Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 328 (2007) 438–451
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Hilbert space-valued forward–backward stochastic differential equations with Poisson jumps and applications

Juliang Yin [a,b,*,1], Yongjin Wang [b]

[a] *Department of Statistics, Jinan University, Guangzhou 510630, PR China*
[b] *School of Mathematics, Nankai University, Tianjin 300071, PR China*

Received 15 April 2005
Available online 23 June 2006
Submitted by J. Rosinski

Abstract

In this paper, we study a class of Hilbert space-valued forward–backward stochastic differential equations (FBSDEs) with bounded random terminal times; more precisely, the FBSDEs are driven by a cylindrical Brownian motion on a separable Hilbert space and a Poisson random measure. In the case where the coefficients are continuous but not Lipschitz continuous, we prove the existence and uniqueness of adapted solutions to such FBSDEs under assumptions of weak monotonicity and linear growth on the coefficients. Existence is shown by applying a finite-dimensional approximation technique and the weak convergence theory. We also use these results to solve some special types of optimal stochastic control problems.

Keywords: Adapted solution; Cylindrical Brownian motion; Forward–backward SDEs; Poisson point process; Optimal stochastic control

* Corresponding author.
E-mail address: yinjuliang@21cn.com (J. Yin).
1 Supported by the Postdoctoral Science Foundation of China Program No. 2003034315 and the NSF of Guangdong Province Program No. 04300573.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.05.033

1. Introduction

In finite-dimensional spaces, the existence and uniqueness of adapted solutions to forward–backward stochastic differential equations (FBSDEs for short) with Poisson jumps were established by SiTu [6], Yin and SiTu [9,10] via purely probabilistic approaches. They deal with the FBSDEs with Lipschitz continuous coefficients or with continuous coefficients in the case of fixed terminal time as well as in the case of random terminal time. Obviously, it is impossible to generalize these results to the infinite-dimensional spaces case for FBSDEs with non-Lipschitz continuous coefficients by using the smoothing technique (see, e.g., Yin and SiTu [10]). So we need to find another approach to solve such FBSDEs.

Hu and Peng [2] investigated a class of infinite-dimensional semi-linear backward stochastic evolution equations, and the so-called "mild solution" was given with the help of the Riesz representation theorem and an extended martingale representation theorem. Thereafter Hu and Peng [3] discussed semi-linear backward stochastic evolution equations and stochastic partial differential equations, and also proved the existence and uniqueness of adapted solutions by utilizing the extended martingale representation theorem and the stochastic Fubini theorem. These results were proved to be very useful in discussing stochastic Hamilton–Bellman–Jacobi equations (cf. [5]). Moreover, SiTu [8] considered a class of backward stochastic differential equations which have jumps and are driven by a K-valued Brownian motion and a Poisson random measure. The existence and uniqueness results were established, and some results were used to solve some optimal stochastic control problems with respect to certain BSDEs with jumps in Hilbert spaces.

In this paper, we are concerned with a class of FBSDEs with bounded random terminal times in an infinite-dimensional space driven by a cylindrical Brownian motion and a Poisson random measure. We give the existence and uniqueness results for such FBSDEs when the coefficients are continuous but not Lipschitz continuous, and some applications to optimal stochastic control problems. The proof of existence is based on the theory of weak convergence and the method of finite-dimensional approximation. It should be mentioned that Yor [11] showed the existence and uniqueness of strong solutions for finite horizon forward SDEs in a Hilbert space when the coefficient satisfies linear growth condition and Lipschitz condition. The meaning of the strong solution is actually identical with that of the adapted solution. Indeed, this result can be generalized to the case of forward SDEs with Poisson jumps. Furthermore, under some suitable conditions (for example, monotonicity condition and Lipschitz condition on the coefficients in [9]), we can use Itô's formula for H-valued cylindrical Brownian motion and Poisson random measure in [8] and the method of continuation given by Hu and Peng [4] to prove the existence and uniqueness theorem of solutions to H-valued FBSDEs with Poisson jumps. But as above mentioned, for those H-valued FBSDEs without Lipschitz continuous coefficients we cannot depend on the smoothing technique of [10] to solve them. Although some ideas of [1,8] are used throughout this work, the differences of framework and studied subject, require more assumptions and more arguments.

The paper is organized as follows: in Section 2, we give the preliminaries, including the definition of Hilbert space-valued cylindrical Brownian motion and the corresponding stochastic integral with respect to it; in Section 3, the existence and uniqueness of adapted solutions to Hilbert space-valued FBSDEs with Poisson jumps and with non-Lipschitz continuous coefficients are proved by partially adopting some ideas of Darling and Pardoux [1] and SiTu [8]. And a priori estimate and a uniqueness theorem of adapted solutions to above FBSDEs are also given. Finally, in Section 4 we use an example, which is a class of special FBSDEs without Lipschitz continuous coefficients, to illustrate that our given assumptions can be fulfilled. We also apply our main results to study optimal stochastic control problems.

2. Preliminaries

Let K and H be two separable Hilbert spaces with inner product $(\cdot,\cdot)_K$ and $(\cdot,\cdot)_H$, respectively. We denote their norms by $|\cdot|_K$ and $|\cdot|_H$. Assume that $\{e_i\}_{i=1}^\infty$ and $\{\bar{e}_i\}_{i=1}^\infty$ are two orthonormal bases of K and H, respectively. Without confusion, we always use $(\cdot,\cdot)$ and $|\cdot|$ to denote the inner product and the norm.

Definition 2.1. We say that $\{w(t), t \in R_+\}$ is a cylindrical Brownian motion on the separable Hilbert space K, if the following three conditions hold:

(i) $\forall t \in R_+$, $w(t)$ is a stochastic linear functional on K;
(ii) $\forall n \in N$, $\forall h_1, h_2, \ldots, h_n \in K$, $\{(w(t)h_1, w(t)h_2, \ldots, w(t)h_n), t \in R_+\}$ is a Brownian motion (not necessary standard) with values in R^n;
(iii) $\forall h_1, h_2 \in K$, $\forall t \in R_+$, $E(w(t)h_1)(w(t)h_2) = t(h_1, h_2)$.

Suppose that $\{k(t), t \in R_+\}$ is a Poisson point process taking values in measurable space $(Z, \mathcal{B}(Z))$ with compensator $\pi(z)\,ds$. $\pi(z)$ is a σ-finite measure on $\mathcal{B}(Z)$. We denote by $N_k(ds, dz)$ the Poisson counting measure induced by $k(\cdot)$, and by $\widetilde{N}_k(ds, dz)$ the martingale measure such that $\widetilde{N}_k(ds, dz) = N_k(ds, dz) - \pi(dz)\,ds$. Throughout this paper we assume that $(\Omega, \mathcal{F}, P; \{\mathcal{F}_t\}_0^T)$ is a complete filtered probability space such that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$, $\mathcal{F}_t = \mathcal{F}_{t^+} = \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}$, $t \geqslant 0$, and $\mathcal{F} = \mathcal{F}_T$. We further assume that the filtration is generated by the cylindrical Brownian motion $w(\cdot)$ and the Poisson point process $k(\cdot)$ and augmented, that is, $\mathcal{F}_t = \sigma[w(s); s \leqslant t] \vee \sigma[N_k(A, (0, s]); s \leqslant t,\ A \in \mathcal{B}(Z)] \vee N$, $t \leqslant T$, where N is all the P-null sets, and $T > 0$ is a real number. The following notation will be used in this paper:

$$S^2_{\mathcal{F}}(H) := \Big\{v(t,\omega):\ v(t,\omega) \text{ is } H\text{-valued, } \mathcal{F}_t\text{-adapted such that } E \sup_{0\leqslant t\leqslant \tau} \big|v(t,\omega)\big|^2 < \infty\Big\};$$

$$L^2_{\mathcal{F}}(H) := \Big\{f(t,\omega):\ f(t,\omega) \text{ is } H\text{-valued, } \mathcal{F}_t\text{-adapted such that } E\int_0^\tau \big|f(t,\omega)\big|^2\,dt < \infty\Big\};$$

$$F^2_{\mathcal{F}}(H) := \Big\{u(t,z,\omega):\ u(t,z,\omega) \text{ is } H\text{-valued, } \mathcal{F}_t\text{-predictable such that } E\int_0^\tau\int_Z \big|u(t,z,\omega)\big|^2 \pi(dz)\,dt < \infty\Big\};$$

$$L^2_{\mathcal{F}}\big(\mathcal{L}(K,H)\big) := \Bigg\{\xi(t,\omega)\in\mathcal{L}(K,H)\colon\ \xi(t,\omega) \text{ is } \mathcal{F}_t\text{-adapted such that}$$
$$E\int_0^\tau \big\|\xi(t,\omega)\big\|^2_{\mathcal{L}(K,H)}\,dt := E\int_0^\tau \sum_{i=1}^\infty \big|\xi(t,\omega)e_i\big|^2_H\,dt < \infty\Bigg\};$$

$$L^2_\pi(H) := \Bigg\{\eta(z)\colon\ \eta(z) \text{ is } \mathcal{B}(Z) \text{ measurable, } H\text{-valued such that}$$
$$\|\eta\|^2 := \int_Z \big|\eta(z)\big|^2\pi(dz) < \infty\Bigg\},$$

where $\tau \leqslant T$ is a bounded stopping time, $\mathcal{L}(K,H)$ is the set of all bounded linear operators from K into H, and $\{e_i\}_{i=1}^\infty$ is the orthonormal basis of K. For any $\varphi(\cdot)\in L^2_{\mathcal{F}}(\mathcal{L}(K,H))$, the stochastic integration $\int_0^\tau \varphi(t)\,dw(t)$ is defined as follows:

$$\int_0^\tau \varphi(t)\,dw(t) = \sum_{i=1}^\infty \int_0^\tau \varphi(t)e_i\,d\big(w(t)e_i\big).$$

Obviously, the above definition has a meaning. It is clear that $L^2_{\mathcal{F}}(H)\times L^2_{\mathcal{F}}(H)\times L^2_{\mathcal{F}}(\mathcal{L}(K,H))\times F^2_{\mathcal{F}}(H)$ is a Hilbert space. We still use $|\cdot|$ to denote the norm on $L^2_{\mathcal{F}}(\mathcal{L}(K,H))$ below.

3. Hilbert space-valued FBSDEs with Poisson jumps

In this section, we consider the following coupled forward–backward stochastic differential equations with Poisson jumps:

$$\begin{cases} dx_t = b(t,x_t,y_t,q_t,p_t,\omega)\,dt + \sigma(t,x_t,y_t,q_t,p_t,\omega)\,dw(t) \\ \qquad + \int_Z c(t,x_{t-},y_{t-},q_t,p_t,z,\omega)\widetilde{N}_k(dt,dz), \quad x_0\in L^2((\Omega,\mathcal{F}_0,P);H), \\ dy_t = -f(t,x_t,y_t,q_t,p_t,\omega)\,dt + q_t\,dw(t) + \int_Z p_t(z)\widetilde{N}_k(dt,dz), \\ \qquad y_\tau = \psi(x_\tau), \quad 0\leqslant t\leqslant \tau, \end{cases} \tag{3.1}$$

where the coefficients b,σ,c,f and ψ are the maps as follows:

$$\begin{aligned}
&b\colon[0,T]\times H\times H\times\mathcal{L}(K,H)\times L^2_\pi(H)\times\Omega\to H,\\
&f\colon[0,T]\times H\times H\times\mathcal{L}(K,H)\times L^2_\pi(H)\times\Omega\to H,\\
&\sigma\colon[0,T]\times H\times H\times\mathcal{L}(K,H)\times L^2_\pi(H)\times\Omega\to \mathcal{L}(K,H),\\
&c\colon[0,T]\times H\times H\times\mathcal{L}(K,H)\times L^2_\pi(H)\times Z\times\Omega\to H,\\
&\psi\colon\Omega\times H\to H.
\end{aligned}$$

Assume further that b and f are all jointly measurable and $\mathcal{F}_t$-adapted; σ and c are all jointly measurable and $\mathcal{F}_t$-predictable; ψ is jointly measurable with respect to $\mathcal{B}(H)\times\mathcal{F}_\tau$.

Definition 3.1. A quadruple $(x_\cdot, y_\cdot, q_\cdot, p_\cdot)$ with values in $H\times H\times\mathcal{L}(K,H)\times H$ is called an adapted solution of FBSDEs (3.1) if and only if

(i) $(x_\cdot, y_\cdot, q_\cdot, p_\cdot) \in S^2_{\mathcal{F}}(H) \times S^2_{\mathcal{F}}(H) \times L^2_{\mathcal{F}}(\mathcal{L}(K,H)) \times F^2_{\mathcal{F}}(H)$;
(ii) $(x_\cdot, y_\cdot, q_\cdot, p_\cdot)$ satisfies (3.1).

For simplicity, we will use the following notation:

$$u := (x, y, q, p), \qquad A(t,u,\omega) := \big(-f(t,u,\omega), b(t,u,\omega), \sigma(t,u,\omega), c(t,u,\cdot,\omega)\big),$$
$$(u, A) := u \cdot A = -(x, f) + (y, b) + \langle\langle \sigma, q \rangle\rangle + \langle p, c \rangle,$$
$$\langle\langle \sigma, q \rangle\rangle := \sum_{i=1}^{\infty} (\sigma e_i, q e_i),$$
$$\langle p, c \rangle := \int_Z \big(p_\cdot(z), c(\cdot, u, z)\big)\pi(dz).$$

We suppose that:

(A1) b, f, σ, c are all continuous with respect to $(x, y, q, p) \in H \times H \times \mathcal{L}(K,H) \times L^2_\pi(H)$.

(A2) $A(t,u,\omega) = A_1(t,u,\omega) + A_2(t,u,\omega)$, where $|A_1(t,u,\omega)| \leqslant u(t)(1+|x|+|y|)$. Furthermore, for any $u_i = (x_i, y_i, q_i, p_i) \in H \times H \times \mathcal{L}(K,H) \times L^2_\pi(H)$, $i = 1,2$, the following conditions are satisfied:

$$\big|A_1(t,x,y,q_1,p_1,\omega) - A_1(t,x,y,q_2,p_2,\omega)\big| \leqslant \alpha\big[|q_1-q_2| + \|p_1-p_2\|\big];$$
$$\big|A_2(t,u_1,\omega) - A_2(t,u_2,\omega)\big| \leqslant u(t)\big[|x_1-x_2| + |y_1-y_2|\big] + \alpha\big[|q_1-q_2| + \|p_1-p_2\|\big],$$

where $\alpha \geqslant 0$, $u(t)$ is a non-negative deterministic function satisfying $\int_0^T u(t)^2\,dt < \infty$.

(A3) $\psi(x,\omega) = \beta_4 x + \psi(0,\omega)$, where $\beta_4 \geqslant 0$, $\psi(0,\omega)$ is measurable with respect to $\mathcal{F}_\tau$ and square integrable.

(A4)

$$E\Bigg[\int_0^\tau \big|f_2(s,0,0,0,0,\omega)\big|^2\,ds + \int_0^\tau \big|b_2(s,0,0,0,0,\omega)\big|^2\,ds + \int_0^\tau \big\|\sigma_2(s,0,0,0,0,\omega)\big\|^2\,ds + \int_0^\tau \big\|c_2(s,0,0,0,0,\cdot,\omega)\big\|^2\,ds\Bigg] = L < \infty.$$

(A5) For any $u_i = (x_i, y_i, q_i, p_i)$, $i = 1,2$, the following inequality holds:

$$\big(A(t,u_1) - A(t,u_2), u_1 - u_2\big) \leqslant -\beta_1 u(t)|x_1-x_2|^2 - \beta_2 u(t)|y_1-y_2|^2 - \beta_3\big[|q_1-q_2|^2 + \|p_1-p_2\|^2\big], \tag{3.2}$$

where $\beta_i \geqslant 0$, $1 \leqslant i \leqslant 3$. To prove the uniqueness of solutions to (3.1), assume further that one of the following conditions is satisfied:
(1) $\beta_1, \beta_4 > 0$, and

$$\big(y_1 - y_2, f_1(t,x_1,y_1,q,p) - f_1(t,x_2,y_2,q,p)\big) \leqslant u(t)\rho\big(|y_1-y_2|^2\big) + u(t)|y_1-y_2||x_1-x_2|; \tag{3.3}$$

(2) $\beta_2, \beta_3 > 0$, and

$$\begin{aligned}&\bigl(x_1 - x_2, b_1(t, x_1, y_1, q, p) - b_1(t, x_2, y_2, q, p)\bigr)\\&\quad \leqslant u(t)\rho\bigl(|x_1 - x_2|^2\bigr) + u(t)|x_1 - x_2||y_1 - y_2|,\end{aligned} \tag{3.4}$$

where $\rho(u), u \geqslant 0$ is a strictly increasing, continuous and concave function with $\rho(0) = 0$ such that $\int_{0+} du/\rho(u) = \infty$.

Lemma 3.1 *(A priori estimate). Assume that* (A1)–(A4) *and* (3.2) *of* (A5) *with* $\beta_1, \beta_4 > 0$ *or* $\beta_2, \beta_3 > 0$ *hold. If* $(x_., y_., q_., p_.)$ *is an adapted solution of FBSDEs* (3.1)*, then*

$$E\left\{\sup_{0\leqslant t\leqslant\tau} |x_t|^2 + \sup_{0\leqslant t\leqslant\tau} |y_t|^2 + \int_0^\tau \|q_t\|^2\,dt + \int_0^\tau \|p_t\|^2\,dt\right\} \leqslant C < \infty,$$

where C *is a positive constant depending on* L, $\int_0^T u(t)^2\,dt$, T, α, β_i, $i = 1, 2, 3, 4$ *and* $E|\psi(0)|^2$ *only.*

Proof. The proof is standard. By combining Itô's formula for H-valued B.M. and Poisson random measure (see [8, Lemma 1]), Gronwall's inequality, Burkhoder–Davis–Gandy's inequality, as well as some elementary algebraic inequalities, the result is obtained easily. □

Lemma 3.2 *(Uniqueness). If* (3.3) *or* (3.4) *is satisfied, then there exists a unique solution* $(x_., y_., q_., p_.) \in S^2_{\mathcal{F}}(H) \times S^2_{\mathcal{F}}(H) \times L^2_{\mathcal{F}}(\mathcal{L}(K, H)) \times F^2_{\mathcal{F}}(H)$ *for FBSDEs* (3.1).

Proof. The proof is the same as that of Theorem 2.1 in [8]. □

Lemma 3.3. *If* K *and* H *are all Euclidean spaces,* $w(t)$ *is a finite-dimensional Brownian motion, then* (3.1) *admits a unique adapted solution under the assumptions of* (A1)–(A5).

Proof. We may prove the lemma in the same way as Theorem 3.2 in [10]. First, we smooth out the coefficient $A_1(t, u, \omega)$ to get $A_1^n(t, u, \omega)$ over components x and y with the same smoother. Then we proceed the same arguments in Theorem 3.2. Indeed, the proof here is even easier, since the assumption of A_1 does not influence on the proof. □

Now let us give the main theorem in this paper:

Theorem 3.1. *Let* (A1)–(A5) *hold. Then FBSDEs* (3.1) *have a unique adapted solution* $(x_., y_., q_., p_.) \in S^2_{\mathcal{F}}(H) \times S^2_{\mathcal{F}}(H) \times L^2_{\mathcal{F}}(\mathcal{L}(K, H)) \times F^2_{\mathcal{F}}(H)$.

Proof. Uniqueness is directly derived by Lemma 3.2. It is sufficient to show the existence of solutions. Let H_m and K_m denote the linear subspaces generated by finite bases $\{\bar{e}_i\}_{i=1}^m$ and $\{e_i\}_{i=1}^m$, respectively. For any $x \in H$, we set

$$Q_m x = \sum_{i=1}^m (x, \bar{e}_i)_H \bar{e}_i, \qquad \tilde{Q}_m w(t) = \sum_{i=1}^m \bigl(w(t) e_i\bigr).$$

Then $\tilde{Q}_m w(t)$ can be seen as an m-dimensional Brownian motion. Consider the following FBSDEs with Poisson jumps:

$$\begin{cases} dx_t^m = Q_m b(t, x_t^m, y_t^m, q_t^m, p_t^m, \omega)\,dt + Q_m \sigma(t, x_t^m, y_t^m, q_t^m, p_t^m, \omega)\tilde{Q}_m\,dw(t) \\ \qquad + \int\limits_Z Q_m c(t, x_{t-}^m, y_{t-}^m, q_t^m, p_t^m, z, \omega)\widetilde{N}_k(dt, dz), \quad x_0^m = Q_m x_0, \\ dy_t^m = -Q_m f(t, x_t^m, y_t^m, q_t^m, p_t^m, \omega)\,dt + q_t^m \tilde{Q}_m\,dw(t) + \int\limits_Z p_t^m(z)\widetilde{N}_k(dt, dz), \\ \qquad y_\tau^m = Q_m \psi(x_\tau^m), \quad 0 \leqslant t \leqslant \tau. \end{cases} \tag{3.5}$$

From Lemma 3.3, we know that FBSDEs (3.5) have a unique adapted solution $(x_t^m, y_t^m, q_t^m, p_t^m) \in S_{\mathcal{F}}^2(H_m) \times S_{\mathcal{F}}^2(H_m) \times L_{\mathcal{F}}^2(\mathcal{L}(K_m, H_m)) \times F_{\mathcal{F}}^2(H_m)$. If we set

$$\begin{aligned} &(\tilde{q}_t^m e_i, \bar{e}_k) = (q_t^m e_i, \bar{e}_k) = q_{ki}(t), \quad 1 \leqslant k, i \leqslant m, \\ &(\tilde{q}_t^m e_{m+j}, \bar{e}_k) = 0, \quad \forall j = 1, 2, \ldots, \ 1 \leqslant k \leqslant m, \\ &b_m(s, x, y, q, p, \omega) = Q_m b(s, x^{(m)}, y^{(m)}, q^{(m)}, p^{(m)}, \omega), \\ &f_m(s, x, y, q, p, \omega) = Q_m f(s, x^{(m)}, y^{(m)}, q^{(m)}, p^{(m)}, \omega), \\ &\sigma_m(s, x, y, q, p, \omega) = Q_m \sigma(s, x^{(m)}, y^{(m)}, q^{(m)}, p^{(m)}, \omega)\tilde{Q}_m, \\ &c_m(s, x, y, q, p, z, \omega) = Q_m c(s, x^{(m)}, y^{(m)}, q^{(m)}, p^{(m)}, z, \omega), \end{aligned}$$

where $\tilde{q}_t^m \in \mathcal{L}(K, H_m)$, $x^{(m)} = \sum_{i=1}^m (x, \bar{e}_i)\bar{e}_i \in H_m$, $y^{(m)} = \sum_{i=1}^m (y, \bar{e}_i)\bar{e}_i \in H_m$, $p^{(m)} = \sum_{i=1}^m (p, \bar{e}_i)\bar{e}_i$, and for any $u \in K$,

$$q^{(m)} u = \sum_{i=1}^m \left(\sum_{j=1}^m (u, e_j) q e_j, \bar{e}_i \right) \bar{e}_i \in H_m,$$

then (3.5) can be rewritten as

$$\begin{cases} dx_t^m = b_m(t, x_t^m, y_t^m, \tilde{q}_t^m, p_t^m, \omega)\,dt + \sigma_m(t, x_t^m, y_t^m, \tilde{q}_t^m, p_t^m, \omega)\,dw(t) \\ \qquad + \int\limits_Z c_m(t, x_{t-}^m, y_{t-}^m, \tilde{q}_t^m, p_t^m, z, \omega)\widetilde{N}_k(ds, dz), \quad x_0^m = Q_m x_0, \\ dy_t^m = -f_m(t, x_t^m, y_t^m, \tilde{q}_t^m, p_t^m, \omega)\,dt + \tilde{q}_t^m\,dw(t) + \int\limits_Z p_t^m(z)\widetilde{N}_k(dt, dz), \\ \qquad y_\tau^m = Q_m \psi(x_\tau^m), \quad 0 \leqslant t \leqslant \tau. \end{cases} \tag{3.6}$$

By Lemma 3.1 and the assumptions (A2) and (A4), we have

$$\sup_m E\left\{ \sup_{0 \leqslant t \leqslant \tau} |x_t^m|^2 + \sup_{0 \leqslant t \leqslant \tau} |y_t^m|^2 + \int_0^\tau \|\tilde{q}_t^m\|^2\,dt + \int_0^\tau \|p_t^m\|^2\,dt \right\} < \infty$$

and

$$\begin{aligned} \sup_m E\Bigg\{ \int_0^\tau \big(&|b_m(t, x_t^m, y_t^m, \tilde{q}_t^m, p_t^m, \omega)|^2 + |f_m(t, x_t^m, y_t^m, \tilde{q}_t^m, p_t^m, \omega)|^2 \\ &+ \|\sigma_m(t, x_t^m, y_t^m, \tilde{q}_t^m, p_t^m, \omega)\|^2 + \|C_m(t, x_t^m, y_t^m, \tilde{q}_t^m, p_t^m, z, \omega)\|^2 \big)\,dt \Bigg\} < \infty. \end{aligned}$$

Hence there exists a subsequence $\{m(j)\}$ of $\{m\}$, where we denote it by $\{m\}$ again, and $(x_., y_., q_., p_.) \in L^2_{\mathcal{F}}(H) \times L^2_{\mathcal{F}}(H) \times L^2_{\mathcal{F}}(\mathcal{L}(K,H)) \times F^2_{\mathcal{F}}(H)$, $b_\infty(\cdot)$, $f_\infty(\cdot) \in L^2_{\mathcal{F}}(H)$, $\sigma_\infty(\cdot) \in L^2_{\mathcal{F}}(\mathcal{L}(K,H))$, $c_\infty(\cdot,z) \in F^2_{\mathcal{F}}(H)$ such that as $m \to \infty$,

$$\begin{aligned}
&x_.^m \to x_. \quad \text{weakly in } L^2_{\mathcal{F}}(H),\\
&y_.^m \to y_. \quad \text{weakly in } L^2_{\mathcal{F}}(H),\\
&\tilde{q}_.^m \to q_. \quad \text{weakly in } L^2_{\mathcal{F}}\big((K,H)\big),\\
&p_.^m \to p_. \quad \text{weakly in } F^2_{\mathcal{F}}(H),\\
&b_m\big(\cdot, x_.^m, y_.^m, \tilde{q}_.^m, p_.^m\big) \to b_\infty \quad \text{weakly in } L^2_{\mathcal{F}}(H),\\
&f_m\big(\cdot, x_.^m, y_.^m, \tilde{q}_.^m, p_.^m\big) \to f_\infty \quad \text{weakly in } L^2_{\mathcal{F}}(H),\\
&\sigma_m\big(\cdot, x_.^m, y_.^m, \tilde{q}_.^m, p_.^m\big) \to \sigma_\infty \quad \text{weakly in } L^2_{\mathcal{F}}\big(\mathcal{L}(K,H)\big),\\
&c_m\big(\cdot, x_.^m, y_.^m, \tilde{q}_.^m, p_.^m, z\big) \to c_\infty(z) \quad \text{weakly in } F^2_{\mathcal{F}}(H).
\end{aligned}$$

On the other hand, for any $\eta \in L^2((\Omega, \mathcal{F}_\tau, P); H)$, the extended martingale representation theorem in Hilbert space H (see, e.g., [7]) yields that

$$\begin{aligned}
&\eta = E\eta + \int_0^\tau h(t)\,dw(t) + \int_0^\tau \int_Z g(t,z)\widetilde{N}_k(ds,dz),\\
&\quad h(\cdot) \in L^2_{\mathcal{F}}\big(\mathcal{L}(K,H)\big), \quad g(\cdot,z) \in F^2_{\mathcal{F}}(H).
\end{aligned}$$

Therefore

$$\begin{aligned}
E\left(\eta, \int_0^\tau \tilde{q}_t^m\,dw(t)\right) &= E\left[\int_0^\tau \langle\!\langle h(t), \tilde{q}_t^m\rangle\!\rangle\,dt\right] \to E\left[\int_0^\tau \langle\!\langle h(t), q_t\rangle\!\rangle\,dt\right]\\
&= E\left(\eta, \int_0^\tau q_t\,dw(t)\right)
\end{aligned}$$

and

$$\begin{aligned}
E\left(\eta, \int_0^\tau \int_Z p_t^m(z)\widetilde{N}_k(dt,dz)\right) &= E\left[\int_0^\tau \int_Z \big\langle g(t,z), p_t^m(z)\big\rangle\,dt\right]\\
&\to E\left[\int_0^\tau \int_Z \big\langle g(t,z), p_t\big\rangle\,dt\right]\\
&= E\left(\eta, \int_0^\tau \int_Z p_t(z)\widetilde{N}_k(dt,dz)\right).
\end{aligned}$$

Also

$$E\big(\eta, y_\tau^m\big) = E\big(\eta, \beta_4 x_\tau^m + Q_m\psi(0)\big) \to E\big(\eta, \beta_4 x_\tau + \psi(0)\big) = E\big(\eta, \psi(x_\tau)\big).$$

Now we take weak limits on both sides of the stochastic integral equations determined by (3.6). Above results and the fact of $x_0^m \to x_0$ (strongly in $L^2((\Omega, \mathcal{F}_0, P); H)$) will give

$$\begin{cases} x_t = x_0 + \int_0^t b_\infty(s)\,ds + \int_0^t \sigma_\infty(s)\,dw(s) + \int_0^t \int_Z c_\infty(s,z)\widetilde{N}_k(ds,dz) \\ y_t = \psi(x_\tau) + \int_t^\tau f_\infty(s)\,ds - \int_t^\tau q_s\,dw(s) - \int_t^\tau \int_Z p_s(z)\widetilde{N}_k(ds,dz), \quad 0 \leqslant t \leqslant \tau. \end{cases} \tag{3.7}$$

In the sequel we will show

$$b_\infty(t) = b(t, x_t, y_t, q_t, p_t, \omega), \qquad f_\infty(t) = f(t, x_t, y_t, q_t, p_t, \omega),$$
$$\sigma_\infty(t) = \sigma(t, x_t, y_t, q_t, p_t, \omega), \qquad c_\infty(t,z) = c(t, x_t, y_t, q_t, p_t, z, \omega).$$

For any $(\bar{x}_t, \bar{y}_t, \bar{q}_t, \bar{p}_t) \in L^2_{\mathcal{F}}(H) \times L^2_{\mathcal{F}}(H) \times L^2_{\mathcal{F}}(\mathcal{L}(K,H)) \times F^2_{\mathcal{F}}(H)$, define

$$u_t^m = (x_t^m, y_t^m, q_t^m, p_t^m), \qquad \bar{u}_t = (\bar{x}_t, \bar{y}_t, \bar{q}_t, \bar{p}_t),$$
$$A_m(t, u_t^m, \omega) = \left(-f_m(t, u_t^m, \omega), b_m(t, u_t^m, \omega), \sigma_m(t, u_t^m, \omega), c_m(t, u_t^m, \cdot, \omega)\right),$$
$$\Delta_m = E\int_0^\tau \left(A_m(t, u_t^m, \omega) - A_m(t, \bar{u}_t, \omega), u_t^m - \bar{u}_t\right) dt$$
$$+ E\int_0^\tau \left(\beta_1 u(t)\left|x_t^m - \bar{x}_t^{(m)}\right|^2 + \beta_2 u(t)\left|y_t^m - \bar{y}_t^{(m)}\right|^2\right.$$
$$\left.+ \beta_3\left[\left|q_t^m - \bar{q}_t^{(m)}\right|^2 + \left\|p_t^m - \bar{p}_t^{(m)}\right\|^2\right]\right) dt.$$

From the definitions of f_m, b_m, σ_m, c_m and assumption (A5), we immediately get

$$\Delta_m = E\int_0^\tau \left(A(t, u_t^m, \omega) - A(t, \bar{u}_t^{(m)}, \omega), u_t^m - \bar{u}_t^{(m)}\right) dt$$
$$+ E\int_0^\tau \left(\beta_1 u(t)\left|x_t^m - \bar{x}_t^{(m)}\right|^2 + \beta_2 u(t)\left|y_t^m - \bar{y}_t^{(m)}\right|^2\right.$$
$$\left.+ \beta_3\left[\left|q_t^m - \bar{q}_t^{(m)}\right|^2 + \left\|p_t^m - \bar{p}_t^{(m)}\right\|^2\right]\right) dt \leqslant 0.$$

By Itô's formula, we obtain

$$E\left[\int_0^\tau \left[\left(y_t^m, b_m(t, u_t^m)\right) - \left(x_t^m, f_m(s, u_t^m)\right) + \left\langle\!\left\langle q_t^m, \sigma_m(t, u_t^m)\right\rangle\!\right\rangle + \left\langle p_t^m(\cdot), c_m(t, u_t^m, \cdot)\right\rangle\right] dt\right]$$
$$= E\left[\beta_4\left|x_\tau^m\right|^2 + Q_m\psi(0)x_\tau^m - Q_m x_0 y_0^m\right].$$

Using weak lower semi-continuity of convex functionals, and noting that $y_0^m \to y_0$ weakly in $L^2((\Omega, \mathcal{F}_0, P); H)$, we can get that

$$\begin{aligned}
&\varliminf_{m\to\infty} E\left[\int_0^\tau \left[\left(y_t^m, b_m\left(t, u_t^m\right)\right) - \left(x_t^m, f_m\left(t, u_t^m\right)\right) + \left\langle\!\left\langle q_t^m, \sigma_m\left(t, u_t^m\right)\right\rangle\!\right\rangle\right.\right. \\
&\qquad \left.\left. + \left\langle p_t^m(\cdot), c_m\left(t, u_t^m, \cdot\right)\right\rangle\right] dt\right] \\
&\quad = \varliminf_{m\to\infty} E\left[\beta_4\left|x_\tau^m\right|^2 + \psi(0)x_\tau^m - Q_m x_0 y_0^m\right] \\
&\quad \geqslant E\left[\beta_4|x_\tau|^2 + \psi(0)x_\tau - x_0 y_0\right] \\
&\quad = E\left[\int_0^\tau \left[\left(y_t, b_\infty(t)\right) - \left(x_t, f_\infty(t)\right) + \left\langle\!\left\langle q_t, \sigma_\infty(t)\right\rangle\!\right\rangle + \left\langle p_t(\cdot), c_\infty(t, \cdot)\right\rangle\right] dt\right]. \qquad (3.8)
\end{aligned}$$

Since $A_m(\cdot, \bar{u}, \omega) \to A(\cdot, \bar{u}, \omega)$ strongly, this gives

$$E\left[\int_0^\tau \left(u_t^m - \bar{u}_t, A_m(t, \bar{u}_t) - A(t, \bar{u}_t)\right) dt\right] \to 0, \quad m \to \infty. \qquad (3.9)$$

Define the following Hilbert space:

$$L^2_{\mathcal{F},u(\cdot)}(H) = \left\{ v(t,\omega)\colon\ v(t,\omega) \text{ is } H\text{-valued}, \mathcal{F}_t\text{-adapted such that} \right. \\
\left. \|v\|^2_{u(\cdot)} = E\int_0^\tau u(t)\left|v(t,\omega)\right|^2 dt < \infty \right\}.$$

It is clear that $S^2_{\mathcal{F}}(H) \subseteq L^2_{\mathcal{F},u(\cdot)}(H)$. By Itô's formula, we have

$$\begin{cases}
|x_t|^2 = |x_0|^2 + \displaystyle\int_0^t 2x_s b_\infty(s)\, ds + \int_0^t 2\left(x_s, \sigma_\infty(s)\, dw(s)\right) + \int_0^t \left\|\sigma_\infty(s)\right\|^2 ds \\
\qquad + \displaystyle\int_0^t \int_Z 2\left(x_s, p_\infty(s,z)\right)\widetilde{N}_k(ds,dz) + \int_0^t \int_Z \left\|c_\infty(s,z)\right\|^2 N_k(ds,dz), \\
|y_t|^2 = \left|\psi(x_\tau)\right|^2 + \displaystyle\int_t^\tau 2y_s f_\infty(s)\, ds - \int_t^\tau 2\left(x_s, q_s\, dw(s)\right) \\
\qquad - \displaystyle\int_t^\tau 2\left(x_s, p_s(z)\right)\widetilde{N}_k(ds,dz) - \int_t^\tau \|q_s\|^2 ds + \int_t^\tau \left\|p_s(z)\right\|^2 N_k(ds,dz).
\end{cases}$$

From Gronwall's inequality, elementary algebraic inequalities and B–D–G's inequality, it is not difficult to prove that $E\sup_{0\leqslant t\leqslant\tau}|x_t|^2 < \infty$ and $E\int_0^\tau |x_t|^2 < \infty$. Indeed, we also have $S^2_{\mathcal{F}}(H) = L^2_{\mathcal{F}}(H)$ under the assumptions of the present paper. By the uniqueness of weak convergence, we conclude that

$$x_\cdot^m \to x_\cdot \quad \text{weakly in } L^2_{\mathcal{F},u(\cdot)}(H); \qquad y_\cdot^m \to y_\cdot \quad \text{weakly in } L^2_{\mathcal{F},u(\cdot)}(H).$$

From these and (3.8) and (3.9), we immediately obtain

$$\begin{aligned}
0 &\geqslant \varliminf_{m\to\infty} \Delta_m \\
&\geqslant E\int_0^\tau \big(A_\infty(t,\omega) - A(t,\bar u_t,\omega), u_t - \bar u_t\big)\,dt \\
&\quad + E\int_0^\tau \big(\beta_1 u(t)|x_t - \bar x_t|^2 + \beta_2 u(t)|y_t - \bar y_t|^2 + \beta_3\big[|q_t - \bar q_t|^2 + \|p_t - \bar p_t\|^2\big]\big)\,dt.
\end{aligned} \tag{3.10}$$

Set

$$\bar y_t = y_t, \qquad \bar q_t = q_t, \qquad \bar p_t = p_t, \qquad x_t - \bar x_t = \varepsilon\phi(t),$$

where $\varepsilon > 0$, $\phi(t) := f(t, x_t, y_t, q_t, p_t) - f_\infty(t)$. As $\varepsilon \to 0$, with probability one

$$f_\infty(t,\omega) = f(t, x_t, y_t, q_t, p_t, \omega)$$

follows. Similarly, if we respectively set

$$\begin{aligned}
&\bar x_t = x_t, \qquad \bar q_t = q_t, \qquad \bar p_t = p_t, \qquad y_t - \bar y_t = -\varepsilon\big(b(t,x_t,y_t,q_t,p_t) - b_\infty(t)\big);\\
&\bar x_t = x_t, \qquad \bar y_t = y_t, \qquad \bar p_t = p_t, \qquad q_t - \bar q_t = -\varepsilon\big(\sigma(t,x_t,y_t,q_t,p_t) - \sigma_\infty(t)\big);\\
&\bar x_t = x_t, \qquad \bar y_t = y_t, \qquad \bar q_t = q_t, \qquad p_t - \bar p_t = -\varepsilon\big(c(t,x_t,y_t,q_t,p_t,z) - c_\infty(t,z)\big),
\end{aligned}$$

then with probability one we have

$$\begin{aligned}
&b(t,x_t,y_t,q_t,p_t) = b_\infty(t), \qquad \sigma(t,x_t,y_t,q_t,p_t) = \sigma_\infty(t),\\
&c(t,x_t,y_t,q_t,p_t,z) = c_\infty(t,z).
\end{aligned}$$

These complete the proof. □

4. Some applications to optimal stochastic control problems

In this section, we first give an example to illustrate that our given assumptions can be fulfilled, then we apply our main results to solve a class of special optimal stochastic control problems in a Hilbert space.

Example 4.1. Consider the following forward–backward stochastic differential equations with Poisson jumps taking values in a separable Hilbert space H:

$$\begin{cases}
dx_t = b(t,x_t,y_t,q_t,p_t,\omega)\,dt + \sigma(t,x_t,y_t,q_t,p_t,\omega)\,dw(t) \\
\qquad\quad + \displaystyle\int_Z c(t,x_{t-},y_{t-},q_t,p_t,z,\omega)\widetilde N_k(dt,dz), \quad x_0 = a \in H,\\
dy_t = -f(t,x_t,y_t,q_t,p_t,\omega)\,dt + q_t dw(t) + \displaystyle\int_Z p_t(z)\widetilde N_k(dt,dz),\\
\qquad\quad y_\tau = x_\tau + \psi(0,\omega), \quad 0 \leqslant t \leqslant \tau \leqslant T,
\end{cases} \tag{4.1}$$

where $\psi(0,\omega)$ is measurable with respect to $\mathcal{F}_\tau$ and square integrable. Moreover,

$$\begin{cases} b(s,x,y,q,p) = b_1(s,x,y,q,p) + b_2(s,x,y,q,p), \\ b_1(s,x,y,q,p) = -I_{s\neq 0}s^{-\alpha}y/|y|^{1-\beta}, \quad 0<\beta<1, \\ b_2(s,x,y,q,p) = -I_{s\neq 0}s^{-\alpha}y, \\ \sigma(s,x,y,q,p) = -q, \\ c(s,x,y,q,p,z) = -p, \\ f(s,x,y,q,p) = I_{s\neq 0}s^{-\alpha}x, \end{cases}$$

where $u(s) = I_{s\neq 0}s^{-\alpha}$, $\alpha < 1$. It is easy to check (4.1) satisfies all assumptions of Theorem 3.1, then there exists a unique adapted solution. However, it is clear that $b_1(s,x,y,q,p)$ is not Lipschitz continuous in y.

Consider now the following Hilbert space-valued feedback control systems:

$$\begin{cases} dx_t^f = b\big(t,x_t^f,y_t^f,q_t^f,p_t^f,\omega\big)\,dt + \sigma\big(t,x_t^f,y_t^f,q_t^f,p_t^f,\omega\big)\,dw(t) \\ \qquad + \int_Z c\big(t,x_{t-}^f,y_{t-}^f,q_t^f,p_t^f,z,\omega\big)\widetilde{N}_k(dt,dz), \quad x_0^f = a^f \in H, \\ dy_t^f = -f\big(t,x_t^f,y_t^f,q_t^f,p_t^f,\omega\big)\,dt + q_t^f\,dw(t) + \int_Z p_t(z)^f\,\widetilde{N}_k(dt,dz), \\ \qquad y_\tau^f = \xi \in L^2\big((\Omega,\mathcal{F}_\tau,P);H\big), \quad 0\leqslant t\leqslant \tau\leqslant T, \end{cases} \tag{4.2}$$

where f belongs to $\mathfrak{U}$ defined by

$$\mathfrak{U} = \big\{u\colon\ u(t,x,y,q,p) \text{ is jointly measurable such that (4.2) has a unique solution}\big\}.$$

Our optimal stochastic control problem is to find f in $\mathfrak{U}$ so as to maximize the following cost functional:

$$\begin{aligned} J(f) = E\Bigg[& h\big(0,y_0^f\big) + \int_0^\tau \frac{\partial h}{\partial t}\big(t,y_t^f\big)\,dt + \frac{1}{2}\int_0^\tau \operatorname{tr}\big[\big(q_t^f\big)^* D^2h\big(t,y_t^f\big)q_t^f\big]\,dt \\ & - \int_0^\tau \big\langle Dh\big(t,y_t^f\big), f^0\big(t,x_t^f,y_t^f,q_t^f,p_t^f\big)\big\rangle\,dt \\ & + \int_0^\tau\int_Z \big[h\big(t,y_{t-}^f + p_t(z)^f\big) - h\big(t,y_{t-}^f\big) - \big\langle Dh\big(t,y_{t-}^f\big), p_t(z)^f\big\rangle\big]\pi(dz)\,dt\Bigg], \end{aligned}$$

where $h(t,y):[0,T]\times H\to R$ with the following properties:

(D1) $h(\cdot,\cdot)$ has continuous $\partial h/\partial t$, and continuous first and second order Fréchet differentials given by $Dh = D_yh$ and D^2h, which are bounded in any bounded subset of $[0,T]\times H$;
(D2) there exists $f^0 \in \mathfrak{U}$ such that $\sup_{f\in\mathfrak{U}}\langle D_yh, f(t,x,y,q,p)\rangle = \langle D_yh, f^0(t,x,y,q,p)\rangle$.

Theorem 4.1. *Let* (D1) *and* (D2) *hold, then*

$$J\big(f^0\big) = \sup_{f\in\mathfrak{U}} J(f). \tag{4.3}$$

Proof. By Itô's formula in [8], for any $f \in \mathfrak{U}$, we have

$$\begin{aligned} Eh(\tau,\xi) = E\Bigg[& h\big(0, y_0^f\big) + \int_0^\tau \frac{\partial h}{\partial t}\big(t, y_t^f\big)\,dt + \frac{1}{2}\int_0^\tau \mathrm{tr}\big[\big(q_t^f\big)^* D^2 h\big(t, y_t^f\big) q_t^f\big]\,dt \\ & - \int_0^\tau \big\langle Dh\big(t, y_t^f\big), f\big(t, x_t^f, y_t^f, q_t^f, p_t^f\big)\big\rangle\,dt \\ & + \int_0^\tau \int_Z \big[h\big(t, y_{t-}^f + p_t(z)^f\big) - h\big(t, y_{t-}^f\big) - \big\langle Dh\big(t, y_{t-}^f\big), p_t(z)^f\big\rangle\big]\pi(dz)\,dt \Bigg] \\ = J(f) - E\Bigg[& \int_0^\tau \big\langle Dh\big(t, y_t^f\big), f\big(t, x_t^f, y_t^f, q_t^f, p_t^f\big)\big\rangle\,dt \\ & - \int_0^\tau \big\langle Dh\big(t, y_t^f\big), f^0\big(t, x_t^f, y_t^f, q_t^f, p_t^f\big)\big\rangle\,dt \Bigg] = J\big(f^0\big). \end{aligned}$$

Hence by (D2), the conclusion follows. □

Remark 4.1. If we suppose $\inf_{f\in\mathfrak{U}}\langle D_y h, f(t,x,y,q,p)\rangle = \langle D_y h, f^0(t,x,y,q,p)\rangle$, then $J(f^0) = \inf_{f\in\mathfrak{U}} J(f)$. Also we can discuss optimal stochastic control problems with restricted coefficients. For example, let $\mathfrak{U}_\beta = \{f^\beta \in \mathfrak{U}\colon\ f^\beta(t,x,y,q,p) = u(t)[x+\beta y],\ 0 \leqslant \beta \leqslant 1\}$, $h(t,y) = \frac{1}{2}\exp(\alpha t)|y|^2$, where α is a constant, it is easy to see that $J(f^1) = \sup_{f^\beta\in\mathfrak{U}_\beta} J(f^\beta)$. Indeed, for this case, appropriate b, σ and c can guarantee the existence and uniqueness of solutions (for example, replace $I_{s\neq 0}s^{-\alpha}$ by $u(s)$ in Example 4.1).

Acknowledgments

The authors are grateful to the referees for their helpful comments and valuable suggestions.

References

[1] R.W.R. Darling, E. Pardoux, Backward SDE with random terminal time and application to semilinear elliptic PDE, Ann. Probab. 25 (3) (1997) 1135–1159.
[2] Y. Hu, S. Peng, Maximum principle for semilinear stochastic evolution equation control systems, Stoch. Stoch. Rep. 33 (1990) 159–180.
[3] Y. Hu, S. Peng, Adapted solution of a backward semilinear stochastic evolution equation, Stoch. Anal. Appl. 9 (4) (1991) 445–459.
[4] Y. Hu, S. Peng, Solutions of forward–backward stochastic differential equations, Probab. Theory Related Fields 103 (1995) 273–283.
[5] S. Peng, Stochastic Hamilton–Jacobi–Bellman equations, SIAM J. Control Optim. 30 (2) (1992) 284–304.
[6] R. SiTu, Backward Stochastic Differential Equations with Jumps and Applications, Guangdong Science and Technology Press, Guangzhou, China, 2000.
[7] R. SiTu, H. Xu, Adapted solutions for semilinear backward stochastic evolution equations with jumps in Hilbert space (I), Acta Sci. Natur. Univ. Sunyatseni 40 (1) (2001) 1–5;
R. SiTu, H. Xu, Adapted solutions for semilinear backward stochastic evolution equations with jumps in Hilbert space (II), Acta Sci. Natur. Univ. Sunyatseni 40 (2) (2001) 1–5.

[8] R. SiTu, On solutions of backward stochastic differential equations with jumps and with non-Lipschitzian coefficients in Hilbert spaces and stochastic control, Statist. Probab. Lett. 60 (4) (2002) 279–288.
[9] J. Yin, R. SiTu, On solutions of forward–backward stochastic differential equation with Poisson jumps, Stoch. Anal. Appl. 23 (6) (2003) 1419–1448.
[10] J. Yin, R. SiTu, Existence of solutions for forward–backward stochastic differential equations with jumps and with non-Lipschitzian coefficients, J. Math. Res. Exposition 24 (4) (2004) 577–588.
[11] M. Yor, Existence et unicité de diffusions à valeus dans un espace de Hilbert, Ann. Inst. H. Poincaré 10 (1974) 55–88.