J. Math. Anal. Appl. 366 (2010) 153–163

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Biholomorphic mappings on bounded starlike circular domains ☆

Qing-Hua Xu [a,*], Tai-Shun Liu [b]

[a] College of Mathematics and Information Science, JiangXi Normal University, NanChang 330027, China
[b] Department of Mathematics, Huzhou Teacher's College, Huzhou 313000, China

ARTICLE INFO

Article history:
Received 2 February 2009
Available online 16 December 2009
Submitted by R. Timoney

Keywords:
Growth and covering theorem
Starlike mapping
Subclasses of starlike mappings
Zero of order $k+1$

ABSTRACT

Let $\Omega \subset \mathbb{C}^n$ be a bounded starlike circular domain with $0 \in \Omega$. In this paper, we introduce a class of holomorphic mappings $\mathcal{M}_g$ on Ω. Let $f(z)$ be a normalized locally biholomorphic mapping on Ω such that $J_f^{-1}(z)f(z) \in \mathcal{M}_g$ and $z=0$ is the zero of order $k+1$ of $f(z)-z$. We obtain a sharp growth theorem and sharp coefficient bounds for $f(z)$. As applications, sharp distortion theorems for a subclass of starlike mappings are obtained. These results unify and generalize many known results.

1. Introduction

In the case of one complex variable, the following growth, distortion theorem and de Branges theorem are well known [16].

Theorem A. *Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ be a normalized univalent holomorphic function on the unit disc D in $\mathbb{C}$. Then*

$$\begin{aligned} &\frac{|z|}{(1+|z|)^2} \leqslant \left|f(z)\right| \leqslant \frac{|z|}{(1-|z|)^2}, \quad z \in D, \\ &\frac{1-|z|}{(1+|z|)^3} \leqslant \left|f'(z)\right| \leqslant \frac{1+|z|}{(1-|z|)^3}, \quad z \in D, \\ &|a_m| \leqslant m. \end{aligned} \tag{1}$$

However, in the case of several complex variables, Cartan [2] pointed out that the above theorem does not hold.

Since Barnard, Fitzgerald and Gong [1], Chuaqui [3] extended the growth theorem (1) to normalized starlike mappings on the Euclidean unit ball in $\mathbb{C}^n$. Liu and Ren [13] obtained the generalization on the bounded starlike circular domains in $\mathbb{C}^n$. After that, many mathematicians investigate the growth and covering theorems for the subclasses of the starlike mappings on the bounded starlike circular domains in $\mathbb{C}^n$ (see [4,8,11,13–15]).

Concerning the distortion theorem, the situation is quite different. Until now, the distortion theorem for the normalized starlike mappings is still a conjecture. Recently, Pfaltzgraff and Suffridge [17], Hamada and Kohr [9] obtained respectively

☆ This work was supported by the Key Project of Zhejing Provincial Natural Science Foundation of China (Grant No. D7080080), the Jiangxi Provincial Natural Science Foundation of China (Grant No. 2007GZS0177), Natural Science Foundation of Department of Education of Jiangxi Province, China (Grant No. GJJ 09149) and Specialized Research Fund for the Doctoral Program of JiangXi Normal University.

* Corresponding author.
E-mail addresses: xuqhster@gmail.com (Q.-H. Xu), lts@ustc.edu.cn (T.-S. Liu).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.12.025

a distortion result for a subclass of starlike mappings on the Euclidean unit ball in $\mathbb{C}^n$ and on bounded balanced pseudo-convex domains in $\mathbb{C}^n$.

As for the bounds for coefficients of subclasses of normalized biholomorphic mappings, Kohr [7] obtained a sharp bound for the second coefficient of starlike mappings or starlike mappings of α on the Euclidean unit ball in $\mathbb{C}^n$. Gong [5] obtained bounds for the second coefficients of starlike mappings on the unit polydisc in $\mathbb{C}^n$. Recently, considering the zero of order (i.e., $x=0$ is a zero of order $k+1$ of $f(x)-x$ and $f(x)$ defined on the unit ball in a complex Banach space) and using the analytical characterizations of starlike mappings, Xu and Liu [19] obtained the coefficient bounds for the class of biholomorphic mappings, while $z=0$ is a zero of order $k+1$ of $e^{-t}f(z,t)-z$ ($f(\cdot,t)$ defined on the unit ball in $\mathbb{C}^n$ with respect to an arbitrary norm), the coefficient bounds for biholomorphic mappings were studied by Hamada, Honda and Kohr [10] using the method of Loewner chains.

In this paper, inspired by the works of Hamada and Honda [11], Pfaltzgraff and Suffridge [17], Hamada and Kohr [9], we obtain sharp growth theorems and sharp coefficient bounds for a class of biholomorphic mappings defined on bounded starlike circular domain in $\mathbb{C}^n$. Moreover, the sharp distortion theorems for a subclass of starlike mappings are obtained. These results generalize the related works of some authors.

Throughout this article, let $\mathbb{C}^n$ be the space of n complex variables $z=(z_1,z_2,\dots,z_n)'$ with the Euclidean inner product $\langle z,w\rangle=\sum_{j=1}^n z_j\overline{w}_j$ and the norm $\|z\|=\langle z,z\rangle^{\frac{1}{2}}$, $z\in\mathbb{C}^n$, B^n be the Euclidean unit ball in $\mathbb{C}^n$. Let $\Omega\subset\mathbb{C}^n$ be a bounded starlike circular domain with $0\in\Omega$, and its Minkowski functional $\rho(z)\in\mathcal{C}^1$ (see Lemma 1) except for a lower dimensional manifold in $\overline{\Omega}$, where $\overline{\Omega}$ represents the closure of Ω. $\mathbb{N}$ be the set of all positive integers and D be the unit disk in $\mathbb{C}$. Let $\partial\Omega$ be the boundary of Ω and $H(\Omega)$ be the set of all holomorphic mappings from Ω into $\mathbb{C}^n$, $H(\Omega,\Omega)$ be the set of all holomorphic mappings from Ω into Ω. As is known to us, if $f\in H(\Omega)$, then

$$f(w)=\sum_{n=0}^{\infty}\frac{1}{n!}D^nf(z)\big((w-z)^n\big),$$

for all w in some neighborhood of $z\in\Omega$, where $D^nf(z)$ is the nth-Fréchet derivative of f at z, and for $n\geqslant 1$,

$$D^nf(z)\big((w-z)^n\big)=D^nf(z)\underbrace{(w-z,\dots,w-z)}_{n}.$$

Let $J_f(z)$ be the Jacobian of f at $z\in\Omega$, $\det J_f(z)$ be the Jacobian determinant of f at $z\in\Omega$. A holomorphic mapping $f:\Omega\to\mathbb{C}^n$ is said to be biholomorphic if the inverse f^{-1} exists and is holomorphic on the open set $f(\Omega)$. A mapping $f\in H(\Omega)$ is said to be locally biholomorphic if $\det J_f(z)\neq 0$ for each $z\in\Omega$. If $f:\Omega\to\mathbb{C}^n$ is a holomorphic mapping, we say that f is normalized if $f(0)=0$ and $J_f(0)=I$, where I represents the identity matrix.

Firstly, we recall a class of mappings $\mathcal{M}$ which plays the role of the Carathéodory class in several complex variables.

$$\mathcal{M}=\left\{h\in H(\Omega)\colon h(0)=0,\ J_h(0)=I,\ \Re e\,\frac{\partial\rho(z)}{\partial z}h(z)>0,\ z\in\Omega\setminus\{0\}\right\},$$

where $\frac{\partial\rho(z)}{\partial z}=(\frac{\partial\rho(z)}{\partial z_1},\dots,\frac{\partial\rho(z)}{\partial z_n})$.

Now, we introduce the following class $\mathcal{M}_g$ on $\Omega\subset\mathbb{C}^n$, which has been introduced by Kohr [7] on B^n and studied by Graham, Hamada and Kohr [6].

Definition 1. Let $g\in H(D)$ be a biholomorphic function such that $g(0)=1$, $g(\bar{\xi})=\overline{g(\xi)}$, for $\xi\in D$, $\Re e\,g(\xi)>0$ on $\xi\in D$, and assume g satisfies the following conditions for $r\in(0,1)$:

$$\begin{cases}\min\limits_{|\xi|=r}\big|g(\xi)\big|=\min\limits_{|\xi|=r}\Re e\,g(\xi)=g(-r),\\ \max\limits_{|\xi|=r}\big|g(\xi)\big|=\max\limits_{|\xi|=r}\Re e\,g(\xi)=g(r).\end{cases}\tag{2}$$

We define $\mathcal{M}_g$ to be the class of mappings given by

$$\mathcal{M}_g=\left\{h\in H(\Omega)\colon h(0)=0,\ J_h(0)=I,\ \frac{\rho(z)}{2\frac{\partial\rho(z)}{\partial z}h(z)}\in g(D),\ z\in\Omega\setminus\{0\}\right\}.$$

Clearly, if $g(\xi)=\frac{1+\xi}{1-\xi}$, $\xi\in D$, then $\mathcal{M}_g$ becomes the class $\mathcal{M}$. Especially, if $\Omega=B^n$, then

$$\mathcal{M}_g=\left\{h\in H(B)\colon h(0)=0,\ J_h(0)=I,\ \frac{\|z\|^2}{\langle h(z),z\rangle}\in g(D),\ z\in B\setminus\{0\}\right\}.$$

A normalized biholomorphic mapping $f:\Omega\to\mathbb{C}^n$ is said to be starlike if $f(\Omega)$ is a starlike domain with respect to the origin. Let $S_g^*(\Omega)$ denote the subset of the starlike mappings consisting of those normalized locally biholomorphic mappings f such that $J_f^{-1}(z)f(z)\in\mathcal{M}_g$. When $\Omega=D$, $S_g^*(D)$ is denoted by S_g^*.

Definition 2. Let $0 \leqslant \alpha < 1$. A normalized locally biholomorphic mappings $f \in H(\Omega)$ is said to be starlike of order α if

$$[Df(x)]^{-1} f(x) \in \mathcal{M}_g,$$

where $g(\zeta) = \frac{1-(2\alpha-1)\zeta}{1-\zeta}$, $\zeta \in D$.

We denote by $S^*_\alpha(\Omega)$ the set of all starlike mappings of order α on Ω.

Definition 3. Let $0 \leqslant \alpha < 1$. A normalized locally biholomorphic mappings $f \in H(\Omega)$ is said to be almost starlike of order α if

$$[Df(x)]^{-1} f(x) \in \mathcal{M}_g,$$

where $g(\zeta) = \frac{1+\zeta}{1+(2\alpha-1)\zeta}$, $\zeta \in D$.

We denote by $AS^*_\alpha(\Omega)$ the set of all starlike mappings of order α on Ω.

Definition 4. Suppose $f, g \in H(D)$. If there exists a function $\varphi \in H(D, D)$, $\varphi(0) = 0$ such that $f = g \circ \varphi$, then we say that f is subordinate to g (written $f \prec g$).

Definition 5. (See [12].) Suppose Ω is a domain (connected open set) in $\mathbb{C}^n$ which contains 0, $f \in H(\Omega)$. We say that $z = 0$ is the zero of order k of $f(z)$ if $f(0) = 0, \dots, D^{k-1} f(0) = 0$, but $D^k f(0) \neq 0$, where $k \in \mathbb{N}$.

We denote by $S^*_{g,k+1}(\Omega)$ (respectively $S^*_{\alpha,k+1}(\Omega)$, $AS^*_{\alpha,k+1}(\Omega)$) the subset of $S^*_g(\Omega)$ (respectively $S^*_\alpha(\Omega)$, $AS^*_\alpha(\Omega)$) of mappings f such that $z = 0$ is a zero of order $k+1$ of $f(z) - z$. When $\Omega = D$, $S^*_{g,k+1}(D)$ is denoted by $S^*_{g,k+1}$.

2. Preliminaries

In order to prove the desired results, we first give some lemmas.

Lemma 1. *(See [13].) $\Omega \subset \mathbb{C}^n$ is a bounded starlike circular domain if and only if there exists a unique real continuous function $\rho : \mathbb{C}^n \to \mathbb{R}$, called the Minkowski functional of Ω, such that*

(i) $\rho(z) \geqslant 0$, $z \in \mathbb{C}^n$; $\rho(z) = 0 \Leftrightarrow z = 0$;
(ii) $\rho(tz) = |t|\rho(z)$, $t \in \mathbb{C}$, $z \in \mathbb{C}^n$;
(iii) $\Omega = \{z \in \mathbb{C}^n\colon\ \rho(z) < 1\}$.

Furthermore, the function $\rho(z)$ has the following properties.

$$2\frac{\partial \rho(z)}{\partial z} z = \rho(z), \quad z \in \mathbb{C}^n,$$
$$2\frac{\partial \rho(z_0)}{\partial z} z_0 = 1, \quad z_0 \in \partial\Omega,$$
$$\frac{\partial \rho(\lambda z)}{\partial z} = \frac{\partial \rho(z)}{\partial z}, \quad \lambda \in (0, \infty),$$
$$\frac{\partial \rho(e^{i\theta} z)}{\partial z} = e^{-i\theta} \frac{\partial \rho(z)}{\partial z}, \quad \theta \in \mathbb{R},$$

where $\frac{\partial \rho(z)}{\partial z} = (\frac{\partial \rho(z)}{\partial z_1}, \dots, \frac{\partial \rho(z)}{\partial z_n})$.

Lemma 2. *(See [13].) If f is a starlike mapping on Ω, $z \in \Omega \setminus \{0\}$, $z(t) = f^{-1}(tf(z))$ $(0 \leqslant t \leqslant 1)$. Then*

(a) *$\rho(z(t))$ is strictly increasing on $[0, 1]$ with respect to t;*
(b) $\rho(f(z)) = \lim_{t\to 0} \frac{\rho(z(t))}{t}$, $\frac{dz(t)}{dt} = \frac{1}{t} J_f^{-1}(z(t)) f(z(t))$, $t \in (0, 1)$;
(c) $\frac{d\rho(z(t))}{dt} = 2\Re e(\frac{\partial \rho(z(t))}{\partial z} \frac{dz(t)}{dt})$, $t \in (0, 1)$.

Lemma 3. *(See [18].) If $f \in H(D)$, g is a biholomorphic function on D, $f(0) = g(0)$, $f'(0) = \dots = f^{(k-1)}(0) = 0$, and $f \prec g$. Then*

$$f(rD) \subseteq g(r^k D), \quad r \in (0, 1),\ rD = \{\xi \in \mathbb{C}\colon\ |\xi| < r\}.$$

Lemma 4. *(See [19].) If $f \in H(D)$, g is a biholomorphic function on D, $f(0) = g(0)$, $f'(0) = \cdots = f^{(k-1)}(0) = 0$, and $f \prec g$, then*

$$\frac{|f^{(n)}(0)|}{n!} \leqslant |g'(0)|, \quad n = k, \ldots, 2k-1.$$

Using Lemma 3, we can prove the following.

Lemma 5. *Let $g : D \to \mathbb{C}$ satisfy the conditions of Definition 1. If $h \in \mathcal{M}_g$ and $z = 0$ is the zero of order $k+1$ ($k \in \mathbb{N}$) of $h(z) - z$, then*

$$\frac{\rho(z)}{g(\rho^k(z))} \leqslant 2\Re e \frac{\partial\rho(z)}{\partial z} h(z) \leqslant \frac{\rho(z)}{g(-\rho^k(z))} \tag{3}$$

for all $z \in \Omega$.

Proof. Fix $z \in \Omega \setminus \{0\}$, and denote $z_0 = \frac{z}{\rho(z)}$. Let $p : D \to \mathbb{C}$ be given by

$$p(\eta) = \begin{cases} \frac{\eta}{2\frac{\partial\rho(z_0)}{\partial z} h(\eta z_0)}, & \eta \neq 0, \\ 1, & \eta = 0. \end{cases}$$

Then $p \in H(D)$, $p(0) = g(0) = 1$, and since $h \in \mathcal{M}_g$, we deduce that

$$p(\eta) = \frac{\eta}{2\frac{\partial\rho(z_0)}{\partial z} h(\eta z_0)} = \frac{\rho(\eta z_0)}{2\frac{\partial\rho(\eta z_0)}{\partial z} h(\eta z_0)} \in g(D), \quad \eta \in D.$$

Let $\psi(\eta) = \frac{1}{p(\eta)}$. This implies that $\psi(\eta) \in \frac{1}{g}(D)$ for all $\eta \in D$. Since $\psi(0) = \frac{1}{g}(0) = 1$, we have $\psi \prec \frac{1}{g}$.

According to hypothesis of Lemma 5, we deduce that

$$\psi(\eta) = 1 - 2\frac{\partial\rho(z_0)}{\partial z} \frac{(D^{k+1}h(0)(z_0^{k+1}))}{(k+1)!}\eta^k + \cdots.$$

It is easy to see that the function $\psi(\eta)$ satisfies the conditions of Lemma 3, hence we obtain

$$\psi(rD) \subseteq \frac{1}{g}\left(r^k D\right), \quad r \in (0,1),\ rD = \left\{\eta \in \mathbb{C}\colon |\eta| < r\right\}.$$

On the other hand, combining the maximum and minimum principles for harmonic functions with (2), we deduce that

$$\frac{1}{g(|\eta|^k)} \leqslant \Re e\, \psi(\eta) \leqslant \frac{1}{g(-|\eta|^k)}, \quad \eta \in D.$$

Setting $\eta = \rho(z)$ in the above relation, we obtain (3), as desired. This completes the proof of Lemma 5. □

Lemma 6. *Let $g : D \to \mathbb{C}$ satisfy the conditions of Definition 1. If $h \in \mathcal{M}_g$ and $z = 0$ is the zero of order $k+1$ ($k \in \mathbb{N}$) of $h(z) - z$, then*

$$\left|2\frac{\partial\rho(z)}{\partial z} \frac{D^m h(0)(z^m)}{m!}\right| \leqslant |g'(0)|\rho^m(z), \quad z \in \Omega,\ m = k+1, \ldots, 2k. \tag{4}$$

Proof. Fix $z \in \Omega \setminus \{0\}$, and denote $z_0 = \frac{z}{\rho(z)}$. Let $p : D \to \mathbb{C}$ be given by

$$p(\eta) = \begin{cases} \frac{\eta}{2\frac{\partial\rho(z_0)}{\partial z} h(\eta z_0)}, & \eta \neq 0, \\ 1, & \eta = 0. \end{cases}$$

Let $\psi(\eta) = \frac{1}{p(\eta)}$. From the proof of Lemma 5, we have

$$\psi(\eta) = 1 - 2\frac{\partial\rho(z_0)}{\partial z} \frac{D^{k+1}h(0)(z_0^{k+1})}{(k+1)!}\eta^k + \cdots. \tag{5}$$

It is easy to see that the function $\psi(\eta)$ satisfies the conditions of Lemma 4, hence we deduce that

$$\frac{|\psi^{(n)}(0)|}{n!} \leqslant |g'(0)|, \quad n = k, \ldots, 2k-1. \tag{6}$$

Combining the relations (5) and (6), we deduce that

$$\left|2\frac{\partial\rho(z)}{\partial z}\frac{D^m h(0)(z^m)}{m!}\right| \leqslant \left|g'(0)\right|\rho^m(z), \quad z\in\Omega,\ m=k+1,\ldots,2k.$$

This completes the proof of Lemma 6. □

Let $b\in S_g^*$ be defined by $b(0)=b'(0)-1=0$ and

$$\frac{\zeta b'(\zeta)}{b(\zeta)}=g(\zeta), \quad \zeta\in D.$$

For a positive integer k, let

$$b_k(\zeta)=\zeta\left[\varphi\left(\zeta^k\right)\right]^{\frac{1}{k}}, \tag{7}$$

where

$$\varphi(\zeta)=\frac{b(\zeta)}{\zeta}.$$

The branches of the power functions are chosen so that

$$\left(\varphi\left(\zeta^k\right)\right)^{\frac{1}{k}}\Big|_{\zeta=0}=1.$$

Since $\Omega\subset\mathbb{C}^n$ is a bounded starlike circular domain with $0\in\Omega$, by the definition of bounded starlike circular domain, it is not difficult to check that $U_j=\{z_j\in\mathbb{C}\colon\ (0,\ldots,0,z_j,0,\ldots,0)'\in\Omega\}$ $(j=1,\ldots,n)$ is a disk with center at the origin. Let

$$f(z)=\frac{rb_k(\frac{z_1}{r})}{z_1}z, \tag{8}$$

where r is the radius of the disk $U=\{z_1\in\mathbb{C}\colon\ (z_1,0,\ldots,0)'\in\Omega\}$. Then, we obtain the following lemma by direct computations.

Lemma 7. *Let b_k be as in* (7), *and f be as in* (8). *Then*:

(i) $b_k(\zeta)=\zeta-\frac{1}{k}g'(0)\zeta^{k+1}+\cdots$, *and*

$$\frac{\zeta b_k'(\zeta)}{b_k(\zeta)}=g\left(\zeta^k\right), \quad \zeta\in D.$$

Thus, $b_k\in S_{g,k+1}^*$ and $b_k(0)=b_k'(0)-1=0$.

(ii) $f(z)\in S_{g,k+1}^*(\Omega)$ *and*

$$f(\zeta u)=b_k(\zeta)u=\left(\zeta-\frac{1}{k}g'(0)\zeta^{k+1}+\cdots\right)u, \quad \zeta\in D,\ u=(u_1,\ldots,u_n)'\in\partial\Omega,\ u_1=r.$$

3. Main results and their proofs

In this section, we give the main results and their proofs. In the case of the unit ball in a complex Banach space, Theorems 1 and 2 were obtained by Hamada and Honda [11].

Theorem 1. *Let $g: D\to\mathbb{C}$ satisfy the conditions of Definition* 1 *and $f\in S_{g,k+1}^*(\Omega)$. Then*

$$\rho(z)\exp\int_0^{\rho(z)}\left[g\left(-y^k\right)-1\right]\frac{dy}{y}\leqslant\rho\left(f(z)\right)\leqslant\rho(z)\exp\int_0^{\rho(z)}\left[g\left(y^k\right)-1\right]\frac{dy}{y}, \quad z\in\Omega. \tag{9}$$

These estimates are sharp.

Proof. Since $f\in S_{g,k+1}^*(\Omega)$, we deduce from Lemma 5 that

$$\frac{\rho(z)}{g(\rho^k(z))}\leqslant 2\Re e\,\frac{\partial\rho(z)}{\partial z}J_f^{-1}(z)f(z)\leqslant\frac{\rho(z)}{g(-\rho^k(z))} \tag{10}$$

for all $z \in \Omega$. Obviously, according to the assumption of Theorem 1, we conclude that f belongs to either starlike mappings class or its subclasses. Fix $z \in \Omega \setminus \{0\}$, let $z(t) = f^{-1}(tf(z))$ $(0 \leqslant t \leqslant 1)$. In view of Lemma 2(a), we deduce that $\rho(z(t))$ is strictly increasing on $[0,1]$. Hence, $\rho(z(t))$ is differentiable on $[0,1]$ a.e. From Lemma 2 and (10), we deduce that for $t \in (0,1]$

$$\frac{\rho(z(t))}{g(\rho^k(z(t)))} \leqslant t\frac{d\rho(z(t))}{dt} \leqslant \frac{\rho(z(t))}{g(-\rho^k(z(t)))}, \tag{11}$$

and we may rewrite (11) as

$$\frac{g(-\rho^k(z(t)))}{\rho(z(t))}\frac{d\rho(z(t))}{dt} \leqslant \frac{1}{t} \leqslant \frac{g(\rho^k(z(t)))}{\rho(z(t))}\frac{d\rho(z(t))}{dt}.$$

Integrating both sides of the above inequalities with respect to t and making a change of variable, we obtain

$$\int_{\rho(z(\varepsilon))}^{\rho(z)} \frac{g(-y^k)\,dy}{y} = \int_{\varepsilon}^{1} \frac{g(-\rho^k(z(t)))}{\rho(z(t))}\frac{d\rho(z(t))}{dt}\,dt \leqslant \int_{\varepsilon}^{1} \frac{1}{t}\,dt,$$

and

$$\int_{\rho(z(\varepsilon))}^{\rho(z)} \frac{g(y^k)\,dy}{y} = \int_{\varepsilon}^{1} \frac{g(\rho^k(z(t)))}{\rho(z(t))}\frac{d\rho(z(t))}{dt}\,dt \geqslant \int_{\varepsilon}^{1} \frac{1}{t}\,dt,$$

where $0 < \varepsilon < 1$. It is elementary to verify that

$$\log\frac{\rho(z(\varepsilon))}{\varepsilon} \geqslant \int_{\rho(z(\varepsilon))}^{\rho(z)} \big[g\big(-y^k\big) - 1\big]\frac{dy}{y} + \log\rho(z), \tag{12}$$

and

$$\log\frac{\rho(z(\varepsilon))}{\varepsilon} \leqslant \int_{\rho(z(\varepsilon))}^{\rho(z)} \big[g\big(y^k\big) - 1\big]\frac{dy}{y} + \log\rho(z). \tag{13}$$

If we now let $\varepsilon \to 0+$ in the above inequalities (12), (13) and use Lemma 2(b), we have

$$\rho(z)\exp\int_0^{\rho(z)} \big[g\big(-y^k\big) - 1\big]\frac{dy}{y} \leqslant \rho\big(f(z)\big) \leqslant \rho(z)\exp\int_0^{\rho(z)} \big[g\big(y^k\big) - 1\big]\frac{dy}{y}, \quad z \in \Omega, \tag{14}$$

as claimed. This completes the proof of Theorem 1. □

Later, we will show that the above estimations are sharp. To end this, we give the following theorem.

Theorem 2. *Let $g : D \to \mathbb{C}$ satisfy the conditions of Definition 1 and $f \in S^*_{g,k+1}(\Omega)$. Then*

$$e^{-\frac{\pi i}{k}} b_k\big(e^{\frac{\pi i}{k}}\rho(z)\big) \leqslant \rho\big(f(z)\big) \leqslant b_k\big(\rho(z)\big), \quad z \in \Omega. \tag{15}$$

These estimations are sharp.

Proof. From (9) and Lemma 7(i), we obtain

$$\exp\int_0^{\rho(z)} \left[\frac{y\tilde{b}_k'(y)}{\tilde{b}_k(y)} - 1\right]\frac{dy}{y} \leqslant \frac{\rho(f(z))}{\rho(z)} \leqslant \exp\int_0^{\rho(z)} \left[\frac{yb_k'(y)}{b_k(y)} - 1\right]\frac{dy}{y}$$

for $z \in \Omega$, where $\tilde{b}_k(\zeta) = e^{-\frac{\pi i}{k}} b_k(e^{\frac{\pi i}{k}}\zeta)$. Then, we obtain

$$\exp\left[\log\frac{\tilde{b}_k(\rho(z))}{\rho(z)} - \log\tilde{b}_k'(0)\right] \leqslant \frac{\rho(f(z))}{\rho(z)} \leqslant \exp\left[\log\frac{b_k(\rho(z))}{\rho(z)} - \log b_k'(0)\right]$$

for $z \in \Omega$, since $\tilde{b}_k(y)$, $b_k(y)$ for $y > 0$. This implies (15).

Next, we will show that the estimations (15) are sharp. Let $f(z) \in S^*_{g,k+1}(\Omega)$ be as in (8). Since $\rho(f(Ru)) = b_k(R)$ and $\rho(f(e^{\frac{\pi i}{k}} Ru)) = |b_k(e^{\frac{\pi i}{k}} R)|$, where $0 \leqslant R < 1$, $u = (u_1, \ldots, u_n)' \in \partial\Omega$, $u_1 = r$, the equalities of the estimations (15) hold. This completes the proof. □

Remark 1. The equivalence of (9) and (15) implies that the estimations (9) are sharp.

Now, we obtain the following corollaries from Theorem 1.

Corollary 1. *If $f \in S^*_{\alpha,k+1}(\Omega)$, then*

$$\frac{\rho(z)}{(1+\rho^k(z))^{\frac{2(1-\alpha)}{k}}} \leqslant \rho\big(f(z)\big) \leqslant \frac{\rho(z)}{(1-\rho^k(z))^{\frac{2(1-\alpha)}{k}}}, \quad z \in \Omega.$$

The above estimate is sharp.

Proof. Letting $g(\zeta) = \frac{1-(2\alpha-1)\zeta}{1-\zeta}$, $\zeta \in D$, $0 \leqslant \alpha < 1$ in Theorem 1, we obtain the desired result. This completes the proof. □

Corollary 2. *If $f \in AS^*_{\alpha,k+1}(\Omega)$, then*

$$\frac{\rho(z)}{(1+(1-2\alpha)\rho^k(z))^{\frac{2(1-\alpha)}{k(1-2\alpha)}}} \leqslant \rho\big(f(z)\big) \leqslant \frac{\rho(z)}{(1-(1-2\alpha)\rho^k(z))^{\frac{2(1-\alpha)}{k(1-2\alpha)}}}$$

for $z \in \Omega$, $0 \leqslant \alpha < 1$, $\alpha \neq \frac{1}{2}$ and

$$\rho(z)\exp\left(-\frac{\rho^k(z)}{k}\right) \leqslant \rho\big(f(z)\big) \leqslant \rho(z)\exp\left(\frac{\rho^k(z)}{k}\right)$$

for $z \in \Omega$, $\alpha = \frac{1}{2}$. The above estimations are sharp.

Proof. Letting $g(\zeta) = \frac{1+\zeta}{1+(2\alpha-1)\zeta}$, $\zeta \in D$, $0 \leqslant \alpha < 1$ in Theorem 1, we have the desired result. This completes the proof. □

Remark 2. Corollaries 1, 2 were obtained by Liu and Liu [15] using the analytical characterizations of starlike mappings of order α and almost starlike mapping of order α on B. However, taking $g(\zeta) = \frac{1-(2\alpha-1)\zeta}{1-\zeta}, \frac{1+\zeta}{1+(2\alpha-1)\zeta}$, $\zeta \in D$ in Theorem 1, respectively, we easily obtain these results.

Theorem 3. *Let $g: D \to \mathbb{C}$ satisfy the conditions of Definition 1 and $f \in S^*_{g,k+1}(\Omega)$. Then*

$$\left|2\frac{\partial\rho(z)}{\partial z}\frac{D^m f(0)(z^m)}{m!}\right| \leqslant \frac{1}{m-1}\big|g'(0)\big|\rho^m(z), \quad z \in \Omega,\ m = k+1, \ldots, 2k. \tag{16}$$

When $m = k+1$, this estimation is sharp.

Proof. Denote $h(z) = J_f^{-1}(z)f(z)$, $z \in \Omega$. Since $f(z) = J_f(z)h(z)$, it follows that

$$\begin{aligned} z &+ \frac{D^{k+1}f(0)(z^{k+1})}{(k+1)!} + \cdots + \frac{D^m f(0)(z^m)}{m!} + \cdots \\ &= \left(I + \frac{D^{k+1}f(0)(z^k,\cdot)}{k!} + \cdots + \frac{D^m f(0)(z^{m-1},\cdot)}{(m-1)!} + \cdots\right) \\ &\quad \times \left(J_h(0)z + \frac{D^2h(0)(z^2)}{2!} + \frac{D^{k+1}h(0)(z^{k+1})}{(k+1)!} + \cdots + \frac{D^m h(0)(z^m)}{m!} + \cdots\right). \end{aligned}$$

Comparing with the coefficients of two sides of the above equality, we have

$$J_h(0)z = z, \qquad D^j h(0)\big(z^j\big) = 0, \quad j = 2, \ldots, k. \tag{17}$$

Using (17), we deduce that

$$\frac{D^m f(0)(z^m)}{m!} = \frac{D^m h(0)(z^m)}{m!} + \frac{D^m f(0)(z^m)}{(m-1)!}, \quad m = k+1, \ldots, 2k,$$

$$\frac{D^m f(0)(z^m)}{m!} = \frac{-1}{m-1}\frac{D^m h(0)(z^m)}{m!}, \quad m = k+1, \ldots, 2k. \tag{18}$$

Clearly, $h(z)$ satisfies the condition of Lemma 6. In view of Lemma 6 and (18), we obtain

$$\left|2\frac{\partial\rho(z)}{\partial z}\frac{D^m f(0)(z^m)}{m!}\right| = \frac{1}{m-1}\left|2\frac{\partial\rho(z)}{\partial z}\frac{D^m h(0)(z^m)}{m!}\right|$$
$$\leqslant \frac{1}{m-1}\left|g'(0)\right|\rho^m(z), \quad z\in\Omega,\ m=k+1,\dots,2k.$$

The following example shows that estimation (16) is sharp for $m=k+1$.

Example 1. Let f be as in (8). According to Lemma 7, we obtain that $f\in S^*_{g,k+1}(\Omega)$, and

$$f(z)=z-\frac{1}{k}g'(0)\left(\frac{z_1}{r}\right)^k z+\cdots.$$

Taking $z=\xi u$, where $u=(u_1,\dots,u_n)'\in\partial\Omega$, $u_1=r$, we have

$$f(\xi u)=\zeta u-\frac{1}{k}g'(0)\zeta^{k+1}u+\cdots.$$

Therefore,

$$\frac{D^{k+1}f(0)(u^{k+1})}{(k+1)!}=-\frac{1}{k}g'(0)u.$$

By Lemma 1, we have

$$2\frac{\partial\rho(u)}{\partial z}\frac{D^{k+1}f(0)(u^{k+1})}{(k+1)!}=-\frac{1}{k}g'(0)2\frac{\partial\rho(u)}{\partial z}u=-\frac{1}{k}g'(0).$$

Setting $u=\frac{z}{\rho(z)}$, $z\in\Omega$ in the above relation, we obtain

$$\left|2\frac{\partial\rho(z)}{\partial z}\frac{D^{k+1}f(0)(z^{k+1})}{(k+1)!}\right|=\frac{1}{k}\left|g'(0)\right|\rho^{k+1}(z),$$

as claimed. This completes the proof. □

The following theorems are associated with the operator F, which was firstly introduced by Pfaltzgraff and Suffridge [17] on B^n.

Theorem 4. *Let $g: D\to\mathbb{C}$ be a convex function which satisfies the conditions of Definition 1. Suppose $f_j\in S^*_g$, $\lambda_j\geqslant 0$, $j=1,2,\dots,n$, and $\sum_{j=1}^n\lambda_j=1$. Then*

$$F\in S^*_g(\Omega),$$

where $F(z)=z\prod_{j=1}^n\left(\frac{r_jf_j(\frac{z_j}{r_j})}{z_j}\right)^{\lambda_j}$, r_j is the radius of the disk U_j, and the branch of the power function is chosen such that $\left(\frac{r_jf_j(\frac{z_j}{r_j})}{z_j}\right)^{\lambda_j}|_{z_j=0}=1$, $j=1,\dots,n$.

Proof. Let $h(z)=\prod_{j=1}^n\left(\frac{r_jf_j(\frac{z_j}{r_j})}{z_j}\right)^{\lambda_j}$. Then $J_F(z)\eta=h(z)(\eta+\frac{(J_h(z)\eta)z}{h(z)})$, $\eta\in\mathbb{C}^n$. Also

$$\frac{J_h(z)z}{h(z)}=\sum_{j=1}^n\lambda_j\left(\frac{z_jf_j'(\frac{z_j}{r_j})}{r_jf_j(\frac{z_j}{r_j})}-1\right)=\sum_{j=1}^n\lambda_j\frac{z_jf_j'(\frac{z_j}{r_j})}{r_jf_j(\frac{z_j}{r_j})}-1.$$

Since $f_j\in S^*_g$ $(j=1,2,\dots,n)$, we have

$$\Re e\left[1+\frac{J_h(z)z}{h(z)}\right]=\sum_{j=1}^n\lambda_j\Re e\left[\frac{z_jf_j'(\frac{z_j}{r_j})}{r_jf_j(\frac{z_j}{r_j})}\right]>0,\quad z\in\Omega.$$

Therefore, $J_h(z)z+h(z)\neq 0$. It is not difficult to check that

$$J_F^{-1}(z)\eta=\frac{1}{h(z)}\left(\eta-\frac{(J_h(z)\eta)z}{J_h(z)z+h(z)}\right),\quad \eta\in\mathbb{C}^n.$$

So $F(z)$ is a normalized locally biholomorphic mapping on Ω.

Straightforward calculation shows that

$$\frac{\rho(z)}{2\frac{\partial\rho(z)}{\partial z}[J_F^{-1}(z)F(z)]} = 1 + \frac{J_h(z)z}{h(z)} = \sum_{j=1}^{n}\lambda_j \frac{z_j f_j'(\frac{z_j}{r_j})}{r_j f_j(\frac{z_j}{r_j})}.$$

On the other hand, since $\frac{z_j f_j'(\frac{z_j}{r_j})}{r_j f_j(\frac{z_j}{r_j})} \in g(D)$, and $g(D)$ is convex, we have

$$\sum_{j=1}^{n}\lambda_j \frac{z_j f_j'(\frac{z_j}{r_j})}{r_j f_j(\frac{z_j}{r_j})} \in g(D),$$

as claimed. This completes the proof. □

Theorem 5. *Let $g: D \to \mathbb{C}$ be a convex function which satisfies the conditions of Definition 1. Suppose $f_j \in S_g^*$, $\lambda_j \geqslant 0$, $j = 1, 2, \ldots, n$, and $\sum_{j=1}^{n}\lambda_j = 1$. Then*

$$g\big(-\rho(z)\big)\exp n\int_0^{\rho(z)}\big[g(-y)-1\big] \leqslant \big|\det J_F(z)\big| \leqslant g\big(\rho(z)\big)\exp n\int_0^{\rho(z)}\big[g(y)-1\big]\frac{dy}{y}, \tag{19}$$

where $z \in \Omega$, $F(z) = z\prod_{j=1}^{n}(\frac{r_j f_j(\frac{z_j}{r_j})}{z_j})^{\lambda_j}$, r_j is the radius of the disk U_j, and the branch of the power function is chosen such that $(\frac{r_j f_j(\frac{z_j}{r_j})}{z_j})^{\lambda_j}|_{z_j=0} = 1$, $j = 1, \ldots, n$.

Proof. As in the proof of Theorem 4, we can easily deduce that

$$\big|\det J_F(z)\big| = \prod_{j=1}^{n}\left|\frac{r_j f_j(\frac{z_j}{r_j})}{z_j}\right|^{n\lambda_j}\left|\sum_{j=1}^{n}\lambda_j\frac{z_j f_j'(\frac{z_j}{r_j})}{r_j f_j(\frac{z_j}{r_j})}\right|. \tag{20}$$

From the maximum (minimum) principle for harmonic functions and the fact that $|\frac{z_j}{r_j}| \leqslant \rho(z)$ $(j = 1, \ldots, n)$, we obtain

$$\left|\sum_{j=1}^{n}\lambda_j\frac{z_j f_j'(\frac{z_j}{r_j})}{r_j f_j(\frac{z_j}{r_j})}\right| \geqslant \sum_{j=1}^{n}\lambda_j \Re e\,\frac{z_j f_j'(\frac{z_j}{r_j})}{r_j f_j(\frac{z_j}{r_j})} \geqslant g\big(-\rho(z)\big), \tag{21}$$

and

$$\left|\sum_{j=1}^{n}\lambda_j\frac{z_j f_j'(\frac{z_j}{r_j})}{r_j f_j(\frac{z_j}{r_j})}\right| \leqslant \sum_{j=1}^{n}\lambda_j\left|\frac{z_j f_j'(\frac{z_j}{r_j})}{r_j f_j(\frac{z_j}{r_j})}\right| \leqslant g\big(\rho(z)\big). \tag{22}$$

On the other hand, by Theorems 4 and 1, we have

$$\exp\int_0^{\rho(z)}\big[g(-y)-1\big] \leqslant \prod_{j=1}^{n}\left|\frac{r_j f_j(\frac{z_j}{r_j})}{z_j}\right|^{\lambda_j} \leqslant \exp\int_0^{\rho(z)}\big[g(y)-1\big]\frac{dy}{y}. \tag{23}$$

From (20)–(23), we obtain (19), as claimed. This completes the proof. □

Remark 3. The estimations of Theorem 5 are sharp. To see this, let $b \in S_g^*$ be defined by $b(0) = b'(0) - 1 = 0$ and

$$\frac{\xi b'(\xi)}{b(\xi)} = g(\xi), \quad \xi \in D. \tag{24}$$

And let

$$F(z) = \frac{rb(\frac{z_1}{r})}{z_1}z, \tag{25}$$

where r is the radius of the disk $U = \{z_1 \in \mathbb{C}\colon\ (z_1, 0, \ldots, 0)' \in \Omega\}$. In view of Lemma 7, $F \in S_g^*(\Omega)$.

From (24), we obtain the following equivalent formulation of Theorem 5.

$$\frac{\rho(z)\tilde{b}'(\rho(z))}{\tilde{b}(\rho(z))}\exp n\int_0^{\rho(z)}\left[\frac{y\tilde{b}'(y)}{\tilde{b}(y)}-1\right]\frac{dy}{y}\leqslant\left|\det J_F(z)\right|\leqslant\frac{\rho(z)b'(\rho(z))}{b(\rho(z))}\exp n\int_0^{\rho(z)}\left[\frac{yb'(y)}{b(y)}-1\right]\frac{dy}{y}$$

for $z\in\Omega$, where $\tilde{b}(\xi)=-b(-\xi)$. Then, we have

$$\frac{\rho(z)\tilde{b}'(\rho(z))}{\tilde{b}(\rho(z))}\exp n\left[\log\frac{\tilde{b}(\rho(z))}{\rho(z)}-\log\tilde{b}'(0)\right]\leqslant\left|\det J_F(z)\right|\leqslant\frac{\rho(z)b'(\rho(z))}{b(\rho(z))}\exp n\left[\log\frac{b(\rho(z))}{\rho(z)}-\log b(0)\right]$$

for $z\in\Omega$, since $\tilde{b}(y),b(y)>0$ for $y>0$. We deduce that

$$\frac{-\rho(z)b'(-\rho(z))}{b(-\rho(z))}\left(\frac{-b(-\rho(z))}{\rho(z)}\right)^n\leqslant\left|\det J_F(z)\right|\leqslant\frac{\rho(z)b'(\rho(z))}{b(\rho(z))}\left(\frac{b(\rho(z))}{\rho(z)}\right)^n,\quad z\in\Omega. \tag{26}$$

Now, we show that the estimations (26) are sharp. Let $F\in S_g^*(\Omega)$ be as in (25). Taking $z=Ru$ or $z=-Ru$ ($0\leqslant R<1$, $u=(u_1,\dots,u_n)'\in\partial\Omega$, $u_1=r$), then the equalities of the estimations (26) hold for $\lambda_1=1$, $\lambda_j=0$ ($j=2,\dots,n$), and $f_j=b$ ($j=1,\dots,n$). The equivalence of (19) and (26) implies that the estimations (19) are sharp. This completes the proof.

Theorem 6. *Let $g:D\to\mathbb{C}$ be a convex function which satisfies the conditions of Definition 1. Suppose $f_j\in S_g^*$, $\lambda_j\geqslant 0$, $j=1,2,\dots,n$, and $\sum_{j=1}^n\lambda_j=1$. Then*

$$g\bigl(-\rho(z)\bigr)\exp\int_0^{\rho(z)}\bigl[g(-y)-1\bigr]\leqslant\rho\bigl(J_F(z)z\bigr)\leqslant g\bigl(\rho(z)\bigr)\exp\int_0^{\rho(z)}\bigl[g(y)-1\bigr]\frac{dy}{y},$$

where $z\in\Omega$, $F(z)=z\prod_{j=1}^n(\frac{r_jf_j(\frac{z_j}{r_j})}{z_j})^{\lambda_j}$, r_j is the radius of the disk U_j, and the branch of the power function is chosen such that $(\frac{r_jf_j(\frac{z_j}{r_j})}{z_j})^{\lambda_j}|_{z_j=0}=1$, $j=1,\dots,n$.

Proof. Denote

$$h(x)=\prod_{j=1}^n\left(\frac{r_jf_j(\frac{z_j}{r_j})}{z_j}\right)^{\lambda_j},\quad z\in\Omega.$$

Straightforward computation shows that

$$J_F(z)z=h(z)z+\bigl(J_h(z)z\bigr)z=z\prod_{j=1}^n\left(\frac{r_jf_j(\frac{z_j}{r_j})}{z_j}\right)^{\lambda_j}\sum_{j=1}^n\lambda_j\frac{z_jf_j'(\frac{z_j}{r_j})}{r_jf_j(\frac{z_j}{r_j})}. \tag{27}$$

From (21)–(23) and (27), we have the desired result. This completes the proof. □

Remark 4. The estimations of Theorem 6 are sharp. The proof of sharpness is similar to that of Theorem 5, so we omit it.

Acknowledgments

The authors are grateful to the anonymous for their helpful suggestions and comments.

References

[1] R.W. Barnard, C.H. Fitzgerald, S. Gong, The growth and 1/4 theorems for starlike mappings in $\mathbb{C}^n$, Pacific J. Math. 150 (1991) 13–22.
[2] H. Cartan, Sur la possibilité d'éntendre aux fonctions de plusieurs variables complexes la théorie des fonctions univalents, in: P. Montel (Ed.), Lecons sur les Fonctions Univalents on Mutivalents, Gauthier–Villars, 1933, pp. 129–155.
[3] M. Chuaqui, Applications of subordination chains to starlike mappings in $\mathbb{C}^n$, Pacific J. Math. 168 (1995) 33–48.
[4] S.X. Feng, K.P. Li, The growth theorem for almost starlike mappings of order α on bounded starlike circular domains, Chinese Quart. J. Math. 15 (2) (2000) 50–56.
[5] S. Gong, The Bieberbach Conjecture, Amer. Math. Soc. Intern. Press, Cambridge, MA, 1999.
[6] I. Graham, H. Hamada, G. Kohr, Parametric representation of univalent mappings in several complex variables, Canad. J. Math. 54 (2) (2002) 324–351.
[7] G. Kohr, On some best bounds for coefficients of subclasses of univalent holomorphic mappings in $\mathbb{C}^n$, Complex Var. 36 (1998) 261–284.
[8] H. Hamada, Starlike mappings on bounded balanced domains with $\mathcal{C}^1$-plurisubharmonic defining functions, Pacific J. Math. 194 (2000) 359–371.

[9] H. Hamada, G. Kohr, Subordination chains and univalence of holomorphic mappings on bounded balanced pseudoconvex domains, Ann. Univ. Mariae Curie-Sklodowska Sect. A 55 (2001) 61–80.
[10] H. Hamada, T. Honda, G. Kohr, Growth theorems and coefficient bounds for univalent holomorphic mappings which have parametric representation, J. Math. Anal. Appl. 317 (2006) 302–319.
[11] H. Hamada, T. Honda, Sharp growth theorems and coefficient bounds for starlike mappings in several complex variables, Chin. Ann. Math. Ser. B 29 (4) (2008) 353–368.
[12] Y.Y. Lin, Y. Hong, Some properties of holomorphic maps in Banach spaces, Acta Math. Sinica 38 (1995) 234–241 (in Chinese).
[13] T.S. Liu, G.B. Ren, The growth theorem for starlike mappings on bounded starlike circular domains, Chin. Ann. Math. Ser. B 19 (1998) 401–408.
[14] H. Liu, K.P. Lu, Two subclasses of starlike mappings in several complex variables, Chinese Ann. Math. Ser. A 21 (5) (2000) 533–546.
[15] X.S. Liu, T.S. Liu, On the sharp growth, covering theorems for normalized biholomorphic mappings in $\mathbb{C}^n$, Acta Math. Sci. Ser. B 27 (4) (2007) 803–812.
[16] C. Pommerenke, Univalent Functions, Vandenhoeck & Ruprecht, Göttingen, 1975.
[17] J.A. Pfaltzgraff, T.J. Suffridge, An extension theorem and linear invariant families generated by starlike maps, Ann. Univ. Mariae Curie-Sklodowska Sect. A 53 (1999) 193–207.
[18] Q.H. Xu, T.S. Liu, The study for some subclasses of biholomorphic mappings by an unified method, Chinese Quart. J. Math. 2 (2006) 9–19.
[19] Q.H. Xu, T.S. Liu, Coefficient bounds for biholomorphic mappings which have the parametric representation, J. Math. Anal. Appl. 355 (2009) 126–130.