Ez \right),$$

where

$$C_i = \frac{w_1 w_{i-1} - w_2 w_i}{w_1(1 - w_2)}, \qquad A_i = \frac{w_1 w_i - w_2 w_i}{w_1(1 - w_2)}, \qquad B_i = \frac{w_1 w_{i+1} - w_2 w_i}{w_1(1 - w_2)}, \tag{7}$$

and B_d is not determined (the corresponding intersection is empty). ■

3. Formal self-duality

Let Γ be a distance-regular graph with diameter $d = 3$. Let $\{k, b_1, b_2; 1, c_2, c_3\}$ be its intersection array (hence $a_1 = k - b_1 - 1$, $a_2 = k - b_2 - c_2$, $a_3 = k - c_3$, so there are 5 independent parameters), and let $k = \theta_0 > \theta_1 > \theta_2 > \theta_3$ be its eigenvalues. Next we assume $a_1 = 0$, i.e., that $b_1 = k - 1$, and we are down to parameters k, b_2, c_2 and c_3. We will show below that we can express these four parameters in terms of the eigenvalues of Γ. To keep the notation simple and to handle various cases simultaneously, we introduce labels u, v and t such that $\{u, v, t\} = \{\theta_1, \theta_2, \theta_3\}$. The eigenvalues of Γ are the eigenvalues of the following tridiagonal matrix

$$\begin{pmatrix} 0 & k & 0 & 0 \\ 1 & 0 & k-1 & 0 \\ 0 & c_2 & a_2 & b_2 \\ 0 & 0 & c_3 & a_3 \end{pmatrix},$$

i.e., the zeros of the following characteristic polynomial

$$(\theta - k)(\theta^3 + \theta^2(c_3 - a_2) - \theta(c_2(a_3 - 1) + k) + ka_2 + c_3(c_2 - k)). \tag{8}$$

Therefore, $u + v + t = a_2 - c_3$, $uv + vt + tu = -c_2(k - c_3 - 1) - k$ and $-uvt = ka_2 - c_3(k - c_2)$. By solving this system for c_2, c_3 and a_2, we obtain:

$$c_2 = -\frac{k(1 + u + v + t) + uv + vt + tu + uvt}{k - 1}, \tag{9}$$

$$c_3 = -\frac{k(u + v + t) + uvt}{c_2}, \tag{10}$$

$$a_2 = u + v + t + c_3. \tag{11}$$

Thus we have also:

$$b_2 = -\frac{(k+vt+v+t)(k+v+u+uv)(k+tu+u+t)}{c_2(k-1)^2} \quad \text{and}$$
$$n = \frac{(k-u)(k-v)(k-t)}{c_2c_3}.$$

For a later purpose, we express the intersection number c_2 also in the following ways:

$$c_2 = \frac{(t+1)(u+1)(v+1)}{1-k} - (t+1+u+v) = \frac{(t+1)(k+u+v+uv)}{1-k} - (u+v). \tag{12}$$

Furthermore, by solving the following system of equations: $m_u\,u + m_v\,v + m_t\,t = -k$, $m_u\,u^2 + m_v\,v^2 + m_t\,t^2 = k(n-k)$ and $m_u\,u^3 + m_v\,v^3 + m_t\,t^3 = -k^3$, we obtain

$$m_u = \frac{k(k-v)(k-t)(k+v+t+vt)}{(u-v)(u-t)c_2c_3},$$
$$m_v = \frac{k(k-u)(k-t)(k+u+t+ut)}{(v-u)(v-t)c_2c_3},$$
$$m_t = \frac{k(k-u)(k-v)(k+v+u+vu)}{(t-v)(t-u)c_2c_3}.$$

Now let us assume that Γ has an eigenvalue of multiplicity k. Without loss of generality, we assume that $m_t = k$, i.e.,

$$(k-u)(k-v)(k+u+v+uv) = -(t-v)(t-u)(k(u+v+t)+uvt). \tag{13}$$

We first consider the case $t = -1$.

Proposition 3.1 (*The Antipodal Case*). *Let Γ be a triangle-free distance-regular graph with diameter $d = 3$, valency k, and nontrivial eigenvalues u, v and t. Suppose t has multiplicity k. Then $t = -1$ if and only if Γ is antipodal. If $t = -1$ then its eigenvalues are integral, $c_2 = -(u+v)$, $k = -uv$, and the size of antipodal classes is $r = (u+1)(v+1)/(u+v)$ and*

$$m_u = (k+1)(r-1)v/(v-u), \qquad m_v = (k+1)(r-1)u/(u-v). \tag{14}$$

Proof. Setting $t = -1$ in (13), we obtain $(k^2-1)(k+uv) = 0$, i.e., $k = -uv$. Now, using (10)–(12), we obtain $c_2 = -(u+v)$, $c_3 = k$, $a_3 = 0$, $a_2 = -(u-1)(v-1)$ and $b_2 = 1$. Therefore, the graph Γ is antipodal, and it can be parametrized by $(k+1, r, c_2)$, see [5, p. 209]. Of course, in this case we have $m_{-1} = k$. By (12) and $0 = a_1 = k-1-(r-1)c_2$, we obtain also:

$$r = 1 + \frac{k-1}{c_2} = 1 + v + \frac{(1-v)(1+v)}{u+v} = \frac{(u+1)(v+1)}{u+v}$$

and (14). We have $u, v \in \mathbb{Z}$ by [5, p. 209], since $0 \neq -c_2\ (= \delta)$. The converse is immediate. ∎

Remark 3.2. (i) Note that there is a known infinite family of examples in this case: $(2^{2s}, 2^{2s-1}, 2)$, where $s \in \mathbb{N}$, constructed by de Caen, Mathon and Moorhouse [3].

(ii) Let Γ be a distance-regular graph with diameter $d = 3$. If Γ is bipartite and antipodal, then $k^1,\ 1^k,\ (-1)^k,\ (-k)^1$ are the eigenvalues of Γ, and Γ is the complete bipartite graph $K_{k+1,k+1}$

with a perfect matching deleted. Among the triangle-free antipodal graphs of diameter three, the bipartite and antipodal graphs are the only graphs which are formally self-dual. If Γ is bipartite and not antipodal, then none of its eigenvalue multiplicities is equal to its valency, see [2, p. 432].

From now on we assume Γ is primitive, which means $t \neq -1$, and we set

$$g(k,t) := k^2 - 2k + t^3 + t^2 - t \quad \text{and} \quad f(k,t,x) := k^2 + kt - kx + t^2x - x^2 - tx^2.$$

Our aim is to prove that Γ is formally self-dual. We begin by showing that one of its eigenvalues is equal to kw_2.

Lemma 3.3. *Let Γ be a primitive triangle-free distance-regular graph with diameter $d = 3$, valency $k > 2$ and nontrivial eigenvalues u, v and t. Suppose $m_t = k$. Let w_0, w_1, w_2, w_3 be the cosine sequence corresponding to t. Then*

(i) $g(k,t) \neq 0$,
(ii) $f(k,t,u) = 0$ *if* $u \neq kw_2$ *and* $f(k,t,v) = 0$ *if* $v \neq kw_2$,
(iii) $kw_2 \in \{u, v\}$.

Proof. By collecting the terms with the same power of v in (13), we obtain

$$(k^2 + t^3 - (t+1)u^2)k + (g(k,t) - (t-1)(k+t+u+tu))\,uv \\ -(1+t)(k - u + tu - u^2)v^2 = 0. \tag{15}$$

Let us now compute the scalar product $\|Ey\|^2 = \langle Ey, Ey\rangle$, for $x \in V\Gamma$ and $y \in \Gamma_2(x)$, where E is the principal idempotent corresponding to the eigenvalue t, by first using Lemma 2.1(i) and then using Lemma 2.2. After multiplication of the obtained relation with n/k, and by noting that $c_2 = |\Gamma(x) \cap \Gamma_1(y)|$, $a_2 = |\Gamma(x) \cap \Gamma_2(y)|$ and $b_i = |\Gamma(x) \cap \Gamma_3(y)|$, we obtain the relation $1 = (U + V)w_2 + V(1 - w_2)$, where $V = c_2C_2^2 + a_2A_2^2 + b_2B_2^2$, and $U + V = (c_2C_2 + a_2A_2 + b_2B_2)^2$. Since we can express the cosines w_1, w_2, w_3 corresponding to the eigenvalue t in terms of the eigenvalues k, u, v, t by using (5) and (6), the same is true also for the coordinates C_2, A_2 and B_2 from Lemma 2.2:

$$C_2 = \frac{t^2-1}{t(k-1)}, \qquad A_2 = \frac{(t+1)(t^2-k)}{(t+k)t(k-1)}, \qquad B_2 = \frac{Xv+Y}{(k+t)t(k-1)(k+v+u+uv)},$$

where $X = k(t^2 + kt - 2t - 1) + t^3 + (k+t)(t^2-1)u$ and $Y = k(k+t)(t^2-1) + u(k^2t + t^3 + k(t^2 - 2t - 1))$. The scalar product relation renders

$$0 = v^2\,(k-1)(1+t)(k+t+u+tu) \\ -v(u(2k - 4k^2 + k^3 - 3kt + u - ku - 2ktu) - k(k+t) + (1+u)t(t^3 + t^2 - k^2 \\ + kt^2 + 2u + tu - ktu)) - (k+t)(k\,g(k,t) - u(1+t)(k - t^2 + u(k-1))). \tag{16}$$

Both Eqs. (15) and (16) are of degree at most 2 in v, but when we subtract a proper multiple of one from the other, the quadratic term on v vanishes, and we obtain the following relation:

$$uk\left(k + t + \frac{g(k,t)}{t+1}\right) + u^2(g(k,t) - k^2 - t^3) + (k(k+t) - u^3(t+1))\frac{k+t}{t+1} \\ + \left[u\left(k(1+t) + \frac{g(k,t)}{t+1}\right) - u^2\left(k + t - \frac{g(k,t)}{t+1}\right) + \left(k(k+t) - u^3(1+t)\right)\right]v = 0, \tag{17}$$

unless $(t+1)(u-kw_2)=0$, i.e., $u=kw_2$, since $t\neq -1$ (as we assumed Γ is primitive).

(i) Suppose $g(k,t)=0$, i.e., $(k-1)^2=(1+t)^2(1-t)$. Let us set $1-t=s^2$ for a positive $s\in\mathbb{R}$. Since $s\leq\sqrt{2}$ would give us too small k, we have $s>\sqrt{2}$ and $k=1+s(s^2-2)$. Thus the relation (16) transforms to $(s-1-u)(s-1-v)(u+v)=0$. We have $u\neq s-1\neq v$, since otherwise we get $k+u+t+ut=0$ or $k+v+t+vt=0$, which implies that $b_2=0$. Therefore, $u+v=0$, and $s=\sqrt{2}$ by the relation (15), and $b_2\neq 0\neq c_2$. This is a contradiction.

(ii) Let us assume $u\neq kw_2$. We can then calculate v from Eq. (17) in the case when the coefficient beside v is not zero. If, on the other hand, this coefficient is zero, then the free coefficient also has to be zero, in which case we subtract the first coefficient multiplied by $(t+1)$ from the second multiplied by $(k+t)$ (the term u^3 and the free term both vanish at the same time), and obtain $g(k,t)=0$ after division by $(k-1)(t-u)u$ (note that $k\neq 1$, $t\neq u$ and $u\neq 0$ by (17) and $b_2\neq 0$). This is obviously not possible, so we have another contradiction. Therefore, v is uniquely determined by the linear equation (17). We solve it, and set the value for v in (16), which gives us $f(k,t,u)=0$ after dividing by

$$(k-1)\,(t-u)\,(t+1)\,g(k,t)\,(k+t+u+tu)\,(k-u^2).$$

This product is nonzero, as $k\neq 1$, $-1\neq t\neq u$, $g(k,t)\neq 0$, $b_2\neq 0$ and $c_2\neq 0$. The remaining part of the statement (ii) follows by symmetry between u and v.

(iii) We can again, by symmetry between u and v, and without loss of generality, continue with the assumption $u\neq kw_2$. So it is enough to verify that the relation (16) is satisfied in the case when $v=kw_2$. Indeed, in this case (16) is equivalent to $f(k,t,u)=0$ by (i), and we are done by (ii). ■

Due to the symmetry between u and v (as we have not ordered the nontrivial eigenvalues of Γ), we can, without loss of generality, assume

$$u=kw_2=\frac{t^2-k}{k-1}=\frac{t^2-1}{k-1}-1, \tag{18}$$

i.e., $(u+1)(k-1)=t^2-1$. The second relation of Lemma 3.3(ii) is a quadratic equation for k (or v): $f(k,t,v)=(k-v)(k+v)-(k+vt)(v-t)=0$.

Theorem 3.4. *Let Γ be a primitive triangle-free distance-regular graph with diameter $d=3$, valency $k>2$ and nontrivial eigenvalues $\{u,v,t\}$. Suppose $m_t=k$. Let w_0,w_1,w_2,w_3 be the cosine sequence corresponding to t. Then*

(i) *All the eigenvalues are integral,*
(ii) *$k>t>u>0>v$ or $k>u>0>v>t$, where $u=kw_2$,*
(iii) *Γ is formally self-dual (in particular, it is also Q-polynomial for the ordering k,t,u,v).*

Proof. Suppose $\{k,u,v,t\}$ are the eigenvalues of Γ, and let $m_t=k$. Without loss of generality we assume, by Lemma 3.3, that $u=kw_2$ and $f(k,t,v)=0$.

(i) The eigenvalues t, u and v are zeros of a cubic monic polynomial (8) with integral coefficients. These eigenvalues are integral unless two of them have the same multiplicity. We show the integrality of the eigenvalues by showing that the following cases (a)–(b) are not possible.

(a) Let us assume the eigenvalues t and u are irrational. If the eigenvalue v is also irrational, then $m_t=m_u=m_v=k$ and we have $n=3k+1$, $k_2\leq 2k$, $c_2\geq (k-1)/2$, $a_2>(k-1)/2$, see [8, Theorem 4.2], and finally $1 > k - a_2 - c_2 = b_2$ gives us a contradiction. Hence v is rational (cf. [1] and [2, p. 130]) and we have $m_u = k$. Since v is an algebraic integer, it has to be integer. By Lemma 3.3(iii), we have $(u^2 - k)/(k-1) \in \{t, v\}$. Suppose first that $t = (u^2 - k)/(k-1)$. Since $u = (t^2 - k)/(k-1)$, this equation transforms (after multiplying by $(k-1)^3/((t+1)(k-t)))$ to $t^2 + kt - t + k^2 - 3k + 1 = 0$, i.e., a quadratic with discriminant $-3 + 10k - 3k^2$. It follows that this quadratic does not have any real solutions for $k > 3$, and for $k = 3$ we get the solution $t = -1$, which is not possible by the assumption that Γ is primitive and Proposition 3.1. Therefore, $v = (u^2 - k)/(k-1)$ and, by Lemma 3.3(ii), also $f(k, u, t) = 0$. Hence we have $u^2 \in \mathbb{Q}$, and furthermore, since t is an algebraic conjugate of u, we have $u^2 = t^2$ and $u = v$, which is a contradiction.

(b) Let us now assume t is irrational and u is rational. Then t is a square-root of a rational number, and we have $v = -t$. Hence $u = (v^2 - k)/(k-1)$ and $f(k, v, t) = 0$. By adding $f(k, v, t) = 0$ to $f(k, t, v) = 0$, we obtain $k^2 = v^2$, which is not possible.

Therefore, by (a) and (b), the eigenvalue t is rational, and hence integral. It follows from (18) that also u is rational and so integral. Then also v is integral by (11).

(ii) Since Γ is primitive, we have $t \neq -1$ by Lemma 3.3. Without loss of generality, we assume that $u = kw_2$, i.e., (18) holds. This implies that $u \geq -1$ and $k - 1 \mid t^2 - 1$. If $u = -1$, then $t = 1, k = 2v - 1$ by $f(k, t, v) = 0$ and $c_2 = -k(v+1)/(k-1) < 0$, which is not possible. The case $u = 0$, i.e. $t^2 = k$, is not possible, since $f(k, t, v) = 0$ (by Lemma 3.3) implies $v^2 = t^3$, which means that $t = s^2$ for $s \in \mathbb{N}$, $v < 0$ and $c_2 = s^4/(s+1)$. So $u > 0$ and $|t| > \sqrt{k}$.

If $t > 0$, then $t > u$ (i.e., $(t - k)(t + 1) < 0$), and we have $k > t > u > 0 > v$.

If $t < 0$, then $v < 0$, and $f(k, t, v) = 0$ imply $v > t$, hence $k > u > 0 > v > t$.

The case $t < 0$ and $v > 0$ is not possible since, by $-k < t < -\sqrt{k}$, we have

$$0 < k(k+t) + v(t^2 - k) + v^2(-1 - t) = f(k, t, v) = 0,$$

which is a contradiction.

(iii) With a straightforward calculation (using $m_t = k$, (18) and the relation $f(k, t, v) = 0$), we verify that the following matrices of eigenvalues and dual eigenvalues are equal:

$$P = \begin{pmatrix} 1 & k & k_2 & k_3 \\ 1 & t & \dfrac{t^2-k}{c_2} & -\dfrac{t^2-k}{c_2} - t - 1 \\ 1 & u & \dfrac{u^2-k}{c_2} & -\dfrac{u^2-k}{c_2} - u - 1 \\ 1 & v & \dfrac{v^2-k}{c_2} & -\dfrac{v^2-k}{c_2} - v - 1 \end{pmatrix} \quad \text{and}$$

$$Q = \begin{pmatrix} 1 & m_t & m_u & m_v \\ 1 & tm_t/k & um_u/k & vm_v/k \\ 1 & m_t\dfrac{t^2-k}{c_2k_2} & m_u\dfrac{u^2-k}{c_2k_2} & m_v\dfrac{v^2-k}{c_2k_2} \\ 1 & m_t\left(\dfrac{k-t^2}{c_2k_3} - \dfrac{1+t}{k_3}\right) & m_u\left(\dfrac{k-u^2}{c_2k_3} - \dfrac{1+u}{k_3}\right) & m_v\left(\dfrac{k-v^2}{c_2k_3} - \dfrac{1+v}{k_3}\right) \end{pmatrix}. \quad \blacksquare$$

Remark 3.5. A direct consequence of Theorem 3.4(iii) is that kw_0, kw_1, kw_2 and kw_3 are all the eigenvalues of the graph Γ. We can express k and the cosine w_2 in terms of the cosines w_1

and w_3 as follows:

$$k = -\frac{1 + w_1 - w_3 - w_3^2}{w_1 w_3 (w_1 - w_3)} \quad \text{and} \quad w_2 = \frac{w_1^2 k - 1}{k - 1} = \frac{w_1(w_1 - w_3^2)}{1 + (w_1 - 1)w_3 - w_3^2},$$

where we obtained k from $f(k, t, v) = 0$, and w_2 directly from Lemma 2.1(ii) and $a_1 = 0$.

Using the absolute bound, we also obtain the following result.

Lemma 3.6. *Let Γ be a primitive triangle-free distance-regular graph with valency $k > 3$, diameter $d = 3$, $a_2 \neq 0$ and an eigenvalue of multiplicity k. Then $c_2 \geq 2$.* ■

4. The 1-homogeneous property

Let Γ be a distance-regular graph. Then Γ is 1-*homogeneous* in the sense of Nomura, i.e. where the distance partition corresponding to any pair of adjacent vertices x and y, i.e., the collection of nonempty intersections $D_i^j(x, y) := \Gamma_i(x) \cap \Gamma_j(y)$, is equitable.

Lemma 4.1. *Let Γ be a primitive triangle-free distance-regular graph with diameter $d = 3$, $a_2 \neq 0$, eigenvalues $k > 2$, $u > 0$, $v < 0$, t, and $m_t = k$. Then Γ is 1-homogeneous. Let x and y be adjacent vertices of Γ. Then we have, for all $z \in D_2^2(x, y)$, the following formula for $b_2 - \tau_2 = |\Gamma(z) \cap D_3^2(x, y)|$*

$$b_2 - \tau_2 = \frac{g(k, t)\,(k^2 - 2k + vt^2 + t^2 - v)^2(k^2 - k + tk + t + 1 + vt^2 + v + 2vt + t^2)}{((v + t)g(k, t) + (t^2 - k)(1 - t^2))^2(k - 1)^2}, \tag{19}$$

where the above denominator is nonzero.

Proof. Γ is 1-homogeneous by [8, Cor. 4.7], cf. Miklavič [9]. By [8, Eq. (11)], we obtain (19). The above denominator is nonzero by $w_3 \neq w_2$, since otherwise the recursion relation for cosines gives us $(k - \theta)w_3 = 0$ and therefore $w_2 = w_1 = w_0 = 0$, which is not possible. ■

From the above formula, we can calculate τ_2 in terms of the eigenvalues $\{k, t, u, v\}$. In particular, we calculate the distance partitions corresponding to an edge in those cases where $k \in \{70, 105, 161, 276\}$, see Fig. A.4 and $k \in \{21, 175\}$, see Fig. A.3.

Lemma 4.2. *Let Γ be a primitive triangle-free distance-regular graph with diameter $d = 3$, valency $k > 2$, nontrivial eigenvalues $\{u, v, t\}$ and the multiplicity $m_t = k$. Then $b_2 = \tau_2$ if and only if*

$$b_2 = q^2(2 - q - q^2 + q^3), \qquad a_2 = (q - 1)^2(q + 1), \qquad c_2 = q(q - 1),$$
$$c_3 = q^2(q^2 - q + 1) \tag{20}$$

for some integer $q \geq 2$. Also, if $b_2 = \tau_2$, then we have

$$k_2 = (1 - q + q^2)(2 + q^3)(1 - q + q^3) \quad \text{and}$$
$$k_3 = (2 + q^3)(1 - q + q^3)(2 - q - q^2 + q^3). \tag{21}$$

Proof. Let us assume $b_2 = \tau_2$. Without loss of generality, we assume, by Lemma 3.3, $u = (t^2 - k)/(k-1)$ and thus we have also $f(k, t, v) = 0$. The expression $b_2 - \tau_2$ is zero if and only if at least one of the three factors in the numerator of (19) is zero. The first factor is nonzero by Lemma 3.3.

Suppose the second factor is zero, i.e., $k^2 - 2k + vt^2 + t^2 - v = 0$. By subtracting the relation $f(k, t, v) = 0$, we obtain $k(2 + t - v) = t^2 - v + v^2 + tv^2$. If $v = t + 2$ then $t = -5$ and $v = -3$, which is not possible by $f(k, t, v) = 0$. So we obtained a linear equation on k, i.e., $k = (t^2 - v + v^2 + tv^2)/(2 + t - v)$. The relation $f(k, t, v) = 0$ now transforms, by $t \neq -1$, to $(2t + v + tv)(t^2 - v - v^2 + v^3) = 0$. Since both t and v are integers by Theorem 3.4, the equation $2t + v + tv = 0$, i.e., $t = 2/(2+v) - 1$, has only four solutions, namely 0, -1, -3 or -4; however, none of them gives $k \geq 3$. Therefore, $t^2 - v - v^2 + v^3 = 0$, i.e., $t^2 = -v(v^2 - v - 1)$. Since v is a negative integer and these two factors are relatively prime, they both have to be perfect squares. This is not possible, because $v^2 - v - 1$ lies between the two consecutive perfect squares v^2 and $v^2 - 2v + 1$. This gives us our contradiction.

So the third factor $k^2 - k + tk + t + 1 + vt^2 + v + 2vt + t^2$ in the numerator of (19) is zero. Again, by subtracting $f(k, t, v)$ from it, we obtain a linear equation on k (note that $v \neq 1$ by Theorem 3.4), i.e.,

$$k = \frac{1 + t + t^2 + (1 + 2t)v + (1 + t)v^2}{1 - v}. \tag{22}$$

The relation $f(k, t, v) = 0$ now transforms, by $v \neq 1$ and $t \neq -1$, to

$$2t + 1 + 3v + 2v^2 = \pm(v - 1)\sqrt{-3 - 4v}. \tag{23}$$

Therefore, the expression $-3 - 4v$ must be a square, i.e., $-3 - 4v = s^2$ for some $s \in \mathbb{N}$. Then $v = -1 + (1 - s^2)/4$, which is an integer only when s is odd, so $s = 2q - 1$ for an integer $q > 1$. Now $v = -q^2 + q - 1$. In the case of the positive sign for the solution of t in the quadratic (23), we do not get a positive integer u. So we assume the negative sign and obtain, by (23) and (22), $t = 1 - 2q + q^3 - q^4 < 0$, $k = (1 - q + q^2)(1 - q + q^3)$, $u = 1 - q^2 + q^3$. We obtain (20) and (21) from formulas (9)–(11), and $k_2 = k(k-1)/c_2$, $k_3 = k(k-1)b_2/(c_2c_3)$.

The converse is a straightforward calculation by Lemma 4.1. ■

Proposition 4.3. *Let Γ be a triangle-free 1-homogeneous distance-regular graph with diameter $d \geq 2$ (if $d = 2$ we assume $b_2 = 0 = \tau_2$). Then*

(i) $c_2 \mid \gcd(a_2(2b_2 - \tau_2) - b_2\rho_3, a_2(a_2 - 1 - b_2 + \tau_2) + b_2\rho_3, b_2(b_2 - 1) - a_2(b_2 - \tau_2) + b_2\rho_3)$,

(ii) *If $\tau_2 = b_2$, then the size of a connected component of a second subconstituent of Γ is $c_2 - 1 + 2a_2 + a_2(a_2 - 1)/c_2$, and it divides $k - 1 + a_2(k-1)/c_2$.*

Proof. We will consider the distance partition corresponding to two distinct pairs of adjacent vertices (y, z_0) and (y, z_1).

(i) Note that for $h \in \{0, 1\}$, $|D_1^2(z_h, y)| = k - 1$, $|D_2^2(z_h, y)| = (k-1)a_2/c_2$ and $|D_3^2(z_h, y)| = (k-1)b_2/c_2$, see Fig. 4.1. These numbers are also the sums of rows and columns in the table of Fig. 4.2. The vertex z_h has, in the set $D_1^2(z_{1-h}, y)$, $c_2 - 1$, a_2 and b_2 vertices respectively at distances 1, 2 and 3. Furthermore, z_h has $(a_2(2b_2 - \tau_2) - b_2\rho_3)/c_2$ vertices at distance 2 in the set $D_3^2(z_{1-h}, y)$. Note that this table is symmetric. In order to understand this, we switch our

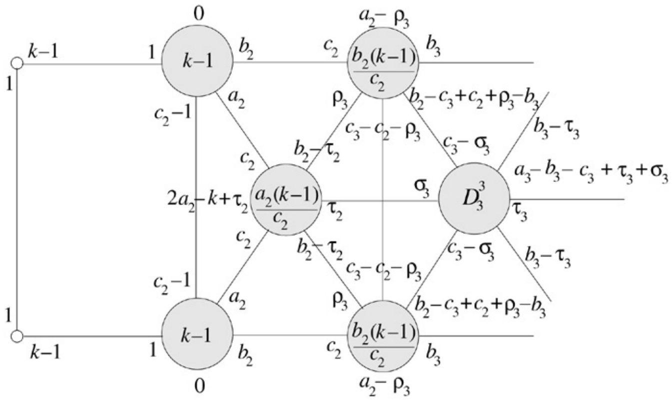

Fig. 4.1. The distance partition corresponding to an edge of a triangle-free 1-homogeneous distance-regular graph with diameter 3. Remember $k_2 = k(k-1)/c_2$, $k_3 = k(k-1)b_2/(c_2c_3)$. Once the parameter τ_2 is computed, we obtain $\rho_3 = (b_2 - \tau_2)a_2/b_2$, and in the case when the diameter of the graph equals 3, also $|D_3^3| = k_3 - (k-1)b_2/c_2 = (k-1)b_2a_3/(c_2c_3)$ and $\sigma_3 = (a_2 - \rho_3)c_3/a_3$.

$\Gamma_2(y)$

(z_1,y) \ (z_0,y)	D_1^2	D_2^2	D_3^2	Σ
D_1^2	c_2-1	a_2	b_2	$k-1$
D_2^2	a_2	$\frac{a_2(a_2-1-b_2+\tau_2)+b_2\rho_3}{c_2}$	$\frac{a_2(2b_2-\tau_2)-b_2\rho_3}{c_2}$	$\frac{(k-1)a_2}{c_2}$
D_3^2	b_2	$\frac{a_2(2b_2-\tau_2)-b_2\rho_3}{c_2}$	$\frac{b_2(b_2-1)-a_2(b_2-\tau_2)+b_2\rho_3}{c_2}$	$\frac{(k-1)b_2}{c_2}$
Σ	$k-1$	$\frac{(k-1)a_2}{c_2}$	$\frac{(k-1)b_2}{c_2}$	$\frac{(k-1)k}{c_2}$

Fig. 4.2. The second subconstituent of the vertex y is split corresponding to the distance from z_0 and z_1.

view from the distance partition corresponding to vertices y and z_h to the distance partition corresponding to vertices y and z_{1-h}, and the vertices from the sets that correspond to the collided numbers (the missing two were a_2, b_2 and $(a_2(2b_2 - \tau_2) - b_2\rho_3)/c_2$) "exchange" their positions in the distance distribution diagram. Therefore, by knowing the sums of columns/rows, it is easy to complete all the entries in the table of Fig. 4.2, and to conclude that they are integral.

(ii) Now we assume $\tau_2 = b_2$, see Figs. 4.3 and 4.4. Then there are no edges between the vertices of the sets $F_h := D_1^2(z_h, y) \cup D_2^2(z_h, y)$ and $D_3^2(z_h, y)$ for $h \in \{0, 1\}$.

Let us denote by H the subgraph induced by the vertices of the intersection $F_0 \cap F_1$, and let s be the size of a connected component in H. Since $|V(H)| = c_2 - 1 + 2a_2 + a_2(a_2 - 1)/c_2$, and

	$(z_1,y) \backslash (z_0,y)$	D_1^2	D_2^2	D_3^2	Σ
	D_1^2	c_2-1	a_2	b_2	$k-1$
$\Gamma_2(y)$	D_2^2	a_2	$\frac{a_2(a_2-1)}{c_2}$	$\frac{a_2 b_2}{c_2}$	$\frac{(k-1)a_2}{c_2}$
	D_3^2	b_2	$\frac{a_2 b_2}{c_2}$	$\frac{b_2(b_2-1)}{c_2}$	$\frac{(k-1)b_2}{c_2}$
	Σ	$k-1$	$\frac{(k-1)a_2}{c_2}$	$\frac{(k-1)b_2}{c_2}$	$\frac{(k-1)k}{c_2}$

Fig. 4.3. The second subconstituent of the vertex y is split corresponding to the distance from z_0 and z_1 in the case $b_2 = \tau_2$.

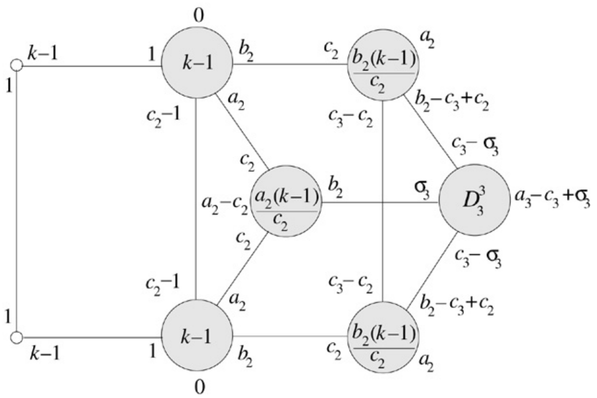

Fig. 4.4. The distance partition corresponding to an edge in the case $b_2 = \tau_2$ and $d = 3$.

a_2 is the valency of the second subconstituent graph that is also triangle-free, we have

$$1 + a_2 + \frac{a_2(a_2-1)}{c_2} \le s \le c_2 - 1 + 2a_2 + \frac{a_2(a_2-1)}{c_2}.$$

But the above lower bound is greater than half of the above upper bound:

$$1 + a_2 + \frac{a_2(a_2-1)}{c_2} - \frac{1}{2}\left(c_2 - 1 + 2a_2 + \frac{a_2(a_2-1)}{c_2}\right)$$
$$= 1 + \frac{(a_2-c_2)(a_2+c_2-1)}{2c_2} > 0,$$

so the subgraph H is connected and $s = c_2 - 1 + 2a_2 + a_2(a_2-1)/c_2$. Finally, the size of F_1 is divisible by s. ■

Theorem 4.4. *Let Γ be a primitive triangle-free distance-regular graph with diameter $d = 3$, valency $k > 2$, nontrivial eigenvalues $\{u, v, t\}$ and the multiplicity $m_t = k$. Then $b_2 = \tau_2$ if and*

Table A.1
The list of feasible parameters of primitive distance-regular graphs with $d = 3$, $a_1 = 0$, $a_2 \neq 0$ and $m_t = k$, where $k < 4000$ and t is the biggest nontrivial eigenvalue

n	k	t	u	v	$k_2 = m_u$	$k_3 = m_v$	a_2	c_2	c_3	τ_2	σ_3	ρ_3	$2a_2 + \tau_2 - k$
4,096	70	22	6	−10	2,415	1,610	48	2	30	10	18	24	36
3,200	105	25	5	−15	2,184	910	75	5	60	10	40	45	55
4,394	161	31	5	−21	3,220	1,012	120	8	105	11	75	80	90
237,276	1425	177	21	−57	169,100	66,750	1053	12	912	120	624	702	801
396,750	2668	253	23	−92	309,372	84,709	2116	23	1932	138	1449	1564	1702

Table A.2
The list of feasible parameters of primitive distance-regular graphs with $d = 3$, $a_1 = 0$, $a_2 \neq 0$ and $m_t = k$, where $k < 4000$ and t is the least eigenvalue

n	k	u	v	t	$k_2 = m_u$	$k_3 = m_v$	a_2	c_2	c_3	τ_2	σ_3	ρ_3	$2a_2 + \tau_2 - k$
! 512	21	5	−3	−11	210	280	3	2	12	16	4	0	1
// 17,576	175	19	−7	−59	5075	12,325	16	6	63	153	9	0	10
49,152	276	20	−12	−76	18,975	29,900	64	4	132	156	44	16	8
93,312	345	21	−15	−87	39,560	53,406	99	3	180	162	72	33	15
// 238,328	793	49	−13	−199	52,338	185,196	45	12	208	736	16	0	33
// 1,815,848	2541	101	−21	−509	322,707	1,490,599	96	20	525	2425	25	0	76

The nonexistence of three cases is proved in Theorem 4.4. The regularity of the graph induced by D_2^2 is equal to $2a_2 + \tau_2 - k$.

only if Γ is the coset graph of the doubly truncated binary Golay code. In particular, there are no examples of graphs with parameters from Table A.2 for $k \in \{175, 793, 2541\}$.

Proof. Without loss of generality, we assume (18) and that $f(k, t, v) = 0$ by Lemma 3.3, and obtain the parametrization (20) by Lemma 4.2. Therefore, by Proposition 4.3, the following expression

$$\left(k - 1 + \frac{(k-1)a_2}{c_2}\right) \frac{1}{c_2 - 1 + 2a_2 + a_2(a_2 - 1)/c_2} = q^2 - q + 2 + \frac{2}{q^2 - 2}$$

is integral. But this is possible only for $q = 2$. ■

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Appendix

See Tables A.1 and A.2 and Figs. A.3 and A.4.

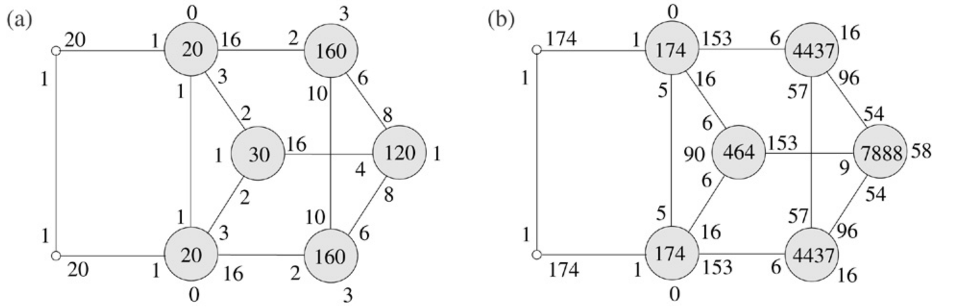

Fig. A.3. The distance partition corresponding to an edge of (a) the coset graph of the doubly truncated binary Golay code (the second subconstituent of the second graph consists of 21 disjoint Petersen graphs), (b) the case $k = 175$, with $n = 17{,}576$ ruled out.

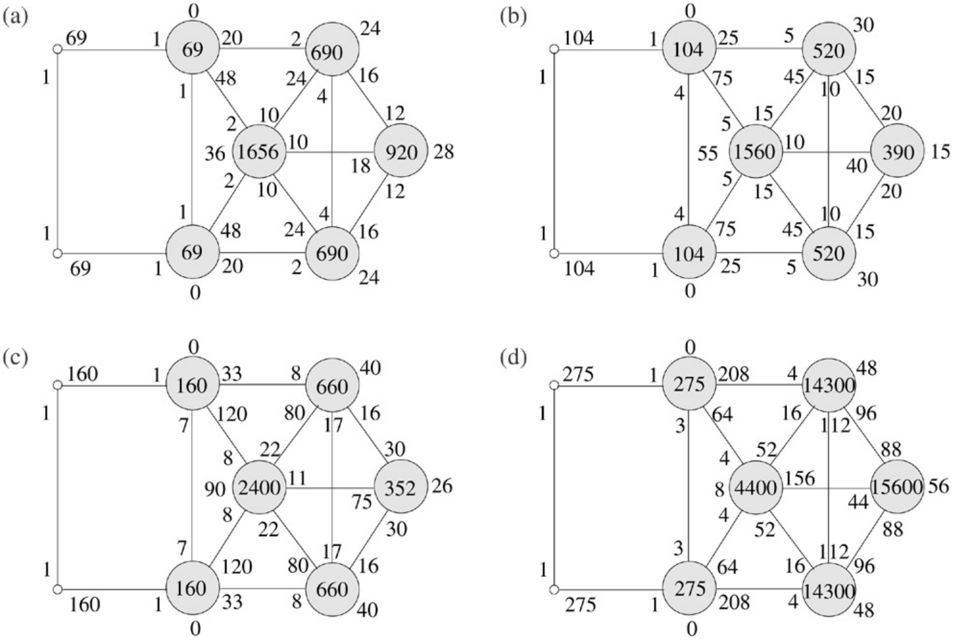

Fig. A.4. The distance partitions corresponding to an edge in the following cases: (a) $k = 70$, $n = 4096$; (b) $k = 105$, $n = 3200$; (c) $k = 161$, $n = 4394$. (d) $k = 276$, $n = 49{,}152$.

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