Europ. J. Combinatorics (1987) **8**, 341–350

Partitions with Prescribed Hook Differences

GEORGE E. ANDREWS,* R. J. BAXTER, D. M. BRESSOUD,† W. H. BURGE, P. J. FORRESTER AND G. VIENNOT

We investigate partition identities related to off-diagonal hook differences. Our results generalize previous extensions of the Rogers–Ramanujan identities. The identity of the related polynomials with constructs in statistical mechanics is discussed.

1. INTRODUCTION

I. Schur's polynomial proof of the Rogers–Ramanujan identities [15] has been the starting point for several diverse studies. The polynomials Schur introduced were generalized to provide partition identities related to successive ranks of partitions [1], [2], [5], [10], [11], [12], [13]. Subsequently these same polynomials were found to be the key element in providing Rogers–Ramanujan type identities for Regime II of the hard hexagon model [4], [9; ch. 14].

In [6], three of us considered an extensive generalization of the hard hexagon model, and we found that polynomial approximations to certain sums of eigenvalues played an essential role. Again these polynomials were clearly of the same general structure as those introduced by Schur.

In this paper we consider in full generality the families of polynomials that arose in [6]. The statistic of partitions that plays the main role is that of 'off-diagonal hook difference', a topic introduced in [13]. In Section 2 we provide the necessary combinatorial preliminaries. In Section 3 we prove a general partition theorem which generalized the results in [1], [2], [5], [10], [11]. Section 4 applies our main theorem to situations in which the products simplify. In Section 5 we identify the polynomials with those that arose in the generalization of the hard hexagon model [6], [14].

The proof we give of our main theorem is via recurrence relations patterned after the work in [5]. This approach lacks the elegance of the more purely combinatorial work in [10] and [13]. However such an approach perfectly parallels the work on the generalized hard hexagon model [6], [14]; indeed one can, by comparing the two projects, see clearly how the polynomials from the hard hexagon work are in fact being formed according to the hook-difference rules on partitions that we describe in Section 2.

2. PRELIMINARIES ON HOOK DIFFERENCES

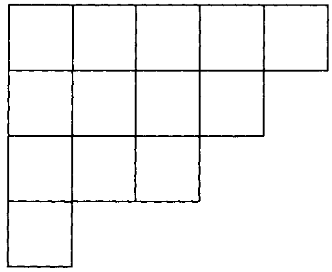
We shall now extend the concept of successive ranks introduced in [1] and [7] to every node of the Ferrers graph of a partition [3; ch. 1].

DEFINITION 1. Let Π be a partition whose Ferrers graph has a node in the i-th row and j-th column; we call this node the (i, j)th node. We define the hook difference at the (i, j)th node to be the number of nodes in the ith row of Π minus the number of nodes in the jth column of Π.

*Partially supported by National Science Foundation Grant DMS 85-03324
†Partially supported by National Science Foundation Grant DMS 84-04083

0195-6698/87/040341 + 10 $02.00/0

For example, if Π is $5 + 4 + 3 + 1$, then the Ferrers graph is



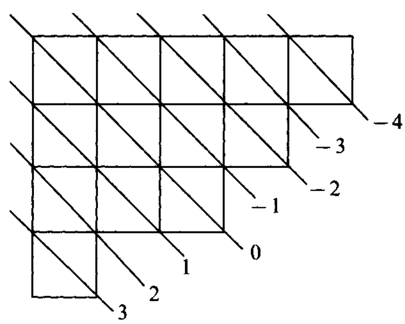
and the hook differences at each node are

1	2	2	3	4
0	1	1	2	
−1	0	0		
−3				

DEFINITION 2. We say that the (i, j)th node lies on diagonal c if $i - j = c$.

We remark that the successive ranks given by Atkin [7] are just the hook differences on the diagonal 0.

Referring again to $5 + 4 + 3 + 1$, we have indicated below the various diagonals.



The original 'successive rank' theorem ([2], [10]) may be revised as follows:

THEOREM. *Let $Q_{K,i}(n)$ denote the number of partitions of n such that on the diagonal 0 we have all hook differences $\geqslant -i + 2$ and also $\leqslant K - i - 2$. Let $A_{K,i}(n)$ denote the number of partitions of n into parts $\not\equiv 0, \pm i (\mathrm{mod}\ K)$. Then for $1 \leqslant i < K/2$ and all n*

$$Q_{K,i}(n) = A_{K,i}(n).$$

We remark that

$$\sum_{n=0}^{\infty} Q_{K,i}(n)q^n = \sum_{n=0}^{\infty} A_{K,i}(n)q^n$$

$$= \prod_{\substack{m=1 \\ m \not\equiv 0, \pm i (\mathrm{mod}\ K)}}^{\infty} (1 - q^m)^{-1}$$

$$= \frac{\sum_{\lambda=-\infty}^{\infty} (-1)^{\lambda} q^{K\binom{\lambda}{2}+i}}{\prod_{m=1}^{\infty} (1-q^m)} \quad \text{(by [3; p. 22, eq. (2.2.11)])}$$

$$= \frac{\sum_{\lambda=-\infty}^{\infty} q^{2K\lambda^2+(2i-K)\lambda} - q^i \sum_{\lambda=-\infty}^{\infty} q^{2K\lambda^2+(2i+K)\lambda}}{\prod_{m=1}^{\infty} (1-q^m)}.$$

Thus the generating function for $Q_{K,i}(n)$ may be written as a difference of two theta series with positive terms divided by $\Pi(1 - q^n)$. In the next section we shall show that this same phenomenon occurs when we fix two arbitrary diagonals (not necessarily distinct) together with a lower bound on hook differences on the first diagonal and an upper bound on the second. This, of course, will reduce to the above when both diagonals are the diagonal 0.

3. The Main Theorems

Definition 3. Let α, β be positive integers. Define $p_{k,i}(N, M; \alpha, \beta; n)$ to be the number of partitions of n into at most M parts each $\leqslant N$ such that the hook differences on diagonal $1 - \beta$ are $\geqslant -i + \beta + 1$ and on diagonal $\alpha - 1$ are $\leqslant K - i - \alpha - 1$.

The related generating function is, of course, a polynomial:

$$D_{K,i}(N, M; \alpha, \beta; q) \equiv D_{K,i}(N, M; \alpha, \beta) \equiv \sum_{n\geqslant 0} p_{K,i}(N, M; \alpha, \beta; n)q^n. \tag{3.1}$$

In order to facilitate the proof of our main theorem, we shall first record some observations in short lemmas.

Lemma 1. *Suppose Π is a partition. Let Π^{λ} denote the partition obtained from Π by deleting the largest part of Π, and let $\Pi^{\#}$ be the partition obtained from Π by deleting all* 1's *and subtracting* 1 *from each of the remaining parts. Let P be a node of the Ferrers graph of Π, and suppose P lies on diagonal Δ and has hook difference d. If P is not in the first row of Π, then in Π^{λ} P lies on diagonal $\Delta - 1$ and has hook difference $d + 1$. If P is not in the first column of Π, then in $\Pi^{\#}$, P lies on diagonal $\Delta + 1$ and has hook difference $d - 1$.*

Proof. Let us assume P is in the ith row and jth column of Π. Thus $\Delta = i - j$. If there are r_i nodes in row i and c_j nodes in column j, then $d = r_i - c_j$.

In Π^{λ}, P is in the $(i - 1)$st row and jth column. Hence the hook difference is now $r_i - (c_j - 1) = d + 1$, and the diagonal is now $(i - 1) - j = \Delta - 1$.

In $\Pi^{\#}$, P is in the ith row and the $(j - 1)$st column. Hence the hook difference is now $(r_i - 1) - c_j = d - 1$ and the diagonal is $i - (j - 1) = \Delta + 1$.

Definition 4.

$$\delta_{K,i}(N, M; \alpha, \beta) = \sum_{\mu=-\infty}^{\infty} q^{\mu(K\mu+i)(\alpha+\beta)-K\beta\mu} \begin{bmatrix} N+M \\ N-K\mu \end{bmatrix} - \sum_{\mu=-\infty}^{\infty} q^{\mu(K\mu-i)(\alpha+\beta)-K\beta\mu+\beta i} \begin{bmatrix} N+M \\ N-K\mu+i \end{bmatrix}, \tag{3.2}$$

where $\left[{A \atop B}\right]$ is the Gaussian polynomial or q-binomial coefficient defined by

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{cases} \dfrac{(1-q^A)(1-q^{A-1})\dots(1-q^{A-B+1})}{(1-q^B)(1-q^{B-1})\dots(1-q)}, & B \text{ a nonnegative integer} \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Lemma 2. *For* $0 < i \leqslant K/2$, K *and* i *integral,*

$$\delta_{K,i}(0, 0; \alpha, \beta) = 1. \tag{3.4}$$

Proof. When $M = N = 0$ and $0 < i \leqslant K/2$, the only nonvanishing term in (3.2) arises for $\mu = 0$ in the first sum, and this term is 1.

Lemma 3.

$$\delta_{K,i}(N, M; \alpha, \beta) = \delta_{K,i}(N-1, M; \alpha, \beta) + q^N \delta_{K,i}(N, M-1; \alpha - 1, \beta + 1), \tag{3.5}$$

$$\delta_{K,i}(N, M; \alpha, \beta) = \delta_{K,i}(N, M-1; \alpha, \beta) + q^M \delta_{K,i}(N-1, M; \alpha + 1, \beta - 1). \tag{3.6}$$

Proof. These recurrences are merely straightforward term-by-term applications of the following well-known recurrences for the Gaussian polynomials [3; p.35, eq.(3.3.4)]

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A-1 \\ B-1 \end{bmatrix} + q^B \begin{bmatrix} A-1 \\ B \end{bmatrix}, \tag{3.7}$$

and [3; p. 35, eq. (3.3.3)]

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A-1 \\ B \end{bmatrix} + q^{A-B} \begin{bmatrix} A-1 \\ B-1 \end{bmatrix}. \tag{3.8}$$

Lemma 4.

$$\delta_{K,i}(M + K - i, M; 0, \beta) = 0, \tag{3.9}$$

$$\delta_{K,i}(M - i, M; \alpha, 0) = 0. \tag{3.10}$$

Proof. First

$$\begin{aligned} \delta_{K,i}(M + K - i, M; 0, \beta) &= \sum_{\mu=-\infty}^{\infty} q^{\mu(K\mu+i)\beta - K\mu\beta} \begin{bmatrix} 2M + K - i \\ M + K - i - K\mu \end{bmatrix} \\ &\quad - \sum_{\mu=-\infty}^{\infty} q^{\mu(K\mu-i)\beta - K\mu\beta + \beta i} \begin{bmatrix} 2M + K - i \\ M + K - K\mu \end{bmatrix} \\ &= 0, \end{aligned}$$

since the second summation becomes identical with the first if we replace μ by $1 - \mu$ and note that $\left[{A \atop B}\right] = \left[{A \atop A-B}\right]$. Second

$$\begin{aligned} \delta_{K,i}(M - i, M; \alpha, 0) &= \sum_{\mu=-\infty}^{\infty} q^{\mu(K\mu+i)\alpha} \begin{bmatrix} 2M - i \\ M - i - K\mu \end{bmatrix} \\ &\quad - \sum_{\mu=-\infty}^{\infty} q^{\mu(K\mu-i)\alpha} \begin{bmatrix} 2M - i \\ M - K\mu \end{bmatrix} \\ &= 0, \end{aligned}$$

since the second summation becomes identical with the first if we replace μ by $-\mu$ and note again that $\left[{A \atop B}\right] = \left[{A \atop A-B}\right]$.

We are now prepared to prove our main result:

THEOREM 1. *Let* $1 \leqslant i \leqslant K/2, \alpha > 0, \beta > 0, \alpha + \beta < K$ *with* $M, N, K, i, \alpha, \beta$ *integers. For* $-i + \beta \leqslant N - M \leqslant K - i - \alpha$,

$$D_{K,i}(N, M; \alpha, \beta) = \delta_{K,i}(N, M; \alpha, \beta). \tag{3.11}$$

PROOF. Let us consider the following set of recurrences and initial conditions:

$$\varepsilon_{K,i}(0, 0; \alpha, \beta) = 1. \tag{3.12}$$

$$\varepsilon_{K,i}(N, M; \alpha, \beta) = \varepsilon_{K,i}(N-1, M; \alpha, \beta) + q^N \varepsilon_{K,i}(N, M-1; \alpha - 1, \beta + 1),$$
$$\alpha > 0, \ -i + \beta < N - M \leqslant K - i - \alpha, \tag{3.13}$$

$$\varepsilon_{K,i}(N, M; \alpha, \beta) = \varepsilon_{K,i}(N, M-1; \alpha, \beta) + q^M \varepsilon_{K,i}(N-1, M; \alpha + 1, \beta - 1),$$
$$\beta > 0, \ -i + \beta \leqslant N - M < K - i - \alpha, \tag{3.14}$$

$$\varepsilon_{K,i}(M + K - i, M; 0, \beta) = 0, \tag{3.15}$$

$$\varepsilon_{K,i}(M - i, M; \alpha, 0) = 0, \tag{3.16}$$

where throughout $K, i, \alpha, \beta, N, M$ are nonnegative integers with $\alpha + \beta < N$, $1 \leqslant i \leqslant K/2$.

Note that (3.12)–(3.16) uniquely define (if not overdefine) $\varepsilon_{K,i}(N, M; \alpha, \beta)$ for $-i + \beta \leqslant N - M \leqslant K - i - \alpha$. To see this we observe that by Lemmas 2–4 the $\delta_{K,i}(N, M; \alpha, \beta)$ do satisfy this set of recurrences. Hence we wish to show that these are the only solutions of (3.12)–(3.16). We proceed by induction on $H = M + N$.

For $H = 0$, we see that $\varepsilon_{K,i} = \delta_{K,i}$ by (3.12). Now assume that we have uniqueness up to but not including a particular H:

If both α and β are greater than 0 then we may use our induction hypothesis together with (3.13) and (3.14) to show that we have uniqueness at $H = N + M$; we emphasize the fact that the union of the conditions on (3.13) and (3.14) is $-i + \beta \leqslant N - M \leqslant K - i - \alpha$ so that at least one of (3.13) and (3.14) is applicable in this instance.

If $\alpha = 0$, then we must apply (3.14) which is possible unless $N - M = K - i$, i.e. $N = M + K - i$. If $\alpha = 0$ and $N = M + K - i$, then we apply (3.15) and get uniqueness.

If $\beta = 0$, then we must apply (3.13) which is possible unless $N - M = -i$, i.e. $N = M - i$. If $\beta = 0$ and $N = M - i$, then we apply (3.16) and get uniqueness.

In passing we mention that the interval of validity for both (3.13) and (3.14) is chosen so that each term in the recurrence, say $\varepsilon_{K,i}(N', M'; \alpha', \beta')$ satisfies $\alpha' + \beta' < K$, $-i + \beta' \leqslant N' - M' \leqslant K - i - \alpha'$.

Thus our induction is complete, and we see that the only solution of (3.12)–(3.16) with $-i + \beta \leqslant N - M \leqslant K - i - \alpha$ is $\varepsilon_{K,i}(N, M; \alpha, \beta) = \delta_{K,i}(N, M; \alpha, \beta)$.

Therefore to conclude our proof we need only show that the $D_{K,1}(N, M; \alpha, \beta)$ fulfill (3.12)–(3.16).

Since $p_{K,i}(N, M; \alpha, \beta; n)$ is only defined for positive α and β, we first extend its definition to the cases $\alpha = 0$ and $\beta = 0$.

DEFINITION 5. Let $p_{K,i}(N, M; 0, \beta; n)$ denote the number of partitions of n subject to the conditions of Definition 3 (with $\alpha = 0$) with the added condition that the number of parts of the partition lies in the closed interval $[N - K + i + 1, M]$.

DEFINITION 6. Let $p_{K,i}(N, M; \alpha, 0; n)$ denote the number of partitions of n subject of Definition 3 (with $\beta = 0$) with the added condition that the largest part of the partition lies in the closed interval $[M - i + 1, N]$.

In light of these two definitions we now have $D_{K,i}(N, M; \alpha, \beta)$ defined for $\alpha \geqslant 0$ and $\beta \geqslant 0$. Let us now examine (3.12)–(3.16) for the $D_{K,i}(N, M; \alpha, \beta)$.

With regard to (3.12), we note that the empty partition of 0 is the only partition admissable for $N = M = 0$. Hence

$$D_{K,i}(0, 0; \alpha, \beta) = 1. \tag{3.17}$$

Next comes (3.13) with $\alpha > 1$ and $\beta > 0$. The expression $p_{K,i}(N, M; \alpha, \beta; n) - p_{K,i}(N - 1, M; \alpha, \beta; n)$ counts those partitions enumerated by $p_{K,i}(N, M; \alpha, \beta; n)$ that have largest part exactly equal to N. Delete this part from each of these partitions. By Lemma 1, the diagonal numbers all drop by 1 and the hook differences all increase by 1. Hence we are now considering a partition of $n - N$ into at most $M - 1$ parts subject to the diagonal and hook difference requirements of Definition 1 wherein α is replaced by $\alpha - 1$ and $\beta + 1$. Therefore the set of partitions under consideration has as generating function $q^N D_{K,i}(N, M - 1; \alpha - 1, \beta + 1)$ and (3.13) is verified in this instance.

If we repeat the preceding argument with $\beta = 0$ and $\alpha > 1$, then we must take into account the fact that the partitions we start with have largest part in $[M - i + 1, N]$. This constraint becomes vacuous after the removal of the largest part.

Finally for (3.13) we look at what happens when $\alpha = 1$. Let ν denote the number of parts of one of the partitions enumerated by $p_{K,i}(N, M; \alpha, \beta; n) - p_{K,i}(N - 1, M; \alpha, \beta; n)$. Clearly $\nu \leqslant M$ and since $\alpha = 1$, we see by Definition 3 that $N - \nu \leqslant K - i - 2$. Hence $N - K + i + 1 \leqslant \nu - 1 \leqslant M - 1$, and this condition is precisely what is needed to apply Definition 5 for $p_{K,i}(N, M - 1; 0, \beta + 1; N - M)$.

In any event, (3.13) holds for $D_{K,i}(N, M; \alpha, \beta)$.

Equation (3.14) follows in a similar manner. Now we note that $p_{K,i}(N, M; \alpha, \beta; n) - p_{K,i}(N, M - 1; \alpha, \beta; n)$ enumerates appropriate partitions with exactly M parts. Delete all 1's and subtract 1 from the remaining parts. Lemma 1 takes care of everything except the case $\beta = 1$. When $\beta = 1$ it follows from Definition 3 that for the partitions under consideration $N - 1 \geqslant l - 1 \geqslant M - i + 1$ where l is the largest part. This, of course, allows us to invoke Definition 6; hence (3.14) is valid in all instances.

Finally (3.15) and (3.16) are trivial by Definitions 5 and 6, respectively; namely there are no numbers in $[M + 1, M]$ or $[M - i + 1, M - i]$ and so there are no partitions enumerated by $D_{K,i}(M + K - i, M; 0, \beta)$ or $D_{K,i}(M - 1, M; \alpha, 0)$.

Therefore $D_{K,i}(N, M; \alpha, \beta)$ fulfills (3.12)–(3.16) for $-i + \beta \leqslant N - M \leqslant K - i - \alpha$, and so $D_{K,i}(N, M; \alpha, \beta) = \delta_{K,i}(N, M; \alpha, \beta)$ as desired.

Definition 7. Let $p_{K,i}(\alpha, \beta; n)$ denote the number of partitions of n such that hook differences on the diagonal $1 - \beta$ are $\geqslant -i + \beta + 1$ and on the diagonal $\alpha - 1$ are $\leqslant K - i - \alpha - 1$.

Theorem 2. *For $1 \leqslant i \leqslant K/2$, $\alpha > 0$, $\beta > 0$, $\alpha + \beta < K$ all integers*

$$\sum_{n=0}^{\infty} p_{K,i}(\alpha, \beta; n) q^n \tag{3.18}$$

$$= \frac{1}{(q)_\infty} \sum_{\mu=-\infty}^{\infty} q^{\mu(K\mu - i)(\alpha+\beta) + K\beta\mu}(1 - q^{\beta i + 2i(\alpha+\beta)\mu})$$

$$= \frac{1}{(q)_\infty}\Bigg(\prod_{n=0}^{\infty}(1 - q^{2K(\alpha+\beta)(n+1)})(1 + q^{K\beta + (\alpha+\beta)(2Kn+K-i)})(1 + q^{-K\beta+(\alpha+\beta)(2Kn+K+i)})$$

$$- q^{\beta i}\prod_{n=0}^{\infty}(1 - q^{2K(\alpha+\beta)(n+1)})(1 + q^{K\beta+(\alpha+\beta)(2Kn+K+i)})(1 + q^{-K\beta+(\alpha+\beta)(2Kn+K-i)})\Bigg)$$

where $(q_\infty) = \Pi_{n=1}^{\infty}(1 - q^n)$.

PROOF. The first portion of (3.18) follows immediately from Theorem 1 if we let N and M both tend to infinity in such a way that all times $-i + \beta \leqslant N - M \leqslant K - i - \alpha$ and we invoke

$$\lim_{N,M\to\infty} \begin{bmatrix} N + M \\ N - A \end{bmatrix} = \frac{1}{(q)_\infty},$$

by (3.3).

The last portion of (3.18) follows immediately by application of Jacobi's triplet product identity [3; p.21, eq.(2.2.10)].

THEOREM 3. *For* $1 \leqslant i, \alpha \leqslant K/2$, $2i \neq K$,

$$\sum_{n=0}^{\infty} p_{K,i}(\alpha, \alpha; n)q^n = \prod_{\substack{n=1 \\ n \not\equiv 0, \pm\alpha i (\bmod \alpha K)}}^{\infty} (1 - q^n)^{-1}. \tag{3.19}$$

PROOF. By the first portion of (3.18) with $\alpha = \beta$

$$\begin{aligned}
\sum_{n=0}^{\infty} p_{K,i}(\alpha, \alpha; n)q^n &= \frac{1}{(q)_\infty} \sum_{\mu=-\infty}^{\infty} q^{2\alpha\mu(K\mu - i) + K\alpha\mu}(1 - q^{\alpha i + 4i\alpha\mu}) \\
&= \frac{1}{(q)_\infty} \sum_{\mu=-\infty}^{\infty} (-1)^\mu q^{\alpha K\binom{\mu+1}{2} - i\alpha\mu} \\
&= \frac{1}{(q)_\infty} \prod_{n=0}^{\infty} (1 - q^{K\alpha(n+1)})(1 - q^{i\alpha + K\alpha n})(1 - q^{-i\alpha + K\alpha(n+1)}) \\
&\qquad\qquad \text{(by [3; p. 21, eq. (2.2.10)])} \\
&= \prod_{\substack{n=1 \\ n \not\equiv 0, \pm i\alpha (\bmod K\alpha)}}^{\infty} (1 - q^n)^{-1}.
\end{aligned}$$

4. REDUCTIONS

It is well-known that in certain circumstances the difference of two theta series is representable as an infinite product, The first instance we have ssen of this is the theorem quoted in Section 2. Another nice application of this idea concerns the quintuple product identity [16; p. 205, eq. (7.4.7)]:

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} z^{3n}(1 + zq^n) \\
&= \prod_{n=1}^{\infty} (1 - q^n)(1 + zq^{n-1})(1 + z^{-1}q^n)(1 - z^2q^{2n-1})(1 - z^{-2}q^{2n-1}).
\end{aligned} \tag{4.1}$$

We may apply (4.1) in Theorem 2 to obtain:

THEOREM 4. *Let* K, α, β *be positive integers with* $\alpha + \beta < 3K$. *Let* $\mathscr{A}_K(\alpha, \beta, n)$ *denote the number of partitions of n into parts not congruent with* 0, $\pm K\beta (\bmod 2K(\alpha + \beta))$ *nor to* $\pm 2K\beta (\bmod 4K(\alpha + \beta))$. *Then*

$$p_{3K,K}(\alpha, \beta, n) = \mathscr{A}_K(\alpha, \beta, n).$$

REMARK. If $\alpha = \beta = 1$ this result reduces to a special case of the theorem given in Section 2.

PROOF. By Theorem 2,

$$\sum_{n=0}^{\infty} p_{3K,K}(\alpha, \beta, n)q^n$$

$$= \frac{\sum_{\mu=-\infty}^{\infty} q^{\mu(3K\mu-K)(\alpha+\beta)+3K\beta\mu}(1 - q^{K\beta+2K(\alpha+\beta)\mu})}{\prod_{m=1}^{\infty}(1 - q^m)}$$

$$= \frac{\prod_{n=1}^{\infty}(1 - q^{Rn})(1 - q^{K\beta+R(n-1)})(1 - q^{Rn-K\beta})(1 - q^{2K\alpha+2R(n-1)})(1 - q^{-2K\alpha+2Rn})}{\prod_{m=1}^{\infty}(1 - q^m)}$$

(where $R = 2K(\alpha + \beta)$)

$$= \prod_{\substack{m=1 \\ m\not\equiv 0, \pm K\beta (\mathrm{mod}\ R) \\ m\not\equiv \pm 2K\alpha (\mathrm{mod}\ 2R)}}^{\infty} (1 - q^m)^{-1}$$

$$= \sum_{n=0}^{\infty} \mathscr{A}_K(\alpha, \beta, n)q^n.$$

Comparing coefficients in the extremes of this identity we obtain the desired result.

W. N. Bailey has given a second identity [8; p. 220, eq. (4.1)] that resembles the quintuple product identity:

$$\prod_{n=1}^{\infty}(1 - q^{4n})(1 + q^{4n-3}z^2)(1 + q^{4n-1}z^{-2})$$

$$-z\prod_{n=1}^{\infty}(1 - q^{4n})(1 + q^{4n-1}z^2)(1 + q^{4n-3}z^{-2})$$

$$= \prod_{n=1}^{\infty}(1 - zq^{n-1})(1 - z^{-1}q^n)(1 - q^n). \tag{4.3}$$

Invoking (4.3) instead of (4.1) we immediately have

THEOREM 5. Let K, α, β be positive integers with $\alpha + \beta < 2K$. Let $\mathscr{B}_K(\alpha, \beta, n)$ denote the number of partitions of n into parts not contruent to 0, $\pm K\beta$ mod $(K(\alpha + \beta))$. Then

$$p_{2K,K}(\alpha, \beta, n) = \mathscr{B}_K(\alpha, \beta, n). \tag{4.4}$$

PROOF. Replace q by $q^{K(\alpha+\beta)}$ and z by $q^{K\beta}$ in (4.2) and then apply Theorem 2.

5. The Generalized Hard Hexagon Model

The point of this section is to show that the $D_{K,i}(N, M; \alpha, \beta)$ have in fact arisen before in full generality in a quite different setting. As we noted in the introduction, our study has its genesis in Schur's work on the Rogers–Ramanujan identities [15], and Schur's ideas have been extensively generalized in recent work in statistical mechanics [4], [6], [14]. Not surprisingly then we find that our combinatorial interpretation and extension of Schur's polynomials is an altered form of the polynomials arising in the generalizations of the hard hexagon model.

Two distinct families were treated in [6]: $X_m(a, b, c)$ [6; p. 211] and $Y_m(a, b, c)$ [6; p. 217]. In our notation,

$$X_m(a, b, c) = q^{(a-b)(a-c)/4} D_{2n+1,a}\left(\frac{m-a+b}{2}, \frac{m+a-b}{2}; 2n - \frac{(b+c-1)}{2}, \frac{b+c-1}{2}\right), \tag{5.1}$$

and

$$Y_m(a, b, c) = q^{a/2+\beta_m(a,b,c)} D_{2n+1,a}\left(\frac{m-a+b}{2}, \frac{m+a-b}{2}; \frac{1-b+c}{2} + \chi(c); \frac{b-c+3}{2} - \chi(c)\right), \tag{5.2}$$

where

$$\beta_m(a, b, c) = \tfrac{1}{4}(b-c+1)(m+b-a) - \frac{a}{2} + \frac{\chi(c)}{2}(mc - mb + a - b), \tag{5.3}$$

$$\chi(c) = \begin{cases} 1, & \text{if } c \leqslant n \\ 0, & \text{if } c > n, \end{cases} \tag{5.4}$$

with $c = b \pm 1$ and $m - a + b$ an even integer.

The families (5.1) and (5.2) are clearly not the $D_{K,i}(N, M; \alpha, \beta)$ in full generality; the polynomials in (5.1) have $\alpha + \beta = K - 1$, and those in (5.2) have $\alpha + \beta = 2$. However in a subsequent paper [14] of still greater generality, the entire sets of $D_{K,i}(N, M; \alpha, \beta)$ polynomials arises. Namely, the polynomial $D_m^{(k)}(a, b, c)$ defined in (1.6.4) of [14] is shown to be [14; Th.2.3.1]:

$$D_{r,a}\left(\frac{m-a+b}{2}, \frac{m+a-b}{2}; \frac{b+c-1}{2} - k; r - \mu + k - \frac{b+c-1}{2}\right) \tag{5.5}$$

where $c = b \pm 1$.

The identifications given by (5.1), (5.2) and (5.5) are immediate once the index of summation in the sums for the $D_{K,i}$ is replaced by its negative and each $\left[{A \atop B}\right]$ is replaced by $\left[{A \atop A-B}\right]$.

Finally we note that a comparison of the polynomial recurrences in [6], [14] with the recurrences we have used shows clearly how to pass from our partition-theoretic interpretation to the truncated eigenvalue sums and back.

References

1. G. E. Andrews, Sieves for theorem and Euler, Rogers and Ramanujan, from *The Theory of Arithmetic Functions* (A. A. Gioia and D. L. Goldsmith, eds.), *Lecture Notes in Math.*, No. 251, Springer, Berlin, 1971, pp. 1–20.
2. G. E. Andrews, Sieves in the theory of partitions. *Amer. J. Math.*, **94** (1972), 1214–1230.
3. G. E. Andrews, *The Theory of Partitions, Vol. 2, Encyclopedia of Math. and Its Applications*, G.-C. Rota ed., Addison-Wesley, Reading, 1976.
4. G. E. Andrews, The hard-hexagon model and Rogers–Ramanujan type identities, *Proc. Nat. Acad. Sci. U.S.A.*, **78** (1981), 5290–5292.
5. G. E. Andrews, Uses and extensions of Frobenius' representation of partitions, from *Enumeration and Design*, D. M. Jackson and S. A. Vanstone eds., Academic Press, Toronto, 1984, pp. 51–65.
6. G. E. Andrews, R. J. Baxter and P. J. Forrester, Eight-vertex SOS model and generalized Rogers–Ramanujan-type identities, *J. Stat. Phys.*, **35** (1984), 193–266.

7. A. O. L. Atkin, A note on ranks and conjugacy of partitions, *Quart. J. Math., Oxford Ser.* **2**, 17 (1966), 335–338.
8. W. N. Bailey, On the simplification of some identities of the Rogers–Ramanujan type, *Proc. London Math. Soc.* (3) **1** (1951), 217–222.
9. R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, London, 1982.
10. D. M. Bressoud, Extension of the partition sieve, *J. Number Theory* **12** (1980), 87–100.
11. W. H. Burge, A correspondence between partitions related to generalizations of the Rogers–Ramanujan, *Discrete Math.* **34** (1981), 9–15.
12. W. H. Burge, A three way correspondence between partitions, *Euro. J. Combinatorics* **3** (1982), 195–213.
13. W. H. Burge, Combinatorial interpretations of some identities of the Rogers–Ramanujan type, I.B.M. Research Report, 1982, RC 9329 (#41101), 10 pp.
14. P. J. Forrester and R. J. Baxter, Further exact solutions of the eight-vertex SOS model and generalizations of the Rogers–Ramanujan identities, *J. Stat. Phys.* **38** (1985), 435–472.
15. I. Schur, Ein Beitrag zur additiven Zahlentheorie und zur Theorie der Kettenbrüche, *Sitz.-Ber. Preuss. Akad. Wiss. Phys.-Math.* **Kl**. (1917), 302–321. (Reprinted in I. Schur, *Gesammelte Abhandlungen, Vol. 2*, pp. 117–136, Springer, Berlin, 1973).
16. L. J. Slater, *Generalized Hypergeometric Functions*, Cambridge University Press, London, 1966.

Received 2 September 1985

G. E. Andrews and D. M. Bressoud
Department of Mathematics
The Pennsylvania State University
University Park, PA 16802, U.S.A.

R. J. Baxter and P. J. Forrester
Research School of Physical Science
Australian National University Canberra, ACT, 2601, Australia

W. H. Burge
Mathematical Sciences Department
IBM, Thomas J. Watson Research Center
Box 218, Yorktown Heights, NY 10598, U.S.A.

G. Viennot
Department Mathematics et Informatiques
Université de Bordeaux I,
33405 Talence Cedex, France