Nonlinear Differential-Difference and Difference Equations: Integrability and Exact Solvability

R. SAHADEVAN
Ramanujan Institute for Advanced Study in Mathematics
University of Madras, Chepauk, Chennai - 600 005, Tamil Nadu, India

Abstract—A brief review on the recent results of nonlinear differential-difference and difference equations toward its complete integrability and exact solvability is presented. In particular, we show how Lie's theory of differential equations can be extended to differential-difference and pure difference equations and illustrate its applicability through the discrete Korteweg-deVries equation as an example. Also, we report that an autonomous nonlinear difference equation of an arbitrary order with one or more independent variables can be linearised by a point transformation if and only if it admits a symmetry vector field whose coefficient is the product of two functions, one of the dependent variable and of the independent variables. This is illustrated by linearising several first- and second-order ordinary nonlinear difference equations. A possible connection between the Lie symmetry analysis and the onset of chaos with reference to first-order mappings is explored. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords—Differential-difference equations, Difference equations, Integrability, Exact solvability, Discrete systems.

1. INTRODUCTION

Many interesting dynamical problems in applied science such as ladder type electric circuits, vibration of particles and lattices in physics [1], spin lattices in statistical mechanics, phenomena in crystals, molecular chains [2,3], to cite a few, can be modeled by nonlinear differential-difference and pure difference equations. Also, differential-difference equations occur naturally whenever discrete phenomena are studied. Difference equations and their q-analogs play a crucial role in the representation theory of quantum groups and, hence, in special function theory [3–5]. Similarly, difference equations occur in studies of essentially continuous phenomena when differential equations are approximated by discrete ones, for instance, in numerical studies. These are some of the reasons of interest to investigate different mathematical and physical aspects of discrete dynamical systems, and hence, there have been hectic activities in this topic in recent years [6–9].

Nonlinear problems are nonintegrable and, hence, not solvable in general. In fact, there exist no well-defined analytical techniques to analyse nonlinear problems systematically. Unfortunately, due to the inherent difficulties of nonlinear problems, most methods of solution apply only in special circumstances [10]. To investigate nonlinear equations, in particular discrete ones, what...
one generally does is look for a stationary solution and study its stability or else make use of a small parameter in a perturbation analysis or else linearize the problem completely. In all the cases, certain important features of the full discrete nonlinear systems are lost. However, recent developments in nonlinear differential-difference and pure difference equations, particularly toward its complete integrability, exact solvability, or linearisability, have clearly demonstrated that this difference can be remedied by analytical methods [6-10]. The aim of this article is two-fold. First, we wish to present a brief salient feature of certain important analytical methods to nonlinear differential-difference and pure difference equations toward its complete integrability. Second, we plan to show how Lie’s theory of differential equations can be extended to differential-difference and difference equations. Also, we wish to demonstrate how Lie’s theory provides an effective analytical method to derive exact solution of nonlinear difference equations.

The plan of the article is as follows. In Section 2, we present a brief account of certain important analytical methods used to study integrability nature of differential-difference and difference equations. We then show how to extend the method of Lie symmetries applicable to differential equations to differential-difference and pure difference equations in Section 3. The usefulness of the method is illustrated through a discrete Korteweg-deVries equation as an example. In Section 4, we show that an autonomous nonlinear difference equation of an arbitrary order with one or more independent variables can be linearised by a point transformation if and only if it admits a symmetry vector field whose coefficient is the product of two functions, one of the dependent and of the independent variables. Also, we illustrate the method by linearising several first- and second-order ordinary nonlinear difference equations. Section 5 deals with a brief discussion about the possible connection between the Lie symmetry analysis and the onset of chaos with reference to the first-order mappings.

2. NONLINEAR DIFFERENTIAL-DIFFERENCE AND PURE DIFFERENCE EQUATIONS: METHODS OF INTEGRABILITY

It is well known that the concept of complete integrability of nonlinear evolution equations is not yet well defined, and there exists no unique definition for it. A systematic effort has been spent to investigate complete integrability of nonlinear partial and ordinary differential equations during the past several decades, and a considerable number of ad hoc methods has been developed [10-12]. For differential-difference and difference equations, the situation is completely unclear [6-13]. However, the nonlinear differential-difference and difference equation is considered to be integrable (working definition) if it satisfies one of the following criteria (the list is not exhaustive):

1. the nonlinear system is solvable through inverse scattering transform method [6] after finding suitable a Lax pair or eigenvalue problems;
2. the system possesses the required number of independent integrals of motion [14];
3. the system passes the singularity confinement criterion [9]—a discrete version of Painlevé property;
4. the system is linearized through suitable variable transformation [15], etc.

In the following, we present a brief account of the above methods which have been widely used in recent years.

2.1. Lax Method

A nonlinear differential equation is said to be completely integrable if it arises from the compatibility condition of a system of linear differential equations. This method originally developed by Lax for differential equations was extended by Ablowitz and Ladik [6] to differential-difference equations. Here, one starts with a discrete eigenvalue problem

\[
\psi(x + 1, t) - L(x, t, \lambda)\psi(x, t), \quad x \in N, \tag{1}
\]
where $L(x, t, \lambda)$ is an unknown matrix and $\lambda$ is the spectral parameter. Then one looks for an evolution equation in the form

$$\frac{\partial}{\partial t} \psi(x, t) = A(x, t, \lambda) \psi(x, t),$$

where $A(x, t, \lambda)$ is an unknown matrix to be determined. The compatibility condition of equations (1) and (2)

$$\frac{\partial}{\partial t} (S \psi) = S \frac{\partial}{\partial t} (\psi),$$

where $S$ is the shift operator $S \psi(x, t) = \psi(x + 1, t)$, implies a relation between $L(x, t, \lambda)$ and $A(x, t, \lambda)$:

$$\frac{d}{dt} L(x, t, \lambda) = A(x + 1, t, \lambda) L(x, t, \lambda) - L(x, t, \lambda) A(x, t, \lambda).$$

Given the matrix $L(x, t, \lambda)$ and assuming for $A(x, t, \lambda)$ an appropriate expansion in $\lambda$, equation (3) yields the explicit form of $A(x, t, \lambda)$. Then equations (1) and (2) define the Lax pair for a solvable differential-difference equation. For example, a discrete Korteweg-de Vries equation

$$\frac{\partial}{\partial t} u(x, t) = \frac{1}{u(x + 1, t) - u(x - 1, t)}, \quad x \in \mathbb{N},$$

admits a Lax pair

$$L(x) = \left[\begin{array}{cc}
-u' & 1 \\
\lambda - u(x)u' & u(x)
\end{array}\right], \quad M(x) = \lambda^{-1} \left[\begin{array}{cc}
u' \frac{\partial u}{\partial t} - 1 & \frac{\partial u}{\partial t} \\
\nu^2 \frac{\partial u}{\partial t} - u' & -\nu' \frac{\partial u}{\partial t}
\end{array}\right],$$

where $u' = u(x + 1)$ and hence is solvable by the inverse scattering transform method.

Recently, this method was extended to difference equations both for ordinary and partial difference equations [8]. Here again, one starts with the eigenvalue (spectral) problem

$$L(x, \lambda) \psi(x, \lambda) = \lambda \psi(x, \lambda),$$

where $L(x, \lambda)$ is a given matrix and $\lambda$ is a spectral parameter. Then one looks for an equation in the form

$$\psi(x + 1, \lambda) = A(x, \lambda) \psi(x, \lambda),$$

where $A(x, \lambda)$ is an unknown matrix to be determined. The compatibility condition of equations (6) and (7) yields an equation

$$L(x + 1, \lambda) A(x, \lambda) = A(x, \lambda) L(x, \lambda).$$

Proceeding as before for the differential-difference equation, one can find the Lax pair for a given integrable difference equation. For example, the well-known two-dimensional mapping [16]

$$u(x + 1) + u(x - 1) = \frac{2au(x)}{1 - u^2(x)}, \quad a \text{ - parameter},$$

admits a Lax pair

$$L(x) = \left[\begin{array}{ccc}
a & t + u' & 0 \\
0 & 0 & t - u \\
\lambda & 0 & 1 - u'
\end{array}\right], \quad M(x) = \lambda^{-1} \left[\begin{array}{ccc}
a(1 + u')^{-1} & 1 & 0 \\
0 & 0 & t \\
\lambda & 0 & 0
\end{array}\right],$$

where $u = u(x)$ and $u' = u(x + 1)$, and hence is completely integrable. In fact, the solution of equation (10) can be expressed in terms elliptic functions [16].
2.2. Integrals of Motion

A nontrivial function \( I(x) \)
\[
I(x) = I(x, u(x), u(x + 1), \ldots, u(x + N - 1))
\] (11)
is said to be an integral for an \( N \)-th-order ordinary difference equation
\[
u(x + N) = f(x, u(x), u(x + 1), \ldots, u(x + N - 1))
\] (12a)
if
\[
I(x + 1) = I(x).
\] (12b)

An \( N \)-th-order ordinary difference equation (12a) is said to be completely integrable if it admits \((N - 1)\) independent integrals of motion [14]. Given a nonlinear difference equation, it is not clear what form of integral \( I(x) \) one has to choose. However, by considering different ansatz, for example, \( I(x) \) is a ratio of two biquadratics, one can isolate the integrable mappings. For example, the second-order difference equation
\[
u(x + 1) = \frac{2u(x + 1) - u(x)}{1 - u^2(x + 1) + 2u(x)u(x + 1)}
\] (13a)
admits an integral
\[
I(x) = \frac{u(x + 1) - u(x)}{1 + u(x)u(x + 1)}
\] (13b)
and hence is integrable. The general solution of equation (13b) is
\[
u(x) = \tan \left( \frac{px^2}{2} + xc_1(x) + c_2(x) \right),
\] (14)
where \( c_1(x) \) and \( c_2(x) \) are arbitrary functions. For more applications, refer to [17,18].

2.3. Discrete Painlevé Property or Singularity Confinement Criterion

A nonlinear differential equation both ordinary and partial is said to pass the Painlevé property if its general solution admits only pole type singularities in the complex plan/manifold of independent variable(s). If a nonlinear differential equation passes the Painlevé property, then it is expected to be integral. The Painlevé analysis originally devised by Painlevé and his school and later on by Ablowitz et al. [10] provides a systematic and effective test to determine the integrability nature of differential equations. Several integrable nonlinear differential equations were identified [11] using the Painlevé analysis. Recently, the Painlevé analysis was extended to differential-difference and difference equations by Grammaticos et al. [9]. The method is known as a singularity confinement criterion which says that nonlinear differential-difference and difference equations are said to pass the singularity confinement criterion if its singularity is confined after a finite number of iterations. For applications of this method, refer to [19].

3. LIE SYMMETRY ANALYSIS OF DIFFERENTIAL-DIFFERENCE AND DIFFERENCE EQUATIONS

Lie symmetry analysis originally advocated by Sophus Lie in the beginning of the 19th century provides an efficient technique for solving differential equations and identifies the underlying symmetry group of it. The symmetry group of a differential equation can be used for many purposes [20]. For example, it will transform given solutions into new ones, often trivial solutions into interesting ones. Similarly, the symmetry group can be used to perform symmetry reduction that reduces the order of an ordinary differential equation, or the number of independent variables in a partial differential equation. Isomorphies between symmetry groups of differential equations can be used to identify equivalent equations. Symmetry groups can also serve as indicators of integrability by Lax pair techniques. In the following, we show how Lie symmetry analysis can be extended to differential-difference and pure difference equations.
3.1. Preliminary

For notational simplicity, let us restrict ourselves to one real-valued scalar function $u(x)$ with $n$-independent variables $x = (x_1, x_2, \ldots, x_n)$. We wish to remind the readers that although usually the independent variables $(x_1, x_2, \ldots, x_n)$ are integers, here we allow them to be real numbers. An equation for $u(x)$ is said to possess a Lie point symmetry if it is invariant under one-parameter continuous transformations

$$x^* = x + \epsilon \xi(x, u(x)) + O(\epsilon^2), \quad (15a)$$

$$u^* = u(x) + \epsilon v(x, u(x)) + O(\epsilon^2). \quad (15b)$$

A fundamental question now arises what the expression like $u^*(x^* + \omega)$ is, where $\omega \in \mathbb{R}^n$ is some given fixed span appearing in the discrete equation. To evaluate $u^*(x^* + \omega)$, we express the right-hand side of $(15b)$ in terms of $x^*$ using $(15a)$:

$$u^*(x^* + \omega) = u(x + \epsilon \xi(x^*, u(x^*))) + \epsilon v(x^*, u(x^*)) + O(\epsilon^2). \quad (16)$$

We now shift, and then expanding the right-hand side, we get

$$u^*(x^* + \omega) = u(x + \omega) + \epsilon v(x + \omega, u(x + \omega))$$

$$+ \sum_{i=1}^{n} \left[ \xi_i(x, u(x)) - \xi_i(x + \omega, u(x + \omega)) \right] \frac{du(x + \omega)}{dx_i} + O(\epsilon^2). \quad (17)$$

3.2 Generalization of Lie Symmetry Analysis: Differential-Difference Equations

It would be appropriate to mention that the similarity method for discrete systems was initiated by Maeda [21] and later on by Levi and Winternitz [22] and Quispel et al. [23,24]. Below, we present brief details of the method.

Consider, for example, a first-order partial differential-difference equation with one dependent variable $u(x, t)$ and two independent variables $t$-continuous and $x$-discrete

$$u(x, t) = F(u(x - 1, t), u(x, t), u(x + 1, t), \ldots). \quad (18)$$

Let a one-parameter ($\epsilon$) Lie group of continuous point transformations be taken as

$$x^* = x(x, t, u(x, t)), \quad t^* = T(x, t, u(x, t)), \quad u^* = U(x, t, u(x, t)). \quad (19)$$

Expanding equation (19) about the identity $\epsilon = 0$, we get

$$x^* = x + \epsilon \xi(x, t, u) + O(\epsilon^2), \quad t^* = t + \epsilon \tau(x, t, u) + O(\epsilon^2), \quad u^* = u + \epsilon v(x, t, u) + O(\epsilon^2), \quad (20)$$

where $\epsilon$, $\tau$, and $v$ are infinitesimals of the variables $x$, $t$, and $u$, respectively. Thus, the symmetry generator becomes

$$G = \xi(x, t, u) \frac{\partial}{\partial x} + \tau(x, t, u) \frac{\partial}{\partial t} + v(x, t, u) \frac{\partial}{\partial u}. \quad (21)$$

Now equation (18) is invariant under the transformations (19) or (20) if

$$\frac{\partial}{\partial t^*} u^*(x^*, t^*) = F(x^*, t^*, \ldots, u^*(x^* - 1, t^*), u^*(x^*, t^*), u^*(x^* + 1, t^*), \ldots), \quad (22)$$

whenever $u(x, t)$ is a solution of equation (18). Making use of equation (17) along with group transformations (20) as well as expression for derivatives of $u^*$ in (21), we obtain an invariant equation from which one can determine the infinitesimals $\xi(x, t, u)$, $\tau(x, t, u)$, and $v(x, t, u)$. The remaining analysis proceeds exactly along the similar lines as that of differential equations. Below, we demonstrate the effectiveness of the method through a discrete Korteweg-deVries equation, equation (4), which is a partial differential-difference equation solvable by Lax pair techniques.
3.2.1. Application

Let us assume that the discrete Korteweg-deVries equation (4) is invariant under one-parameter continuous transformations, equation (20). Then the invariant equation reads

\[
\frac{\partial}{\partial t} v(x) + \left[ \frac{\partial}{\partial u(x)} v(x) \frac{\partial}{\partial \tau(x)} \right] \frac{d}{dt} u(x) - \frac{\partial}{\partial t} \xi(x) \frac{d}{dx} \frac{[v(x-1) - v(x+1)]}{[u(x+1) - u(x-1)]^2} = 0,
\]

where we have not indicated the \( t \) and \( u \) dependence of \( v \) and \( \tau \) explicitly. Solving the invariance equation (23), we obtain the infinitesimals explicitly:

\[
\xi = k(x), \quad \tau = f(x) + tf(x + 1) + h(x), \quad v = u(x, t)f(x) + g(x),
\]

where \( f(x), g(x) \) are periodic functions of period 2, and \( h(x) \) and \( k(x) \) are unit periodic. Proceeding further with the infinitesimals \( \xi, \tau, \) and \( v \), one can find the similarity variable \( \eta = x - A^{-1} \log [t + b(x)]/[E(x)E(x-1)] \) and the similarity transformation \( V(\eta) \) in \( u(x, t) \sim \exp(Ax/2) \times E(x)V(\eta) \), where \( b(x) \) is unit periodic and \( E(x) \) is periodic function with period 2. Substituting the similarity transformation \( V(\eta) \) in the discrete Korteweg-deVries equation, we find that it reduces into an ordinary differential-difference equation

\[
\frac{V(\eta)}{2} \frac{1}{A} \frac{d}{d\eta} V(\eta) - \frac{1}{V(\eta + 1) \cdot V(\eta - 1)},
\]

which is completely integrable in the sense of Lax [11]. For more applications of this method to differential-difference equations, refer to [22,25,26].

3.3. Lie Symmetry Analysis and Linearization of Difference Equations

Consider a first-order difference equation (that is, an equation with a single span)

\[
u(x + 1) = F(x, u(x)), \quad x \in \mathbb{R},
\]

where \( F \) is a given function. Equation (26) is invariant under the one-parameter continuous transformations

\[
x^* = x + \epsilon \xi(x, u(x)) + O(\epsilon^2), \quad u^* = u(x) + \epsilon v(x, u(x)) + O(\epsilon^2),
\]

if

\[
u(x + 1) = F(x, u(x)), \quad x \in \mathbb{R},
\]

provided \( u(x) \) satisfies (26). For clarity, we consider the autonomous case in the following.

Making use of equation (17) with \( (\omega = 1) \), in equation (27b), we obtain

\[
v(x + 1, u(x + 1)) + \left[ \xi(x, u(x)) - \xi(x + 1, u(x + 1)) \frac{d}{dx} u(x + 1) \right] = v(x, u(x)) \frac{\partial}{\partial u} F.
\]

Equation (28) implies

\[
\xi(x, u(x)) - \xi(x + 1, u(x + 1)) = 0,
\]

\[
v(x - 1, u(x + 1)) = v(x, u(x)) \frac{\partial F}{\partial u}.
\]

Obviously, any unit periodic function \( \alpha(x) \), that is \( \alpha(x + 1) = \alpha(x) \), is a solution of (29a). It is very difficult to solve the functional equation (29b) in general. An attempt has been made to compute \( v(x, u(x)) \) in terms of series expansion by Quispel and Sahadevan [24]. However, it is not clear what form of series expansion one has to choose for a given nonlinear difference equation.
Integrability and Exact Solvability

Next, let us assume that the Lie point symmetry \( v(x, u(x)) \) is factorisable, that is,

\[
v(x, u(x)) = A(x)G(u),
\]

where \( A(x) \) and \( G(u(x)) \) are arbitrary and unknown functions in \( x \) and \( u(x) \), respectively. Using equation (30) in equation (29b) and rearranging, we get

\[
\frac{dF}{G(F)} = \frac{a}{A(x)G(u)},
\]

where \( A(x) = a^x \); "\( a \)" is a constant. Integrating equation (31) and then substituting it in the given autonomous nonlinear difference equation, equation (26), we find that it reduces into a linear difference equation with constant coefficients:

\[
Q(x + 1) = aQ(x) + p,
\]

and \( p \) is a constant. The general solution of equation (32) is

\[
Q(x) = \begin{cases} 
\frac{P}{1-a} + \gamma(x)a^x, & a \neq 1, \\
px + \gamma(x), & a = 1,
\end{cases}
\]

and so the general solution of the given autonomous first-order nonlinear difference equation is

\[
u(x) = \begin{cases} 
Q^{-1}\left[\frac{P}{1-a} + \gamma(x)a^x\right], & a \neq 1, \\
Q^{-1}[px + \gamma(x)], & a = 0,
\end{cases}
\]

where \( \gamma(x) \) is an arbitrary unit periodic function.

For the nonautonomous case, that is equation (26), the reduced linear equation becomes

\[
Q(x + 1) = q(x)Q(x) + p(x),
\]

where \( q(x) = A(x + 1)/A(x) \) and \( p(x) \) is an arbitrary function. The solution of equation (34) is

\[
Q(x) = \begin{cases} 
\gamma(x)q^{x-1} + q^{x-1}\sum_{k=1}^{x-1} \frac{p(k)}{q^k}, & a = \text{constant}, \\
\beta(x)\sum_{k=1}^{x-1} \frac{p(k)}{q^k},
\end{cases}
\]

where \( \gamma(x) \) is unit periodic and \( \beta(x) = \prod_{j=1}^{x-1} q(j) \).

Inverting the above equation, we obtain the general solution of the given difference equation. For generalisation of the above results to \( N^{th} \)-order difference equations and the sufficiency part of linearisation, refer to [15].

4. APPLICATIONS

4.1. First-Order Nonlinear Difference Equations

Consider a first-order difference equation having the form

\[
u(x + 1) = \frac{4u(x)[1 - u(x)][1 - (l - 1)u(x)]}{[1 - (l - 1)u^2(x)]},
\]
where \( l \) is a constant. Following the procedure outlined above, one can find that equation (36) is invariant under one-parameter continuous transformations

\[
x^* = x + \epsilon \alpha(x) + O(\epsilon^2), \quad u^* = u + \epsilon 2^\pi \left[ u(x) \left\{ 1 - lu(x) + (l - 1)u^2(x) \right\} \right]^{1/2} + O(\epsilon^2).
\]

(37)

Thus, the homogenizing variable is

\[
Q(x) = \int_{u(x)}^{u(x)} \frac{du'}{|u'(1 - lu') + (l - 1)u^2|} = s_n^{-2}(u, k),
\]

(38)

where \( s_n \) is the Jacobian elliptic function and the square of the modulus is \( k^2 = (l - 1) \). Inverting (38) and substituting in (36), we find that the nonlinear difference equation (36) reduces into a linear difference equation

\[
Q(x + 1) = 2Q(x),
\]

(39a)

whose solution is \( Q(x) = \gamma(x)2^\pi \), and therefore, the general solution of equation (36) is

\[
u(x) = s_n^2(\gamma(x), 2^\pi, k),
\]

(39b)

where \( \gamma(x) \) is unit periodic.

4.2. Second-Order Nonlinear Difference Equations

**ILLUSTRATION 1.** Consider a second-order difference equation having the form

\[
u(x + 2) = 2\nu(x + 1) - \nu(x) (1 - \nu^2(x + 1))
\]

(40)

It is straightforward to check that the above equation is invariant under one-parameter continuous transformations

\[
x^* = x + \epsilon \alpha(x) + O(\epsilon^2), \quad u^2 = u + \epsilon [x\beta(x) + \delta(x)] (1 + \nu^2(x)) + O(\epsilon^2).
\]

(41)

Here, the homogenizing variable takes

\[
Q(x) = \int \frac{du}{1 + u^2(x)} = \tan^{-1} u(x).
\]

(42)

Inverting equation (42) and substituting in the original equation (40), we find that it is reduced into a second-order linear equation with constant coefficients

\[
Q(x + 2) - 2Q(x + 1) + Q(x) = \rho \pi
\]

(43)

whose solution is

\[
Q(x) = \frac{\rho \pi}{2} x^2 + xc_1(x) + c_2(x)
\]

(44)

where \( c_1(x) \) and \( c_2(x) \) are unit periodic functions. Therefore, the general solution of (40) is

\[
u(x) = \tan \left[ \frac{\rho \pi}{2} x^2 + xc_1(x) + c_2(x) \right].
\]

(45)

**ILLUSTRATION 2.** Consider another second-order difference equation having the form

\[
u(x + 2) = u(x + 1)\sqrt{(1 + u(x)^2)\sqrt{(1 + u(x + 1)^2)} - u(x)\sqrt{(1 + u(x + 1)^2)}}.
\]

(46)
It is easy to check that the above equation is invariant under one-parameter continuous transformations

\[ x^* = x + \epsilon \alpha(x) + O\left(\epsilon^2\right) , \quad u^* = u + \epsilon \left[ \gamma(x) \cos \frac{x\pi}{3} + \beta(x) \sin \frac{x\pi}{3} \right] \sqrt{1 + u^2(x)} + O\left(\epsilon^2\right) . \]  

Here, the homogenizing variable takes

\[ Q(x) = \int \frac{du}{\sqrt{1 + u^2(x)}} = \sinh^{-1} u(x) . \]  

Inverting equation (48) and substituting in the original equation (46), we find that it is reduced into a second-order linear equation with constant coefficients

\[ Q(x + 2) - Q(x + 1) + Q(x) = 0 , \]  

whose solution is

\[ Q(x) = c_1(x) \cos \frac{x\pi}{3} + c_2(x) \sin \frac{x\pi}{3} , \]  

where \( c_1(x) \) and \( c_2(x) \) are unit periodic functions. Therefore, the general solution of (50) is

\[ u(x) = \sinh \left[ c_1(x) \cos \frac{x\pi}{3} + c_2(x) \sin \frac{x\pi}{3} \right] . \]

Brief details of results for some other one-dimensional mappings are given in Table 1.

### Table 1. Lie symmetries and exact solution of certain first- and second-order mappings.

<table>
<thead>
<tr>
<th>Mapping</th>
<th>Lie Symmetries</th>
<th>Homogenizing Variable ( \theta(x) )</th>
<th>Reduced Equation</th>
<th>Exact Solution ( u(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u' = 4u(1 - u) )</td>
<td>( 2^x \sqrt{u(1 - u)} )</td>
<td>( \sin^{-2} u )</td>
<td>( \theta = 2^x )</td>
<td>( \sin^2 \left( \gamma(x)2^x \right) )</td>
</tr>
<tr>
<td>( u' = u(3 - 4u^2) )</td>
<td>( 3^x \sqrt{1 - u^2} )</td>
<td>( \sin^{-1} u )</td>
<td>( \theta = 3^x )</td>
<td>( \sin \left( \gamma(x)3^x \right) )</td>
</tr>
<tr>
<td>( u' = u(3 + 4u^2) )</td>
<td>( 3^x \sqrt{1 + u^2} )</td>
<td>( \sin h^{-1} u )</td>
<td>( \theta = 3^x )</td>
<td>( \sin h \left( \gamma(x)3^x \right) )</td>
</tr>
<tr>
<td>( u' = \frac{4u}{(1 - u)^2} )</td>
<td>( 2^x \sqrt{u(1 - u)} )</td>
<td>( \tan^{-2} u )</td>
<td>( \theta = 2^x )</td>
<td>( \tan^2 \left( \gamma(x)2^x \right) )</td>
</tr>
<tr>
<td>( u' = \frac{2u + p(x)(1 - u^2)}{1 - u^2 - 2p(x)u} )</td>
<td>( 2^x \frac{1}{(1 + u^2)} )</td>
<td>( \tan^{-1} u )</td>
<td>( \theta = 2^x )</td>
<td>( \tan^2 \left( \gamma(x)2^x \right) - 1 )</td>
</tr>
</tbody>
</table>

### 5. LIE SYMMETRIES AND CHAOS

In this section, we wish to explore the possible connection between Lie symmetries of nonlinear difference equations or mappings.

We wish to restrict our discussion for first-order mappings only, for example, logistic mapping

\[ u(x + 1) = 4u(x)[1 - u(x)] , \quad x \in \mathbb{N} . \]  

The concept of chaos is not well defined, and there exist several routes to chaos such as period doubling bifurcations, sensitive dependence on initial conditions, Hopf bifurcation, etc. [27]. In this article, we consider the sensitive dependence on initial condition route to chaos.

It is straightforward to check that the logistic mapping equation (52) is invariant under one-parameter continuous transformations

\[ x^* = x + \epsilon \alpha(x) + O\left(\epsilon^2\right) , \quad u^* = u + \epsilon^2 \left[ u - u^2 \right]^{1/2} + O\left(\epsilon^2\right) . \]
The homogenizing variable \( Q(x) \) is

\[
Q(x) = \int \frac{du}{\sqrt{u - u^2}} = \sin^2 u(x),
\]

and so the logistic equation (54) becomes

\[
Q(x + 1) = 2Q(x).
\]

Thus, the exact solution of the logistic equation is

\[
u(x) = \sin^2 \left[ 2^{-1} \cos^{-1}(1 - u(0)) \right].
\]

Let \( u(0) \) and \( v(0) \) be any two initial values which lie near each other in \((0, 1)\). This means that

\[
Q(1) = \cos^{-1}(1 - u(0)) \quad \text{and} \quad \phi(1) = \cos^{-1}(1 - v(0))
\]

also close together \( \mod \pi \). For \( x \geq 1 \),

\[
\theta(x) - \phi(x) = 2^{-1} [\theta(1) - \phi(1)] \mod \pi.
\]

Equation (58) shows that the difference doubles \( \mod \pi \) with each iteration. In other words, the solutions \( u(x) = \sin^2 \theta(x) \) and \( v(x) = \sin^2 \phi(x) \) will not be near each other for most values of \( x \). Thus, the exact solution (56) of the logistic mapping (52) has sensitive dependence on initial conditions, and hence displays chaos. Similar conclusions can be arrived at for all the one-dimensional mappings given in the table.

REFERENCES

Integrability and Exact Solvability