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Matching edges and faces in polygonal partitions ☆

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Abstract

We define general Laman (count) conditions for edges and faces of polygonal partitions in the plane. Several well-known classes, including k-regular partitions, k-angulations, and rank-k pseudo-triangulations, are shown to fulfill such conditions. As an implication, non-trivial perfect matchings exist between the edge sets (or face sets) of two such structures when they live on the same point set. We also describe a link to spanning tree decompositions that applies to quadrangulations and certain pseudo-triangulations.

Keywords: Polygonal partition; Pseudo-triangulation; Quadrangulation; Laman condition; Perfect matching; Tree decomposition

1. Introduction

Polygonal partitions in the plane are a versatile tool in computational geometry, and much effort has gone into the study of their structural properties. In particular, a wealth of results exists for the class of triangulations of a finite point set in the plane. Less is known for polygonal partitions where faces are of more general shape, like k-angulations, pseudo-triangulations, or k-regular partitions. In the present paper, we intend to generalize results of a particular type from triangulations to the three classes of polygonal partitions mentioned above.

The paper [2] establishes the existence of certain matchings between two given triangulations on top of the same point set S. For instance, for any two triangulations T_1 and T_2 of S we can pair each edge $e_1 \in T_1$ with an edge $e_2 \in T_2$ such that either $e_1 = e_2$ or e_1 crosses e_2. Moreover, each triangle $\Delta_1 \in T_1$ can be paired with a triangle $\Delta_2 \in T_2$ such that either $\Delta_1 = \Delta_2$ or Δ_1 partially overlaps with Δ_2. Perfect matchings of this kind prove to be useful for obtaining lower bounds on the total edge length of the minimum weight triangulation of S; see [2].

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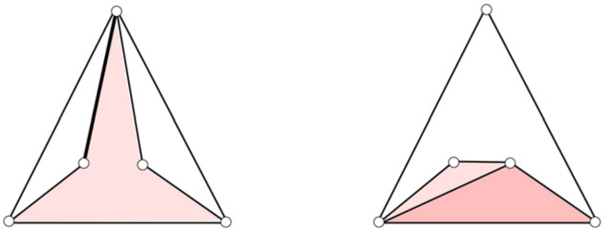

Fig. 1. The matching theorems in [2] fail for pseudo-triangulations.

On the negative side, pseudo-triangulations (see Section 3 for a definition) do not share these properties. Fig. 1 depicts two pseudo-triangulations PT_1 (left) and PT_2 (right) on a set of five points. Note that PT_1 and PT_2 have the same number of edges (and faces). The bold edge in PT_1 neither crosses, nor coincides with, an edge in PT_2. Thus no edge matching as above is possible. Also, the two shaded faces in PT_2 both overlap only with the single shaded face in PT_1. This rules out a face matching.

We intend to show that perfect matchings can be retained when 'crossing' and 'overlap', respectively, are relaxed to 'vertex incidence'. In fact, such incidence matchings also exist for polygonal partitions different from pseudo-triangulations. We define general conditions that guarantee the existence of incidence matchings for edges (and faces) in two polygonal partitions that share the same vertex set. The idea is to generalize the well-known Laman count conditions for graphs [10]—which bound the edge density in subgraphs—to polygonal partitions, and taking into account the extremal properties of the underlying vertex set. These generalized Laman conditions, which may be of interest in their own right, sometimes also imply decomposability into edge-disjoint spanning trees.

2. Generalized Laman property

Throughout, let S be a finite set of (at least three) points in the plane. Let *conv*(S) denote the convex hull of S. A *polygonal partition*, P, *on* S is a partition of *conv*(S) into simple polygons (faces) such that S is the vertex set of P, and such that each edge of P which is not an edge of *conv*(S) is common to exactly two faces. We will exclude the face exterior to *conv*(S) from considerations unless stated otherwise.

Here and in later sections, let the term 'object' consistently stand for either 'edge' or 'face' of a given polygonal partition P on S. Consider an arbitrary subset $S' \subseteq S$. We say that an object x of P is *spanned* by S' if x has all its incident vertices in S'. Denote with $\alpha(S')$ the number of objects of P that are spanned by S'. Further, let $n(S')$ be the cardinality of S', and let $h(S')$ be the number of vertices of *conv*(S'). Note that $\alpha(S)$ expresses the total number of objects of P. As P defines a planar straight line graph on S, $\alpha(S)$ is a linear function of $n(S)$. We call P *object-Laman* if there exist three constants $c_1 \geqslant c_2 \geqslant 0$ and $c_3 \geqslant -1$ such that the following two conditions hold:

$$\alpha(S) = c_1 n(S) - c_2 h(S) - c_3$$

and, for each subset $S' \subset S$ with $n(S') \geqslant 2$,

$$\alpha(S') \leqslant c_1 n(S') - c_2 h(S') - c_3.$$

The second condition is the so-called *hereditary Laman condition*. We term the triple (c_1, c_2, c_3) the (*object*) *characteristic* of P. Classical planar Laman graphs [10] have embeddings as straight line graphs that yield polygonal partitions with edge characteristic $(2, 0, 3)$; see [8]. That is, a Laman graph on n vertices has precisely $2n - 3$ edges, and each subgraph on $n' \geqslant 2$ vertices has at most $2n' - 3$ edges. In [3], the concept of bounded graph density from [10] is extended to general functions of n. Dealing with purely graph-theoretical concepts, they do not consider the number of convex hull points as a parameter.

An object x of P is said to be *covered* by a subset $S' \subseteq S$ if x has at least one incident vertex in S'. Let $\beta(S')$ denote the number of objects of P that are covered by S'. Clearly $\beta(S') \geqslant \alpha(S')$ holds, as each object spanned by S' is also covered by S'. Polygonal partitions that are object-Laman satisfy the following property.

Lemma 1. *Let P be a polygonal partition on S, and let P be object-Laman with characteristic $(c_1, c_2, c_3 \geqslant 0)$. Then $\beta(S') \geqslant c_1 n(S') - c_2 h(S') - c_3$ holds for each $S' \subseteq S$.*

Proof. Each vertex of $conv(S)$ has to be a vertex of one of $conv(S')$ and $conv(S \setminus S')$. Therefore, we have

$$h(S) \leqslant h(S') + h(S \setminus S'). \tag{1}$$

Consider the case $n(S \setminus S') \leqslant 1$ first. All objects of P are covered by S' in this case, and we get

$$\beta(S') = \alpha(S) = c_1 n(S) - c_2 h(S) - c_3. \tag{2}$$

This directly implies the lemma for $S' = S$. For $n(S \setminus S') = 1$, we plug in $n(S) = n(S') + 1$, and $h(S) \leqslant h(S') + 1$ from (1). Then (2) evaluates to the inequality in the lemma by $c_1 \geqslant c_2 \geqslant 0$.

Now consider the case $n(S \setminus S') \geqslant 2$. Observe that $\alpha(S \setminus S')$ counts the number of objects that have no vertex in S'. So we get

$$\beta(S') = \alpha(S) - \alpha(S \setminus S'). \tag{3}$$

Moreover, as $n(S \setminus S') \geqslant 2$, we have the hereditary condition

$$\alpha(S \setminus S') \leqslant c_1 n(S \setminus S') - c_2 h(S \setminus S') - c_3. \tag{4}$$

But $n(S \setminus S') = n(S) - n(S')$, and $h(S \setminus S') \geqslant h(S) - h(S')$ holds by (1). Thus from (4) we get

$$\alpha(S \setminus S') \leqslant c_1\big[n(S) - n(S')\big] - c_2\big[h(S) - h(S')\big] - c_3 \tag{5}$$

because $c_2 \geqslant 0$. Plugging (5) and the right-hand equality from (2) into (3) evaluates to

$$\beta(S') \geqslant c_1 n(S') - c_2 h(S')$$

which gives the inequality in the lemma by $c_3 \geqslant 0$. □

The object-Laman property is strong enough to imply a non-trivial bijection between the edge sets (or face sets) of two polygonal partitions that live on the same configuration of points.

Theorem 1. *Let S be a finite set of points in the plane. Let P_1 and P_2 be any two polygonal partitions on S that are object-Laman with the same characteristic $(c_1, c_2, c_3 \geqslant 0)$. There exists a perfect matching between the set of objects of P_1 and the set of objects of P_2 such that matched objects share a vertex.*

Proof. Let O_i be the set of objects of P_i, for $i = 1, 2$. For a subset $X \subseteq O_1$, let $Y \subseteq O_2$ denote the set of objects that possibly can be matched to some object in X. More precisely, Y contains all objects $y \in O_2$ such that y shares some vertex with an object in X. We show $|Y| \geqslant |X|$. That is, the Hall condition [5] for the marriage theorem is fulfilled, which implies the existence of a perfect matching between O_1 and O_2.

Let S' be the subset of S that consists of all the vertices of the objects in X. That is, X is the set of objects of P_1 that are spanned by S'. If $n(S') \leqslant 1$ then $|X| = 0$, and $|Y| \geqslant |X|$ clearly holds. Let $n(S') \geqslant 2$. By the assumed Laman property for P_1 we have $|X| \leqslant c_1 n(S') - c_2 h(S') - c_3$. On the other hand, Y is precisely the set of objects of P_2 that are covered by S'. By the assumed Laman property for P_2 we now get $|Y| \geqslant c_1 n(S') - c_2 h(S') - c_3$ from Lemma 1. We conclude $|Y| \geqslant |X|$ again. □

The proofs of Lemma 1 and Theorem 1 do not use the fact that the considered partitions live in the plane. The theorem, thus, generalizes for cell complexes in higher dimensions.

In the plane, the Eulerian relation for planar graphs implies a correspondence between the edge-Laman and the face-Laman property. From now on, let us write the number $\alpha(S')$ of objects spanned by a subset $S' \subseteq S$ as $e(S')$ if the objects are edges, and as $f(S')$ if the objects are faces.

Lemma 2. *Let a polygonal partition P on S be given. Assume that P is edge-Laman with characteristic $(c_1 \geqslant 1, c_2 \leqslant c_1 - 1, c_3 \geqslant 1)$. Then P is face-Laman with characteristic $(c_1 - 1, c_2, c_3 - 1)$.*

Proof. Let $S' \subseteq S$. The set of objects spanned by S' defines a planar graph with $n(S')$ vertices, $e(S')$ edges, and $t \geqslant 1$ connected components. For the ith component G_i let n_i, e_i, and f_i, respectively, denote the number of its vertices, edges, and faces (including the exterior face). The Eulerian relation for G_i reads

$$n_i - e_i + f_i = 2.$$

Summing over all components, we get

$$\sum_{i=1}^{t} f_i = 2t - n(S') + \sum_{i=1}^{t} e_i. \tag{6}$$

Let $n(S') \geqslant 2$. As $e_i = 0$ holds for $n_i = 1$, the assumption $n_i \geqslant 2$ for $i = 1, \ldots, t$ maximizes the sum in (6) on the right-hand side. On the other hand, by the edge-Laman property of P we now have

$$e_i \leqslant c_1 n_i - c_2 h_i - c_3 \quad \text{if } n_i \geqslant 2, \tag{7}$$

where h_i is the number of vertices of $conv(G_i)$. Plugging (7) into (6) and rearranging terms gives

$$\sum_{i=1}^{t} f_i \leqslant (c_1 - 1)n(S') - c_2 \sum_{i=1}^{t} h_i - (c_3 - 2)\,t.$$

The left-hand sum is bounded from below by $f(S') + t$, because each component G_i defines at least one face which is not a face of P, namely, the external face. The right-hand sum is bounded from below by $h(S')$; this follows from (1). As $t \geqslant 1$ and $c_3 \geqslant 1$ we can conclude

$$f(S') \leqslant (c_1 - 1)n(S') - c_2 h(S') - (c_3 - 1), \tag{8}$$

the claimed hereditary face-Laman property for P. Finally, it is easy to check that equality holds in (8) for the case $S' = S$. This completes the proof. □

3. Some relevant polygonal partitions

The edge-Laman and the face-Laman properties are quite natural. They are shared by several well-known classes of polygonal partitions. In the sequel, we require $n(S') \geqslant 2$ for the considered subset $S' \subseteq S$. This ensures that the formulas below yield nonnegative values for $e(S')$ and $f(S')$. Let us denote with $A(S')$ the subset of objects (under consideration) spanned by S'.

3.1. Rank-k pseudo-triangulations

A *pseudo-triangulation*, *PT*, of S is a polygonal partition on S whose faces are pseudo-triangles, i.e., simple polygons with exactly three convex vertices. A vertex of *PT* is called *pointed* if its incident edges span a convex angle. Let *PT* contain exactly p pointed vertices. In [1], the (*edge*) *rank* of *PT* is defined as $n(S) - p$, the number of non-pointed vertices. The maximum rank of *PT* is $n(S) - h(S)$, in which case *PT* is a triangulation. The minimum rank of *PT* is zero, and *PT* is commonly called a *minimum* (or *pointed*) pseudo-triangulation in that case.

It is well known that every rank-k pseudo-triangulation of S has exactly

$$e(S) = 2n(S) + k - 3$$

edges. Consider a subset $S' \subseteq S$, and assume that the set $A(S')$ defines a pseudo-triangulation of S'. As each vertex that is non-pointed in $A(S')$ has to be non-pointed in *PT* as well, the rank of $A(S')$ is at most k. If, on the other hand, $A(S')$ is a proper subset of a pseudo-triangulation of S', then $A(S')$ can be completed to one with rank k. This shows $e(S') \leqslant 2n(S') + k - 3$. That is, the hereditary Laman condition is fulfilled. We conclude that *PT* is edge-Laman, provided that $k \leqslant 4$. In conjunction with Lemma 2 we obtain:

Observation 1. *For $k \leqslant 4$, every rank-k pseudo-triangulation of S is edge-Laman with characteristic* $(2, 0, 3 - k)$. *For $k \leqslant 2$, every rank-k pseudo-triangulation of S is face-Laman with characteristic* $(1, 0, 2 - k)$.

It has been known [15] that minimum pseudo-triangulations enjoy the edge-Laman property; in fact, they are planar Laman graphs in the classical sense [8]. A related edge-Laman condition, for general pseudo-triangulations, is used in [12,13] to define their combinatorial abstractions. In Section 3.2 we will observe that triangulations are both edge-Laman and face-Laman. Pseudo-triangulations of arbitrary rank share neither property, in general.

3.2. k-angulations

A *k-angulation* of S, $k \geqslant 3$, is a polygonal partition on S all whose faces are k-gons, i.e., polygons with exactly k vertices. Prominent representatives are triangulations ($k = 3$) and quadrangulations ($k = 4$). Note that convexity of the faces is not required. As a well-known fact, every triangulation of a fixed point set S contains the same number of edges, and this is also true for the number of triangles. This fact generalizes to k-angulations, for $k \geqslant 4$.

The sum of interior angles in any k-gon is $\pi(k-2)$. The sum of angles in all the (interior) faces of a k-angulation of S is thus $\pi(h(S)-2)$ for angles at vertices of $conv(S)$, plus $2\pi(n(S)-h(S))$ for angles at vertices interior to $conv(S)$. Dividing by $\pi(k-2)$ gives the number of (interior) faces,

$$f(S) = \frac{2n(S) - h(S) - 2}{k-2}. \tag{9}$$

Taking into account the exterior face, the Eulerian relation gives $n(S) - e(S) + (f(S) + 1) = 2$. We plug in (9) and get the number of edges,

$$e(S) = \frac{kn(S) - h(S) - k}{k-2}. \tag{10}$$

Consider a subset $S' \subseteq S$. If the set $A(S')$ is a k-angulation of S' then (10) holds with S replaced by S'. But this formula also describes the maximum number of edges possible when k-gons on top of S' are constructed. Therefore, the hereditary Laman condition is fulfilled. Together with Lemma 2 this yields:

Observation 2. *Every k-angulation of S, for $k \geqslant 3$, is object-Laman with edge characteristic* $\frac{1}{k-2}(k, 1, k)$ *and face characteristic* $\frac{1}{k-2}(2, 1, 2)$.

If we predefine the number of j-gons for *different* values of j (for instance, m triangles and the rest quadrangles) for a polygonal partition on S, then its numbers of edges and faces are still fixed linear functions of $n(S)$ and $h(S)$. However, the corresponding linear inequalities need not be hereditary, i.e., they do not carry over to subsets $S' \subset S$. Thus such partitions are not object-Laman, in general.

3.3. k-regular partitions

A polygonal partition P is called *k-regular* if the degree of each vertex of P is exactly k. For $k = 3$, simple partitions (in the classical sense) are obtained. For instance, Schlegel diagrams of simple three-dimensional polytopes [6], and thus power diagrams and Voronoi diagrams [4] in suitable domains, belong to this class. Apart from trivial cases, k-regular partitions only exist for $3 \leqslant k \leqslant 5$.

Let now P be a k-regular partition on S. Each vertex of P is incident to exactly k edges, and each edge of P has two vertices. Consequently,

$$e(S) = \frac{k}{2} n(S). \tag{11}$$

Applying the Eulerian formula gives

$$f(S) = \left(\frac{k}{2} - 1\right) n(S) + 1. \tag{12}$$

Observe that (11) is also the maximum number of edges possible when drawing on top of S a planar straight line graph with vertex degree at most k. But, for any $S' \subseteq S$, each vertex in the set $A(S')$ is of degree at most k, which shows that the hereditary Laman condition holds for the edges of P.

In the edge characteristic of P the constant c_3 is zero and Lemma 2 does not apply. However, by using the arguments above on (12), P is easily seen to fulfill the hereditary Laman condition for faces, too. We summarize:

Observation 3. *Every k-regular polygonal partition on S, for $3 \leqslant k \leqslant 5$, is object-Laman with edge characteristic* $(\frac{k}{2}, 0, 0)$ *and face characteristic* $(\frac{k}{2} - 1, 0, -1)$.

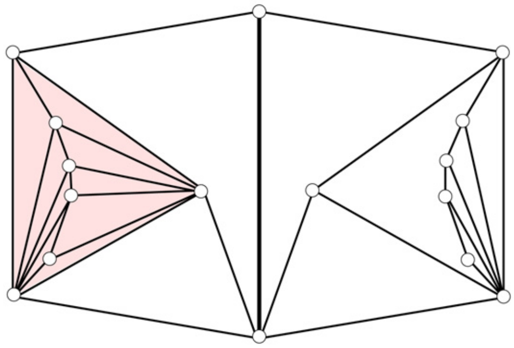

Fig. 2. No edge matching exists for this rank-4 pseudo-triangulation and its reflection.

The face characteristic of k-regular partitions does not satisfy the requirements in Lemma 1 because of $c_3 < 0$. Still, the assertion of the lemma is true for this class unless $S' = \emptyset$, by the following reasoning: For $n(S \setminus S') \leqslant 1$, the condition $c_3 \geqslant 0$ is not used in the proof of the lemma. For $n(S) > n(S \setminus S') \geqslant 2$, formula (4) sharpens to $f(S \setminus S') < (\frac{k}{2} - 1)n(S \setminus S') + 1$, because not all vertices in $A(S \setminus S')$ can achieve degree k. This implies the desired inequality $\beta(S') \geqslant (\frac{k}{2} - 1)n(S') + 1$. As a consequence, the face matching in Theorem 1, that relies on Lemma 1 for $n(S') \geqslant 2$, now has to exist for k-regular partitions.

For straight line graphs on S (as opposed to polygonal partitions on S) the notion of k-regularity is meaningful for general k. For example, for $k = 2$ we obtain vertex-disjoint covering cycles, and for $k = 1$ we obtain perfect matchings. It follows that these structures are edge-Laman with characteristics $(1, 0, 0)$ and $(\frac{1}{2}, 0, 0)$, respectively. Finally, note that any spanning tree of S is edge-Laman with characteristic $(1, 0, 1)$.

3.4. Summary of incidence matchings

Our results in Sections 3.1–3.3 combine with Theorem 1 in Section 2 (the incidence matching theorem). We summarize in the following statement:

Theorem 2. *Let S be a finite set of points in the plane. Let P and Q be two structures on top of S, both belonging to one of the following classes (k fixed): Rank-k pseudo-triangulations for $k \leqslant 3$, k-angulations, k-regular partitions, k-regular straight line graphs, spanning trees. Then there exists a perfect matching between the edge sets of P and Q such that matched edges share a vertex. The same is true for the face sets of P and Q, except for the last two classes and for rank-3 pseudo-triangulations.*

Observe that, although rank-4 pseudo-triangulations are edge-Laman (Observation 1), starting already from rank 4 an edge incidence matching need not exist for general (fixed) rank. The two pseudo-triangulations we use to demonstrate this are the one shown in Fig. 2 (call it PT_1) and the one we obtain when reflecting PT_1 along the bold vertical edge (call this structure PT_2). Note that PT_1 and PT_2 live on the same point set. Let Δ denote the shaded triangle. Consider the restrictions of PT_1 and PT_2, respectively, to Δ, and let E_1 and E_2 be their respective edge sets. The 15 edges of E_1 can only be matched to the 11 edges of E_2 or to the 3 additional edges of PT_2 that are incident to the vertices of Δ. Thus no perfect matching is possible.

Fig. 2 also serves as an example for the following fact: Requiring $c_3 \geqslant -1$ (instead of $c_3 \geqslant 0$) in Theorem 1 is not strong enough to ensure an incidence matching.

For triangulations, vertex incidence of matched triangles *plus* overlap can be satisfied simultaneously [2]. While the overlap condition has to be dropped for general pseudo-triangulations, see Fig. 1, the incidence condition for pseudo-triangles can be retained for rank $k \leqslant 2$, according to Observation 1. In particular, minimum pseudo-triangulations admit such a face matching.

4. Decomposition into spanning trees

Several authors considered the question of whether a given graph is decomposable into disjoint spanning trees; see e.g. [7] and references therein. A basic theorem has been given by Nash-Williams [11] and Tutte [16]:

Theorem 3. *A graph $G = (V, E)$ contains k edge-disjoint spanning trees if and only if for every partition of V into t parts there are at least $k(t-1)$ edges in E connecting these parts.*

For polygonal partitions, Theorem 3 implies:

Theorem 4. *Let P be a polygonal partition on S with $k(n(S)-1)$ edges. The edge set of P can be decomposed into k spanning trees if and only if P is edge-Laman with characteristic $(k, 0, k)$.*

Proof. Assume that P is edge-Laman with characteristic $(k, 0, k)$. Then $e(S') \leqslant k(n(S')-1)$ holds for each subset $S' \subseteq S$ with $n(S') \geqslant 2$. For $n(S') \leqslant 1$ we clearly have $e(S') = 0$. Therefore, for any partition of S into subsets $S_1, \ldots, S_t$, the number of edges of P that do not connect two different subsets is at most

$$\sum_{i=1}^{t} k\big(n(S_i)-1\big) = kn(S) - kt.$$

That is, the number of edges of P between different subsets is at least

$$e(S) - \big(kn(S) - kt\big) = k(t-1).$$

So, by Theorem 3, the set of edges of P can be decomposed into k spanning trees.

Conversely, assume that P enjoys this decomposition property. Let $S' \subseteq S$. We apply Theorem 3 to the partition of S into S' and $n(S) - n(S')$ singleton sets. We get that the number of edges of P between different subsets is at least

$$k\big(1 + n(S) - n(S') - 1\big) = k\big(n(S) - n(S')\big).$$

Consequently, the number of edges that do not connect different subsets, which is precisely the number $e(S')$ of edges spanned by S', is not larger than

$$k\big(n(S)-1\big) - k\big(n(S) - n(S')\big) = k\big(n(S')-1\big).$$

Thus the hereditary Laman condition is fulfilled, and we conclude that P is edge-Laman with characteristic $(k, 0, k)$. □

From Observation 1 we get the following property:

Corollary 1. *Every rank-1 pseudo-triangulation of S can be decomposed into two spanning trees.*

It is well known that, in case *conv*(S) is a triangle, every triangulation of S is decomposable into three spanning trees which are edge-disjoint apart from the three edges of *conv*(S); see, e.g., [9,14]. We obtain the following generalization:

Corollary 2. *Every triangulation of S can be decomposed into three spanning trees if the $h(S)$ edges of conv(S) are duplicated.*

Proof. Consider some triangulation T of S, and let us count edges of *conv*(S) twice. Then Observation 2 implies that T has exactly $3n(S) - 3$ edges, and that each subset $S' \subset S$ spans at most $3n(S') - 3$ edges because no edge in the interior of *conv*(S') is duplicate. That is, the Laman properties are satisfied with edge characteristic $(3, 0, 3)$. □

Corollary 3. *Every quadrangulation of S can be decomposed into two spanning trees if every other edge of conv(S) is duplicated.*

Proof. The number of edges in any quadrangulation of S is $2n(S) - \frac{h(S)}{2} - 2$; see formula (10). Thus $h(S)$ is even. Counting every other edge of *conv*(S) twice, we get from Observation 2 that each subset $S' \subset S$ spans at most $2n(S') - 2$ edges, because no edge in the interior of *conv*(S') is duplicate, and duplicate edges do not share a vertex. Thus the Laman properties are fulfilled with edge characteristic $(2, 0, 2)$. □

The existence of *some* edges in a triangulation (or quadrangulation) whose duplication leads to a decomposition into spanning trees can also be proved using a result in [7]. Duplication of *arbitrary* edges does not suffice, as can be shown by simple examples.

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