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The arc space of a toric variety

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Abstract

The Nash problem on arc families is affirmatively answered for a toric variety by Ishii and Kollár's paper which also shows the negative answer for general case. The Nash problem is one of questions about the relation between arc families and valuations. In this paper, the relation is described clearly for a toric variety. The arc space of a toric variety admits an action of the group scheme determined by the torus. Each orbit on the arc space corresponds to a lattice point in the cone and therefore corresponds to a toric valuation. The dominant relation among the orbits is described in terms of the lattice points. As a corollary, we obtain the answer to the embedded version of the Nash problem for an invariant ideal on a toric variety.

Keywords: Arc space; Toric variety; Nash problem

1. Introduction

The concept of jet schemes and arc space over an algebraic variety or an analytic space was introduced by Nash in his preprint in 1968 which was later published as [12]. These schemes are considered as something to represent the nature of the singularities of the base space. In fact, papers [5,10,11] by Mustaţă, Ein and Yasuda show that geometric properties of the jet schemes determine certain properties of the singularities of the base space. Primarily the Nash problem posed in [12] is based on this idea. The Nash problem asks if the set of arc families through the singularities corresponds bijectively to the set of the essential components of resolutions of the singularities. Here an arc family through the singularities on X is a good component of $\pi^{-1}(\mathrm{Sing}\,X)$ (see Section 3.5 or [8] for the

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definition of a good component), where π is the canonical projection from the arc space to X. The paper [8] proves that if X is a toric variety, the answer to the Nash problem is "yes," while the paper also shows the negative answer for general X.

In this paper, we study the structure of the arc space of a toric variety defined over an algebraically closed field k of arbitrary characteristic. We prove that each jet scheme or arc space admits a canonical action of the jet scheme or arc space of the torus. The arc space of a toric variety becomes an almost homogeneous space by this action, which means that the arc space is the closure of one orbit. A good component turns out to be the closure of a certain orbit and there is no non-good component in the arc space of a toric variety.

Each orbit of the arc space corresponds to a lattice point of the cone, therefore to a toric valuation, and the dominant relation of two orbits is translated to the order relation of the corresponding lattice points. As a corollary, we show the answer to the embedded version of Nash problem posed by Ein, Lazarsfeld and Mustaţă in [6] for an invariant ideal on a toric variety.

This paper is organized as follows: In Section 2 we study some basic properties on jet schemes and arc spaces. The closed points in the arc spaces of varieties are discussed here. In Section 3 we introduce a stratification on the arc space of a toric variety according to the fan. Some basic properties of the arc space of a toric variety (non-existence of non-good components, irreducibility in any characteristic) are proved here. In Section 4 we study the orbits of the arc space of a toric variety by the action of the arc space of the torus. In Section 5 we give the answer to the embedded version of Nash problem for an invariant ideal on a toric variety.

Throughout this paper the base field k is an algebraically closed field of arbitrary characteristic unless otherwise stated.

2. Basic properties of jet schemes and the arc space

Definition 2.1. Let X be a scheme of finite type over k and $K \supset k$ a field extension. For $m \in \mathbb{N}$ a morphism $\operatorname{Spec} K[t]/(t^{m+1}) \to X$ is called an *m-jet* of X and $\operatorname{Spec} K[[t]] \to X$ is called an *arc* of X. We denote the closed point of $\operatorname{Spec} K[[t]]$ by 0 and the generic point by η.

2.2. Let X be a scheme of finite type over k. Let $\mathcal{S}ch/k$ be the category of k-schemes and $\mathcal{S}et$ the category of sets. Define a contravariant functor $F_m : \mathcal{S}ch/k \to \mathcal{S}et$ by

$$F_m(Y) = \operatorname{Hom}_k\bigl(Y \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/\bigl(t^{m+1}\bigr), X\bigr).$$

Then, F_m is representable by a scheme X_m of finite type over k, that is

$$\operatorname{Hom}_k(Y, X_m) \simeq \operatorname{Hom}_k\bigl(Y \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/\bigl(t^{m+1}\bigr), X\bigr).$$

This X_m is called the *m-jet scheme* of X. A K-valued point $\alpha : \operatorname{Spec} K \to X_m$ is regarded as an m-jet $\alpha : \operatorname{Spec} K[t]/(t^{m+1}) \to X$.

Let $X_\infty = \varprojlim_m X_m$ and call it the *arc space* of X. X_∞ is a scheme which is not of finite type over k, see [4]. Denote the canonical projection $X_\infty \to X$ by π. A K-valued point $\alpha : \operatorname{Spec} K \to X_\infty$ is regarded as an arc $\alpha : \operatorname{Spec} K[\![t]\!] \to X$.

Using the representability of F_m we obtain the following universal property of X_∞.

Proposition 2.3. *Let X be a scheme of finite type over k. Then*

$$\operatorname{Hom}_k(Y, X_\infty) \simeq \operatorname{Hom}_k\bigl(Y \mathbin{\widehat{\times}}_{\operatorname{Spec} k} \operatorname{Spec} k[\![t]\!], X\bigr)$$

for an arbitrary k-scheme Y, where $Y \mathbin{\widehat{\times}}_{\operatorname{Spec} k} \operatorname{Spec} k[\![t]\!]$ means the formal completion of $Y \times_{\operatorname{Spec} k} \operatorname{Spec} k[\![t]\!]$ along the subscheme $Y \times_{\operatorname{Spec} k} \{0\}$.

2.4. A morphism $\Phi : X \to Z$ of varieties over k induces a canonical morphism $\Phi_m : X_m \to Z_m$ ($m \in \mathbb{N} \cup \{\infty\}$). Some properties of Φ are inherited by Φ_m; for example, if Φ is a closed immersion, an open immersion or étale, then Φ_m is also a closed immersion, an open immersion or étale. But many properties of Φ are not inherited by Φ_m; for example, properness, projectiveness, closedness, and so on.

Next we study the jet schemes and the arc space of a variety which admits an action of a group scheme.

Proposition 2.5. *Let G be a group scheme of finite type over k. Then G_m ($m \in \mathbb{N} \cup \{\infty\}$) is again a group scheme over k. If G is irreducible, then G_m is also irreducible.*

Proof. Let $\mu : G \times G \to G$ be the multiplication of the group, let $e \in G$ be the unit element of the group and let $\iota : G \xrightarrow{\sim} G$ be the morphism defining the inverse elements. Then, G_m becomes a group scheme with $\mu_m : G_m \times G_m \to G_m$ the multiplication of the group, where μ_m is induced on $(G \times G)_m \simeq G_m \times G_m$ from μ. The scheme $\{e\}_m$ is a k-valued point of G_m and it is the unit element under this multiplication. The morphism $\iota_m : G_m \xrightarrow{\sim} G_m$ induced from ι gives the inverse elements. If G is irreducible, then it is a non-singular irreducible variety which yields that G_m is also non-singular and irreducible. $\square$

Proposition 2.6. *Let G be a group scheme of finite type over k and X a variety admitting an action of G. Then, for $m \in \mathbb{N} \cup \{\infty\}$, X_m admits a canonical action of G_m induced from the action of G on X.*

Proof. Let $\psi : G \times X \to X$ be the morphism defining the action of G on X. Then the morphism $\psi_m : G_m \times X_m \simeq (G \times X)_m \to X_m$ induced from ψ gives an action of G_m on X_m. $\square$

Example 2.7. If G is an n-dimensional torus $T^n \simeq (\mathbb{A}^1_k \setminus \{0\})^n$, then $G_m \simeq T^n \times \mathbb{A}^{nm}_k$. Let

$$x = \bigl(x_1^{(0)}, \dots, x_n^{(0)}, x_1^{(1)}, \dots, x_n^{(1)}, \dots, x_1^{(m)}, \dots, x_n^{(m)}\bigr) \quad \text{and}$$
$$y = \bigl(y_1^{(0)}, \dots, y_n^{(0)}, y_1^{(1)}, \dots, y_n^{(1)}, \dots, y_1^{(m)}, \dots, y_n^{(m)}\bigr)$$

be two k-valued points of G_m, where $(x_1^{(0)}, \dots, x_n^{(0)}), (y_1^{(0)}, \dots, y_n^{(0)}) \in T^n$. Then the multiplication $x \cdot y$ of x and y is $(x_1^{(0)}y_1^{(0)}, \dots, x_n^{(0)}y_n^{(0)}, \sum_{i+j=1} x_1^{(i)}y_1^{(j)}, \dots, \sum_{i+j=1} x_n^{(i)}y_n^{(j)}, \dots, \sum_{i+j=m} x_1^{(i)}y_1^{(j)}, \dots, \sum_{i+j=m} x_n^{(i)}y_n^{(j)})$. The unit element of G_m is

$$(\overbrace{1, \dots, 1}^{n\ \text{times}}, 0, \dots, 0).$$

Example 2.8. Let X be a toric variety with the torus T. Then T_m acts on X_m for every $m \in \mathbb{N} \cup \{\infty\}$.

2.9. As the m-jet scheme X_m of a variety X is of finite type over k, a point of X_m is closed if and only if it is a k-valued point. But X_∞ is not of finite type and the equivalence above does not hold. First we will see the affirmative case under a condition on k.

Proposition 2.10. *Assume that the base field k is uncountable. Then, for every variety X, a point of X_∞ is closed if and only if the point is a k-valued point.*

Proof. As the problem is local, we may assume that X is affine. Therefore we have only to prove the assertion for the case $X_\infty = \operatorname{Spec} R$, $R = k[x_1, x_2, \dots, x_n, \dots]$, where the variables $x_1, x_2, \dots, x_n, \dots$ are countably infinite. For the assertion of the proposition, it is sufficient to prove that every prime ideal $I \subset k[x_1, x_2, \dots, x_n, \dots]$ is contained in a maximal ideal $(x_1 - a_1, x_2 - a_2, \dots, x_n - a_n, \dots)$, $a_1, a_2, \dots, a_n, \dots \in k$. For every n, let R_n be a subring $k[x_1, \dots, x_n]$ of R and I_n be the intersection $I \cap R_n$. For $m < n$ the inclusion $R_m \hookrightarrow R_n$ induces the projection $\operatorname{Spec} R_n \to \operatorname{Spec} R_m$ which induces a dominant map $\psi_{n,m} : Z(I_n) \to Z(I_m)$, $(a_1, \dots, a_m, \dots, a_n) \mapsto (a_1, \dots a_m)$, where $Z(I_n)$ is the set of closed points of the closed subscheme defined by I_n. Fix $r \geqslant 1$. Since $Z(I_n) \neq \emptyset$ for every $n > r$, $\operatorname{Im} \psi_{n,r}$ is a non-empty constructible set and

$$\operatorname{Im} \psi_{r+1,r} \supset \operatorname{Im} \psi_{r+2,r} \supset \cdots$$

is a non-increasing sequence. As k is uncountable, the intersection $\bigcap_{n>r} \operatorname{Im} \psi_{n,r}$ is non-empty by [1, Proposition 6.5]. Take a point p_r from this set. In $Z(I_{r+1})$,

$$\psi_{r+1,r}^{-1}(p_r) \cap \operatorname{Im} \psi_{r+2,r+1} \supset \psi_{r+1,r}^{-1}(p_r) \cap \operatorname{Im} \psi_{r+3,r+1} \supset \cdots$$

is a non-increasing sequence of non-empty constructible sets. Therefore, we can take a point $p_{r+1} \in \psi_{r+1,r}^{-1}(p_r) \cap (\bigcap_{n>r+1} \operatorname{Im} \psi_{n,r+1})$. In the same way, we have points $p_{r+2} \in Z(I_{r+2})$, $p_{r+3} \in Z(I_{r+3}), \dots$ such that $\psi_{n+1,n}(p_{n+1}) = p_n \in Z(I_n)$ for $n \geqslant r$. Therefore, there is a sequence $a_1, a_2, \dots, a_n, \dots \in k$ such that $p_n = (a_1, a_2, \dots, a_n)$. Hence, $I_n \subset (x_1 - a_1, x_2 - a_2, \dots, x_n - a_n)$ for every n. Then, it follows $I = \varinjlim I_n \subset (x_1 - a_1, x_2 - a_2, \dots, x_n - a_n, \dots)$. $\square$

In the proposition above, the condition on k is essential. In fact, we obtain the following.

Proposition 2.11 (Watanabe, Yoshida). *Let k be a countable field. Then there is a closed point which is not a k-valued point in* $\operatorname{Spec} k[x_1, x_2, \dots, x_n, \dots]$.

Proof. Let y be a transcendental element over k. As k is countable, the extension field $k(y)$ is a countably generated k-algebra. Therefore there exists a surjective homomorphism $k[x_1.x_2, \dots, x_n, \dots] \to k(y)$. The kernel of this homomorphism is a maximal ideal which does not give a k-valued point. $\square$

As we assume that the base field is an arbitrary algebraically closed field, a closed point of an arc space is not necessarily a k-valued point. In spite of such a difficulty, we can see the structure of the arc space for a toric variety.

3. Basic properties of the arc space of a toric variety

3.1. We use the notation and terminology of [7]. Let M be the free abelian group $\mathbb{Z}^n$ ($n \geqslant 1$) and N its dual $\operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. We denote $M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N \otimes_{\mathbb{Z}} \mathbb{R}$ by $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$, respectively. The canonical pairing $\langle\, ,\rangle : N \times M \to \mathbb{Z}$ extends to $\langle\, ,\rangle : N_{\mathbb{R}} \times M_{\mathbb{R}} \to \mathbb{R}$. For a linear subspace $W \subset N_{\mathbb{R}}$, the induced pairing $(N_{\mathbb{R}}/W) \times W^{\perp} \to \mathbb{R}$ is also denoted by $\langle\, ,\rangle$. Here, for $v \in N_{\mathbb{R}}$, $u \in W^{\perp}$ we have that $\langle v, u\rangle = \langle \rho(v), u\rangle$, where $\rho : N_{\mathbb{R}} \to N_{\mathbb{R}}/W$ is the projection.

For a finite fan Δ in N, the corresponding toric variety is denoted by $T_N(\Delta)$. If Δ is the fan consisting of all faces of a cone σ, then $T_N(\Delta)$ is affine and sometimes denoted by $T_N(\sigma)$.

For a cone $\tau \in \Delta$ we denote by U_τ the invariant affine open subset which contains $\operatorname{orb}\tau$ as the unique closed orbit. The open set U_τ is isomorphic to $T_N(\tau)$.

We can write $k[M]$ as $k[x^u]_{u \in M}$, where we use the shorthand $x^u = x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n}$ for $u = (u_1, \dots, u_n) \in M$. The torus $\operatorname{Spec} k[M]$ is denoted by T. We also write T for the open orbit of the toric variety.

Proposition 3.2. *Let X be a toric variety over k and $f : Y \to X$ an equivariant resolution of the singularities. Then, the induced morphism $f_\infty : Y_\infty \to X_\infty$ is surjective in a strong sense*; *i.e., for every extension field $K \supset k$ the corresponding morphism $Y_\infty(K) \to X_\infty(K)$ is surjective.*

Proof. Let $\alpha : \operatorname{Spec} K[\![t]\!] \to X$ be an arc of X, then the generic point $\eta \in \operatorname{Spec} K[\![t]\!]$ is mapped to $\operatorname{orb}\tau$ for some cone τ in the defining fan of X. As f is equivariant, $f^{-1}(\operatorname{orb}\tau)$ contains a subscheme isomorphic to $\operatorname{orb}\tau \times T^s$, where T^s is the torus of dimension $0 \leqslant s < n$. Hence the restriction $\operatorname{Spec} K(\!(t)\!) \to X$ of α can be lifted to Y. Therefore, by the properness of f, α can be lifted to Y. $\square$

The irreducibility of the arc space of a variety is known for a base field of characteristic zero [9]. In the positive characteristic case, [8, Example 2.13] gives an example of non-irreducible arc space. But for a toric variety, the characteristic is not a problem.

Corollary 3.3. *The arc space of a toric variety X is irreducible.*

Proof. This follows immediately from the irreducibility of Y_∞ and Proposition 3.2. □

Corollary 3.4. *Since the arc space X_∞ of a toric variety contains T_∞ as an open orbit, X_∞ is an almost homogeneous space by the action of T_∞.*

3.5. An irreducible component of the fiber $\pi^{-1}(\operatorname{Sing} X)$ of the singular locus $\operatorname{Sing} X \subset X$ is called a *good component* if it contains an arc α such that $\alpha(\eta)$ is in the non-singular locus [8]. If the characteristic of the base field is zero, then every component of $\pi^{-1}(\operatorname{Sing} X)$ is a good component, while there is a non-good component for a positive characteristic case [8, Example 2.13]. The following shows that the characteristic does not affect on this problem for a toric variety.

Proposition 3.6. *For a toric variety X, every component of $\pi^{-1}(\operatorname{Sing} X)$ is a good component.*

Proof. Let C be a non-good component of $\pi^{-1}(\operatorname{Sing} X)$. Let $f: Y \to X$ be an equivariant resolution of the singularities and E_i $(i = 1, 2, \dots, r)$ be the irreducible components of $f^{-1}(\operatorname{Sing} X)$. Then, $\pi_Y^{-1}(E_i)$'s are the irreducible components of $f_\infty^{-1}(\pi^{-1}(\operatorname{Sing} X))$, where $\pi_Y: Y_\infty \to Y$ is the canonical projection. By the surjectivity of f_∞ proved in Proposition 3.2, there is a component $\pi_Y^{-1}(E_i)$ mapped to C. However, $\pi_Y^{-1}(E_i)$ contains an arc whose image of the generic point corresponds to a point in the non-singular locus on X, which is a contradiction. □

Now we are going to make a stratification of the arc space of a toric variety according to the fan. From now on we assume that a toric variety X is defined by a fan Δ. Let $X(\tau) \subset X$ be the closure $\overline{\operatorname{orb}\tau}$ for the cone $\tau \in \Delta$. Then $X(\tau)$ is again a toric variety.

Definition 3.7. Let X be a toric variety corresponding to a fan Δ. We define $X_\infty(\tau)$ as follows:

$$X_\infty(\tau) = \bigl\{\alpha \in X_\infty \bigm| \alpha: \operatorname{Spec} K[\![t]\!] \to X \text{ factors through } X(\tau) \\ \text{but does not factor through } X(\gamma) \text{ for } \gamma \not< \tau\bigr\}.$$

Remark 3.8.

(i) By definition, we have:

$$X_\infty(\tau) = \bigl\{\alpha \in X_\infty \mid \alpha(\eta) \in \operatorname{orb}\tau\bigr\}.$$

In particular,

$$X_\infty(0) = \bigl\{\alpha \in X_\infty \mid \alpha(\eta) \in T\bigr\}.$$

(ii) $X_\infty(\tau) = X(\tau)_\infty(0)$, where 0 is the cone consisting of the origin.
(iii) X_∞ is the disjoint union:

$$X_\infty = \bigsqcup_{\tau \in \Delta} X_\infty(\tau).$$

Proposition 3.9. *Let X be a toric variety defined by a fan Δ, T the torus acting on X and τ a cone in Δ. Then, the subset $X_\infty(\tau)$ is a locally closed subset which is invariant under the action of T_∞.*

Proof. As $X(\gamma)$ is closed in X for every cone $\gamma \in \Delta$, $X(\gamma)_\infty$ is considered as a closed subscheme of X_∞. By definition

$$X_\infty(\tau) = X(\tau)_\infty \Big\backslash \Big(\bigcup_{\gamma \not< \tau} X(\gamma)_\infty \Big) \tag{3.9.1}$$

as subsets in X_∞, which shows that $X_\infty(\tau)$ is locally closed.

As $X(\gamma)$ is invariant under the action of T for every $\gamma \in \Delta$, $X(\gamma)_\infty$ is invariant under the action of T_∞. The description of $X_\infty(\tau)$ as above gives the assertion of the invariance. $\square$

Proposition 3.10. *Let X be a toric variety defined by a fan Δ and τ, γ be cones in Δ. Then, $\gamma < \tau$ if and only if $\overline{X_\infty(\gamma)} \supset X_\infty(\tau)$.*

Proof. First note that $X(\gamma)_\infty$ and $X(\tau)_\infty$ are irreducible (Corollary 3.3) and closed in X_∞. Then, the description (3.9.1) gives that $\overline{X_\infty(\gamma)} = X(\gamma)_\infty$ and $\overline{X_\infty(\tau)} = X(\tau)_\infty$. Therefore, the relation $\overline{X_\infty(\gamma)} \supset X_\infty(\tau)$ holds if and only if $X(\gamma)_\infty \supset X(\tau)_\infty$ holds, which is equivalent to $X(\gamma) \supset X(\tau)$. It is well known that the last relation is equivalent to $\gamma < \tau$. $\square$

4. Orbits on the arc space of a toric variety

In this section we associate each T_∞-orbit on X_∞ to a lattice point, and describe the dominant relation of two orbits in terms of the corresponding lattice points.

Theorem 4.1. *Let X be a toric variety defined by a fan Δ. Then,*

(i) *there is a surjective canonical map*

$$\Psi : X_\infty(0) \to |\Delta| \cap N, \quad \alpha \mapsto v_\alpha,$$

(ii) *for every $v \in |\Delta| \cap N$ there exists a k-valued point $\alpha \in X_\infty(0)$ such that*

$$\Psi^{-1}(v) = T_\infty \cdot \alpha,$$

where $T_\infty \cdot \alpha$ is the orbit of α by the action of T_∞, and

(iii) *for* $v \in |\Delta| \cap N$, $\Psi^{-1}(v)$ *is a locally closed subset of* X_∞.

Proof. For a K-valued point $\alpha \in X_\infty(0)$, take a cone $\sigma \in \Delta$ such that $\alpha(0) \in U_\sigma$. Then α is an arc of U_σ with $\alpha(\eta) \in T$, therefore we have a commutative diagram:

$$\begin{tikzcd} k[\sigma^\vee \cap M] \arrow[r, "\alpha^*"] \arrow[d, hook] & K[[t]] \arrow[d, hook] \\ k[M] \arrow[r, "\alpha^*"] & K((t)) \end{tikzcd}$$

Let $v_\alpha : M \to \mathbb{Z}$ be a map defined by $u \mapsto \operatorname{ord} \alpha^*(x^u)$. Then v_α is a group homomorphism, therefore $v_\alpha \in N$ with the pairing $\langle v_\alpha, u\rangle = \operatorname{ord} \alpha^*(x^u)$. For $u \in \sigma^\vee \cap M$, it follows $\langle v_\alpha, u\rangle = \operatorname{ord} \alpha^*(x^u) \geqslant 0$, which implies that $v_\alpha \in \sigma$. Now we obtain a map $\Psi : X_\infty(0) \to |\Delta| \cap N$, $\alpha \mapsto v_\alpha$. To show the surjectivity, take a point $v \in |\Delta| \cap N$. Let σ be a cone containing v. Let $\alpha^* : k[M] \to k((t))$ be a k-algebra homomorphism defined by $\alpha^*(x^u) = t^{\langle v,u\rangle}$ for $u \in M$. Then, $\alpha^*(k[\sigma^\vee \cap M]) \subset k[[t]]$, since $\langle v, u\rangle \geqslant 0$ for $u \in \sigma^\vee$. Hence, α^* gives a k-valued point α in $X_\infty(0)$.

For (ii), we prove the equality $\Psi^{-1}(v) = T_\infty \cdot \alpha$ for a k-valued point $\alpha \in \Psi^{-1}(v)$. For a k-valued point $\alpha \in X_\infty(0)$, take a cone σ such that $\alpha \in (U_\sigma)_\infty$. Then α corresponds to a ring homomorphism $\alpha^* : k[\sigma^\vee \cap M] \to k[[t]]$. On the other hand, a K-valued point $\gamma \in T_\infty$ corresponds to a ring homomorphism $\gamma^* : k[M] \to K[[t]]$. This homomorphism is equivalent to a ring homomorphism $\gamma^* : k[\sigma^\vee \cap M] \to K[[t]]$ such that the order of $\gamma^*(x^u)$ is zero for every $u \in \sigma^\vee \cap M$, because $\sigma^\vee \cap M$ generates M. Then, $\gamma \cdot \alpha$ corresponds to the homomorphism $k[\sigma^\vee \cap M] \to K[[t]]$ which maps x^u to $\gamma^*(x^u)\alpha^*(x^u)$.

Now let $\alpha \in (U_\sigma)_\infty$ be the arc corresponding to v which was constructed in (i). If $\beta \in T_\infty \cdot \alpha$, then there exists a K-valued point $\gamma \in T_\infty$ such that $\beta = \gamma \cdot \alpha$. Then, by the above remark, it follows that $\beta \in (U_\sigma)_\infty$ and β corresponds to $\beta^* : k[\sigma^\vee \cap M] \to K[[t]]$ which maps x^u to $\gamma^*(x^u)t^{\langle v,u\rangle}$ whose order is $\langle v, u\rangle$. Therefore $\beta \in \Psi^{-1}(v)$. Conversely, suppose that $\beta \in \Psi^{-1}(v)$ and let σ be a cone such that $\beta \in (U_\sigma)_\infty$. Then we can define $\gamma \in T_\infty$ by $\gamma^* : k[\sigma^\vee \cap M] \to K[[t]]$, $\gamma^*(x^u) = t^{-\langle v,u\rangle}\beta^*(x^u)$. For this γ we have that $\gamma \cdot \alpha = \beta$.

For the assertion (iii), take a cone $\sigma \in \Delta$ such that $T_\infty \cdot \alpha \subset (U_\sigma)_\infty$. It is sufficient to prove that $T_\infty \cdot \alpha$ is locally closed in $(U_\sigma)_\infty \cap X_\infty(0)$. Denote $(U_\sigma)_\infty$ by $\operatorname{Spec} A$. Let $\Lambda : k[\sigma^\vee \cap M] \to A[[t]]$ be the ring homomorphism induced from the universal family of arcs on $(U_\sigma)_\infty$ (see Proposition 2.3). Let $\Lambda(x^{u_j}) = \sum_{i \geqslant 0} a_{j,i} t^i$ for generators u_j ($j = 1, \dots, r$) of the semigroup $\sigma^\vee \cap M$. Then

$$\begin{aligned} T_\infty \cdot \alpha &= \Psi^{-1}(v) \\ &= \bigl\{\beta \in (U_\sigma)_\infty \cap X_\infty(0) \bigm| a_{j,i}(\beta) = 0 \text{ for } i < \langle v, u_j\rangle, \\ &\qquad a_{j,i}(\beta) \neq 0 \text{ for } i = \langle v, u_j\rangle,\ j = 1, \dots, r\bigr\}. \end{aligned}$$

Hence, $T_\infty \cdot \alpha$ is locally closed in $(U_\sigma)_\infty \cap X_\infty(0)$. $\square$

4.2. For $\tau \in \Delta$, $X(\tau)$ is a toric variety $T_{N_\tau}(\Delta_\tau)$, where Δ_τ consists of the cones $\bar{\sigma} \subset N_{\mathbb{R}}/\tau\mathbb{R}$ which are the images of the cones $\sigma \in \Delta$ such that $\tau < \sigma$ and N_τ is the image of N in $N_{\mathbb{R}}/\tau\mathbb{R}$. The affine open subset $U_{\bar{\sigma}} \subset X(\tau)$ is $\operatorname{Spec} k[\tau^\perp \cap \sigma^\vee \cap M]$.

Since $X_\infty(\tau) = X(\tau)_\infty(0)$ as is seen in Remark 3.8, we obtain the following from Theorem 4.1.

Corollary 4.3. *Let X be a toric variety defined by a fan Δ and $\tau \in \Delta$. Then,*

(i) *there is a surjective canonical map*

$$\Psi : X_\infty(\tau) \to |\Delta_\tau| \cap N_\tau, \quad \alpha \mapsto v_\alpha;$$

(ii) *for every $v \in |\Delta_\tau| \cap N_\tau$ there exists a k-valued point $\alpha \in X_\infty(\tau)$ such that*

$$\Psi^{-1}(v) = T_\infty \cdot \alpha,$$

where $T_\infty \cdot \alpha$ is the orbit of α by the action of T_∞; and

(iii) *for $v \in |\Delta_\tau| \cap N_\tau$, $\Psi^{-1}(v)$ is a locally closed subset of X_∞.*

Corollary 4.4.

(i)
$$X_\infty = \bigcup_{\alpha:\ k\text{-valued point of } X_\infty} T_\infty \cdot \alpha.$$

(ii) *For every cone τ, there is a bijection*:

$$\{T_\infty \cdot \alpha \mid \alpha \text{ is a } k\text{-valued point} \in X_\infty(\tau)\} \simeq |\Delta_\tau| \cap N_\tau.$$

Definition 4.5. As an orbit of a k-valued point α in $X_\infty(\tau)$ is determined by the lattice point $v = v_\alpha \in |\Delta_\tau|$, we sometimes denote the orbit $T_\infty \cdot \alpha$ by $T_\infty(v)$.

Definition 4.6. Let σ be a cone in N and v, v' two points in σ. We denote $v \leqslant_\sigma v'$ if $v' \in v + \sigma$. It is clear that $\leqslant_\sigma$ is an order in σ.

Now we are going to study the dominant relation between orbits.

Proposition 4.7. *Let X be a toric variety defined by a fan Δ. Let $\alpha \in X_\infty(\tau)$ and $\beta \in X_\infty(\gamma)$ be k-valued points for $\tau, \gamma \in \Delta$. If $\overline{T_\infty \cdot \alpha} \supset T_\infty \cdot \beta$, then $\tau < \gamma$ and there exists a cone $\sigma \in \Delta$ containing τ and γ such that $\alpha, \beta \in (U_\sigma)_\infty$.*

Proof. By the condition of the proposition, it follows that $\beta \in \overline{X_\infty(\tau)} = X(\tau)_\infty$. As $\beta(\eta) \in orb\gamma$, we have $orb\gamma \subset X(\tau)$, which implies $\tau < \gamma$. To see the second assertion, take a cone $\sigma \in \Delta$ such that $\beta \in (U_\sigma)_\infty$. Then $\beta(\eta) \in orb(\gamma)$ implies $\gamma < \sigma$. Since $(U_\sigma)_\infty$ is an open subset of X_∞ containing β, there is an arc $\alpha' \in T_\infty \cdot \alpha \cap (U_\sigma)_\infty$. As $(U_\sigma)_\infty$ is T_∞-invariant, it contains both $T_\infty \cdot \alpha$ and $T_\infty \cdot \beta$. $\square$

Hence, in order to interpret the condition of the domination $\overline{T_\infty \cdot \alpha} \supset T_\infty \cdot \beta$ in terms of the corresponding lattice points, we may assume that X is an affine toric variety. If X is an affine toric variety defined by a cone σ and $T_\infty(v) \subset X_\infty(\tau)$ for a face $\tau < \sigma$, then $v \in \bar{\sigma} \cap N_\tau$ by Corollary 4.3, where $\bar{\sigma}$ is the image of $\sigma \subset N_{\mathbb{R}}$ by the projection $N_{\mathbb{R}} \to N_{\mathbb{R}}/\tau\mathbb{R}$.

Proposition 4.8. *Let X be an affine toric variety defined by a cone σ in N. Then, two orbits $T_\infty(v)$ and $T_\infty(v')$ in $X_\infty(0)$ satisfy $\overline{T_\infty(v)} \supset T_\infty(v')$ if and only if $v \leqslant_\sigma v'$.*

Proof. Assume $\overline{T_\infty(v)} \supset T_\infty(v')$. If $\langle v, u\rangle > \langle v', u\rangle$ for some $u \in \sigma^\vee \cap M$, then

$$T_\infty(v) \subset \overline{T_\infty(v)} \cap \bigl\{\alpha \in X_\infty(0) \bigm| \operatorname{ord} \alpha^*(x^u) \geqslant \langle v', u\rangle + 1\bigr\},$$

where the right-hand side is a proper closed subset of $\overline{T_\infty(v)}$. This is a contradiction. Hence, $v \leqslant_\sigma v'$.

Next, assume that $v \leqslant_\sigma v'$ for $v, v' \in \sigma \cap N$. To prove the converse, we divide the proof into two steps.

Step 1. The case X is non-singular.

Let $e_1, e_2, \ldots, e_n$ be the basis of M such that $e_1, \ldots, e_r, e_{r+1}^{\pm 1}, \ldots, e_n^{\pm 1}$ generate $\sigma^\vee$. Define a k-algebra homomorphism $\Phi^* : k[\sigma^\vee \cap M] \to k[\![\lambda, t]\!]$ by

$$\Phi^*(x^{e_i}) = t^{\langle v', e_i\rangle} + \lambda t^{\langle v, e_i\rangle}.$$

Here, note that $\Phi^*(x^{e_i}) = 1 + \lambda$ for $i \geqslant r + 1$, since $\langle v, e_i\rangle = \langle v', e_i\rangle = 0$ for these i's. Then, we obtain a morphism $\Phi : \operatorname{Spec} k[\![\lambda]\!] \to X_\infty(0)$ such that $\Phi(0') \in T_\infty(v')$ and $\Phi(\eta') \in T_\infty(v)$, where $0'$ is the closed point and η' is the generic point of $\operatorname{Spec} k[\![\lambda]\!]$. This implies that $\overline{T_\infty(v)}$ contains a point of $T_\infty(v')$. As $\overline{T_\infty(v)}$ is T_∞-invariant, it follows that $\overline{T_\infty(v)} \supset T_\infty(v')$.

Step 2. The general case.

Define σ' as the cone generated by v and $v' - v$. Then, note that $\sigma' \subset \sigma$ and

$$v \leqslant_{\sigma'} v'.$$

Let N' be the subgroup of N generated by v, $v' - v$ and $v_1, v_2, \ldots, v_s \in N$, where their images $\overline{v_1}, \overline{v_2}, \ldots, \overline{v_s} \in N/N \cap \sigma'\mathbb{R}$ are a basis of $N/N \cap \sigma'\mathbb{R}$. Then, the toric variety $Z = T_{N'}(\sigma')$ is non-singular and there is a canonical equivariant morphism

$$\varphi : Z \to X$$

with the surjective morphism $T' \to T$ of the tori. By Step 1, $\overline{T'_\infty(v)} \supset T'_\infty(v')$ follows from $v \leqslant_{\sigma'} v'$. Take k-valued points $\alpha, \beta \in Z_\infty(0)$ such that $v_\alpha = v$, $v_\beta = v'$, then $T_\infty \cdot \varphi_\infty(\alpha) = \varphi_\infty(T'_\infty \cdot \alpha)$ and $T_\infty \cdot \varphi_\infty(\beta) = \varphi_\infty(T'_\infty \cdot \beta)$. Therefore $\overline{T_\infty(v)} \supset T_\infty(v')$ follows from $v_{\varphi_\infty(\alpha)} = v$, $v_{\varphi_\infty(\beta)} = v'$. $\square$

As $X_\infty(\tau) = X(\tau)_\infty(0)$ we obtain the following as a corollary of Proposition 4.8.

Corollary 4.9. *Let X be an affine toric variety defined by a cone σ in N. Then, for a face $\tau < \sigma$, two orbits $T_\infty(v)$ and $T_\infty(v')$ in $X_\infty(\tau)$ satisfy $\overline{T_\infty(v)} \supset T_\infty(v')$ if and only if $v \leqslant_{\bar\sigma} v'$, where $\bar\sigma \subset N_\mathbb{R}/\tau\mathbb{R}$ is the image of σ.*

Next we will see the relation of the orbits in mutually different strata. To see this we need the following combinatorial lemma:

Lemma 4.10. *Let σ be an n-dimensional cone in N, where $n = \dim N_\mathbb{R}$, and τ an r-dimensional face of σ. Then, there exist a non-singular n-dimensional cone σ_0 in N and its r-dimensional face τ_0 such that $\sigma_0 \subset \sigma$ and $\tau_0 \subset \tau$.*

Proof. First, subdivide τ into non-singular cones and take one of r-dimensional cones as τ_0. Take any n-dimensional cone σ' in N with the face τ_0 inside of σ, and then subdivide σ' into a non-singular fan Σ by Danilov's procedure [3, §8]. As τ_0 is non-singular, it is still in the new fan Σ as a cone. Hence, we can take an n-dimensional non-singular cone σ_0 with the face τ_0 in Σ. □

Proposition 4.11. *Let X be an affine toric variety defined by a cone σ in N. Then, two orbits $T_\infty(v) \subset X_\infty(\tau)$, $T_\infty(v') \subset X_\infty(\gamma)$ satisfy the relation $\overline{T_\infty(v)} \supset T_\infty(v')$ if and only if $\tau < \gamma$ and $\rho(v) \leqslant_{\bar\sigma} v'$, where $\rho : N_\mathbb{R}/\tau\mathbb{R} \to N_\mathbb{R}/\gamma\mathbb{R}$ is the canonical projection and $\bar\sigma$ is the image of σ in $N_\mathbb{R}/\gamma\mathbb{R}$.*

Proof. First assume that $\overline{T_\infty(v)} \supset T_\infty(v')$. Then, we have $\tau < \gamma$ by Proposition 4.7. By the assumption, there is a morphism $\Phi : \operatorname{Spec} k[\![\lambda]\!] \to X_\infty(0)$ such that $\beta := \Phi(0') \in T_\infty(v')$ and $\alpha := \Phi(\eta') \in T_\infty(v)$, where $0'$ is the closed point and η' is the generic point of $\operatorname{Spec} k[\![\lambda]\!]$. As $\alpha \in X_\infty(\tau)$, Φ factors through $X(\tau)_\infty$. This gives the k-algebra homomorphism:

$$\Phi^* : k\bigl[\tau^\perp \cap \sigma^\vee \cap M\bigr] \to k[\![\lambda, t]\!].$$

By using Φ^*, we obtain $\operatorname{ord} \alpha^*(x^u) \leqslant \operatorname{ord} \beta^*(x^u)$ for $u \in \tau^\perp \cap \sigma^\vee \cap M$ in the same way as in the proof of Proposition 4.8. Therefore, for $u \in \gamma^\perp \cap \sigma^\vee \cap M \subset \tau^\perp \cap \sigma^\vee \cap M$ the inequality $\langle v, u\rangle = \langle \rho(v), u\rangle \leqslant \langle v', u\rangle$ holds. Hence, $\rho(v) \leqslant_{\bar\sigma} v'$.

To prove the converse, assume $\rho(v) \leqslant_{\bar\sigma} v'$. Then, it is sufficient to prove that $\overline{T_\infty(v)} \supset T_\infty(\rho(v))$, because $\overline{T_\infty(\rho(v))} \supset T_\infty(v')$ follows from Corollary 4.9. To prove $\overline{T_\infty(v)} \supset T_\infty(\rho(v))$, we may assume that $\gamma = \sigma$, since $X_\infty(\gamma) = X(\gamma)_\infty(0)$. We also can assume that $\dim \sigma = n = \dim N_\mathbb{R}$, because if $\dim \sigma = s < n$, then $T_\infty(v) = T_\infty^{n-s} \times T_\infty^s(v)$, $T_\infty(\rho(v)) = T_\infty^{n-s} \times T_\infty^s(\rho'(v))$, where $\rho' : \sigma\mathbb{R} \to \sigma\mathbb{R}/\tau\mathbb{R}$ is the projection and T^s, T^{n-s} are s and $(n-s)$-dimensional tori, respectively. So the problem is reduced to proving that $\overline{T_\infty^s(v)} \supset T_\infty^s(\rho'(v))$.

Now, for σ and τ, let σ_0 and τ_0 be as in Lemma 4.10. Let $e_1, e_2, \ldots, e_n$ be a basis of M which generate $\sigma_0^\vee$ and $e_1, e_2, \ldots, e_r$ $(r < n)$ generate $\tau_0^\perp \cap \sigma_0^\vee$. Let

$$\Lambda^* : k\bigl[\sigma_0^\vee \cap M\bigr] \to k[\![\lambda]\!](\!(t)\!)$$

be a k-algebra homomorphism defined by

$$\Lambda^*(x^{e_i}) = (\lambda + 1)t^{\langle v, e_i\rangle} \quad \text{for } i = 1, \dots, r,$$

$$\Lambda^*(x^{e_i}) = \lambda t^{\langle v, e_i\rangle} \quad \text{for } i = r+1, \dots, n.$$

It is easy to check that $\Lambda^*(x^u) \in k[[\lambda, t]]$ for every $u \in \sigma^\vee \cap M$, since $v \in \sigma$. Then, we obtain a morphism $\Lambda : \operatorname{Spec}[[\lambda]] \to X_\infty$. For every $u \in \sigma^\vee \cap M$, we have $\operatorname{ord} \Lambda^*(x^u) = \langle v, u\rangle$, therefore $\alpha := \Lambda(\eta') \in T_\infty(v) \subset X_\infty(0)$, where $0'$ is the closed point and η' is the generic point of $\operatorname{Spec} k[[\lambda]]$. Since $\tau^\perp = \tau_0^\perp$, $\beta := \Lambda(0') : \operatorname{Spec} k[[t]] \to X$ factors through $X(\tau)$ by the definition of Λ^*. As the corresponding ring homomorphism β^* is extended to a ring homomorphism $k[\tau^\perp \cap M] \to k((t))$, it follows that $\beta(\eta) \in orb\tau$, which implies $\beta \in X_\infty(\tau)$. For every $u \in \tau^\perp \cap \sigma^\vee \cap M$, we have $\operatorname{ord} \beta^*(x^u) = \langle v, u\rangle = \langle \rho(v), u\rangle$. Therefore $\beta \in T_\infty(\rho(v))$. Hence, it follows that $\overline{T_\infty(v)}$ contains a point of $T_\infty(\rho(v))$. By the T_∞-invariance of $\overline{T_\infty(v)}$, we obtain $\overline{T_\infty(v)} \supset T_\infty(\rho(v))$. □

Summing up Propositions 4.7, 4.11, and Corollary 4.9, we obtain the following.

Theorem 4.12. *Let X be a toric variety and $T_\infty(v)$ and $T_\infty(v')$ two orbits in $X_\infty(\tau)$ and $X_\infty(\gamma)$, respectively. Then the following are equivalent*:

(i) $\overline{T_\infty(v)} \supset T_\infty(v')$,
(ii) $\tau < \gamma$, *there exists a cone $\sigma > \gamma$ such that $T_\infty(v), T_\infty(v') \subset (U_\sigma)_\infty$, and $\rho(v) \leqslant_{\overline{\sigma}} v'$, where $\rho : N_\mathbb{R}/\tau\mathbb{R} \to N_\mathbb{R}/\gamma\mathbb{R}$ is the projection and $\overline{\sigma}$ is the image of σ in $N_\mathbb{R}/\gamma\mathbb{R}$.*

4.13. By now, the dominant relation of orbits is discussed in terms of the order relation of lattice points. This gives a relation between arc families and valuations, which will be discussed in the next section. But the dominant relation of orbits can be more simply described in terms of homomorphisms of semigroups.

If X is an affine toric variety defined by a cone σ and $T_\infty(v) \subset X_\infty(\tau)$ for a face $\tau < \sigma$, then $v \in \overline{\sigma} \cap N_\tau \subset N_\mathbb{R}/\tau\mathbb{R}$, where $\overline{\sigma}$ is the image of σ in $N_\mathbb{R}/\tau\mathbb{R}$. Then, v can be considered as a semigroup homomorphism $v : \tau^\perp \cap \sigma^\vee \cap M \to \mathbb{Z}_{\geqslant 0}$. Here, v can be extended as a semigroup homomorphism $v : \sigma^\vee \cap M \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$, where we define $v(u) = \infty$ for every $u \notin \tau^\perp$.

Conversely, every semigroup homomorphism $v : \sigma^\vee \cap M \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ is obtained by such an extension from an element of $\overline{\sigma} \cap N_\tau \subset N_\mathbb{R}/\tau\mathbb{R}$ for some face τ.

Lemma 4.14. *Let σ be a cone in N and $v : \sigma^\vee \cap M \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ a homomorphism of semigroups. Then, there exists a face $\tau < \sigma$ such that $v^{-1}(\mathbb{Z}_{\geqslant 0}) = \tau^\perp \cap \sigma^\vee \cap M$.*

Proof. Take the minimal face γ of σ containing $C = v^{-1}(\mathbb{Z}_{\geqslant 0})$. Then, C contains a relative interior point u of γ. We will show that $C = \gamma \cap M$. Assume that there exists a point $u_0 \in \gamma \cap M$ such that $v(u_0) = \infty$. Then, note that $u_0 + \sigma^\vee \subset v^{-1}(\infty)$. Let $\sigma^\vee$ be generated by $u_1, u_2, \dots, u_r$. Then, there is a representation $u = \sum_{i=1}^{r} a_i u_i$ with $a_i > 0$ for every i and $u_0 = \sum_{i=1}^{r} b_i u_i$ with $b_i \geqslant 0$ for every i. Then, in the equality:

$$mu = \sum b_i u_i + \sum_i (ma_i - b_i) u_i,$$

the second term of the right-hand side is in $\sigma^\vee$ for $m \gg 0$. Hence, $v(mu) = \infty$, but this contradicts to that $v(mu) = mv(u) \in \mathbb{Z}_{\geqslant 0}$. Now, we obtain that $C = \gamma \cap M$ and γ can be written as $\tau^\perp \cap \sigma^\vee$ for some $\tau < \sigma$. $\square$

By Corollary 4.3 and Theorem 4.12, we obtain the following interpretation.

Theorem 4.15. *Let X be a toric variety defined by a fan Δ, then we obtain the following*:

(i) *There is a bijective map*:

$$\{T_\infty \cdot \alpha \mid \alpha : k\text{-valued point of } X\} \xrightarrow{\sim} \bigsqcup_\sigma \mathrm{Hom}_{s.g.}\big(\sigma^\vee \cap M, \mathbb{Z}_{\geqslant 0} \cup \{\infty\}\big),$$

where σ varies the maximal cones in Δ. Via this map, each $T_\infty \cdot \alpha$ can be written as $T_\infty(v)$ for a suitable element v of the right-hand side.

(ii) *We have the relation $\overline{T_\infty(v)} \supset T_\infty(v')$ if and only if there is a maximal cone σ in Δ such that $v, v' \in \mathrm{Hom}_{s.g}(\sigma^\vee \cap M, \mathbb{Z}_{\geqslant 0} \cup \{\infty\})$ and $v \leqslant v'$, where $v \leqslant v'$ means that $v(u) \leqslant v'(u)$ for every $u \in \sigma^\vee \cap M$.*

5. Contact loci of an invariant ideal

In this section, we will give the answer to the embedded version of Nash problem for an invariant ideal of a toric variety.

Definition 5.1. Let X be a variety over an algebraically closed field k and $k(X)$ the rational function field of X. A *divisorial valuation* of $k(X)$ is a positive integer times discrete valuation val_D associated to a prime divisor D on some normal variety X' which is birational to X. Note that this definition is wider than the definition of "divisorial valuation" in [6].

Definition 5.2. Let X be an affine toric variety defined by a cone σ in N. For every point $v \in \sigma \cap N$ we can associate a valuation val_v on $k(X)$ as follows:

Define

$$val_v(f) := \min_{x^u \in f} \langle v, u \rangle, \quad \text{for } f \in k\big[\sigma^\vee \cap M\big]$$

and extend it on $k(X)$, the quotient field of $k[\sigma^\vee \cap M]$. This valuation is called a *toric valuation*. Here $x^u \in f$ means that the coefficient of the monomial x^u in f is not zero.

Note that the toric valuation defined by a primitive element v is val_{D_v}, where D_v is the irreducible invariant divisor $\overline{orb(\mathbb{R}_{\geqslant 0}v)}$ on some toric variety X' which is birational to X. Since every toric valuation is a positive integer times such a valuation, every toric valuation is a divisorial valuation.

5.3. For a variety X over an algebraically closed field k, let $\psi_m : X_\infty \to X_m$ ($m \in \mathbb{Z}_{\geqslant 0}$) be the truncation morphism. Note that $\psi_0 = \pi$. Recall that a cylinder C in X_∞ is a subset of the form $\psi_m^{-1}(S)$, for some m and some constructible subset $S \subset X_m$.

Example 5.4. Let X be a toric variety. Then an orbit $T_\infty(v)$ of a k-valued point in $X_\infty(0)$ is a cylinder. Indeed, we may assume that X is the affine toric variety defined by a cone σ. The orbit is the subset of X_∞ consisting of arcs α whose corresponding homomorphisms $\alpha^* : k[\sigma^\vee \cap M] \to K[\![t]\!]$ satisfy $\operatorname{ord}\alpha^*(x^{u_i}) = \langle v, u_i\rangle$ for generators $u_1, \ldots, u_s$ of $\sigma^\vee \cap M$. Let $m \geqslant \max_{i=1,\ldots,s}\langle v, u_i\rangle$ and $S_m \subset X_m$ the subset consisting of m-jets γ whose corresponding homomorphisms $\gamma^* : k[\sigma^\vee \cap M] \to \operatorname{Spec} K[t]/(t^{m+1})$ satisfy $\operatorname{ord}\gamma^*(x^{u_i}) = \langle v, u_i\rangle$. Then, S_m is a locally closed subset of X_m and $T_\infty(v) = \psi_m^{-1}(S_m)$.

5.5. Let X be a non-singular variety over $\mathbb{C}$ and C an irreducible cylinder in X_∞. In [6] a valuation val_C corresponding to C is defined as follows: Note first that if $\alpha \in X_\infty$ is a $\mathbb{C}$-valued point, and if f is a rational function on X defined in a neighborhood of $\pi(\alpha)$, then $\operatorname{ord}\alpha^*(f)$ is well defined, where $\alpha^* : \mathcal{O}_X \to \mathbb{C}[\![t]\!]$ is the ring homomorphism corresponding to α. If the domain of f intersects $\pi(C)$, then $val_C(f) := \operatorname{ord}\alpha^*(f)$, for general $\alpha \in C$. Then $val_C(f)$ is well defined and can be extended to a valuation of the function field of X.

Proposition 5.6 [6]. *Let X be a non-singular variety over $\mathbb{C}$ and C an irreducible cylinder in X_∞ which does not dominate X. Then val_C is equal with a divisorial valuation.*

In the proof of Proposition 5.6, the condition that X is non-singular is used. Therefore, this proposition does not imply that for a cylinder $C = T_\infty(v) \subset X_\infty(0)$ on a singular toric variety X, the corresponding valuation val_C is a divisorial valuation. However, the following proposition shows that val_C is a divisorial valuation for $C = T_\infty(v)$.

Proposition 5.7. *Let X be a toric variety over an algebraically closed field k and $C = T_\infty(v) \subset X_\infty(0)$; then $val_C = val_v$. In particular, val_C is a divisorial valuation.*

Proof. We may assume that X is an affine toric variety defined by a cone σ. It is sufficient to prove that $val_C(f) = val_v(f)$ for every element $f \in k[\sigma^\vee \cap M]$. Note that $val_C(f) = \operatorname{ord}\alpha^*(f)$ for the generic point $\alpha \in C$. If f is a monomial x^u ($u \in \sigma^\vee \cap M$), then by the definition of $C = T_\infty(v)$ we have

$$val_C(x^u) = \operatorname{ord}\alpha^*(x^u) = \langle v, u\rangle = val_v(x^u).$$

For general f, we have

$$val_C(f) \geqslant \min_{x^u \in f} val_C(x^u) = \min_{x^u \in f}\langle v, u\rangle = val_v(f).$$

On the other hand, let R_v is the discrete valuation ring of the divisorial valuation val_v. Then there is an indeterminate t such that the composite

$$\beta^* : k[\sigma^\vee \cap M] \hookrightarrow R_v \hookrightarrow \widehat{R}_v \simeq K[\![t^e]\!] \hookrightarrow K[\![t]\!]$$

satisfies $\operatorname{ord} \beta^*(f) = val_v(f)$ for $f \in k[\sigma^\vee \cap M]$. Here, K is the residue field of R_v by the maximal ideal and e is the positive integer such that $v = ev_0$ for a primitive element v_0. As the arc $\beta : \operatorname{Spec} K[\![t]\!] \to X$ corresponding to β^* is a K-valued point of C, we obtain the following inequality by the upper semicontinuity

$$val_C(f) = \operatorname{ord} \alpha^*(f) \leqslant \operatorname{ord} \beta^*(f) = val_v(f).$$

Therefore, we obtain $val_C(f) = val_v(f)$. $\square$

Now we recall the definition of the contact locus of an ideal of a variety X. Let X be an affine variety over an algebraically closed field k with the coordinate ring A and $\mathfrak{a}$ an ideal of A. Then, we define the pth contact locus of $\mathfrak{a}$ by

$$\operatorname{Cont}^p(\mathfrak{a}) = \Big\{\alpha \in X_\infty \;\Big|\; \min_{f \in \mathfrak{a}} \operatorname{ord} \alpha^*(f) = p\Big\}.$$

It is clear that this is a cylinder. If X is non-singular then the irreducible components are also cylinders. Therefore each irreducible component of the contact locus corresponds to a divisorial valuation. Now, we can state the embedded version of Nash problem posed in [6].

Problem 5.8. *Which valuations correspond to the irreducible components of* $\operatorname{Cont}^p(\mathfrak{a})$?

We consider this problem for an invariant ideal $\mathfrak{a}$ on a toric variety X. We should note that for a singular variety X, an irreducible component of a cylinder is not a cylinder in general, therefore an irreducible component does not necessarily correspond to a divisorial valuation. But in our toric case, an irreducible component of the contact locus corresponds to a divisorial valuation.

Lemma 5.9. *Let X be an affine toric variety and $\mathfrak{a}$ an invariant ideal on X. Then, for every integer $p > 0$, an orbit $T_\infty(v)$ is either contained in* $\operatorname{Cont}^p(\mathfrak{a})$ *or disjoint from* $\operatorname{Cont}^p(\mathfrak{a})$.

Proof. Take an arc $\alpha \in T_\infty(v)$. Then α belongs to $\operatorname{Cont}^p(\mathfrak{a})$ if and only if

$$p = \min_{x^u \in \mathfrak{a}} \operatorname{ord} \alpha^*(x^u) = \min_{x^u \in \mathfrak{a}} \langle v, u\rangle,$$

where we define $\langle v, u\rangle = \infty$ if $v \in N_{\mathbb{R}}/\tau\mathbb{R}$ and $u \notin \tau^\perp$ for a cone τ. The assertion of the lemma follows immediately from this. $\square$

By this lemma it follows that $\operatorname{Cont}^p(\mathfrak{a})$ is a union of $T_\infty(v)$'s.

Lemma 5.10. *Let X be an affine toric variety defined by a cone σ in N and $\mathfrak{a}$ an invariant ideal on X. If an orbit $T_\infty(v) \subset \mathrm{Cont}^p(\mathfrak{a})$ is in $\underline{X_\infty(\tau)}$ for $\tau \neq 0$, then there is an orbit $T_\infty(\tilde{v}) \subset X_\infty(0)$ such that $T_\infty(\tilde{v}) \subset \mathrm{Cont}^p(\mathfrak{a})$ and $\overline{T_\infty(\tilde{v})} \supset T_\infty(v)$.*

Proof. Let $\rho : N_\mathbb{R} \to N_\mathbb{R}/\tau\mathbb{R}$ be the projection. As v is in the image $\rho(\sigma \cap N)$, we can take a point $v_0 \in \sigma \cap N$ such that $\rho(v_0) = v$. Then $\langle v, u\rangle = \langle v_0, u\rangle$ for $u \in \sigma^\vee \cap \tau^\perp$. We can naturally define $\langle v, u\rangle = \infty$ for $u \in \sigma^\vee \setminus \tau^\perp$. Let $v_1 \in \tau \cap N$ be in the relative interior of τ. Then $\langle mv_1, u\rangle > p$ for every $u \in (\sigma^\vee \setminus \tau^\perp) \cap N$ and an integer $m > p$. Let $\tilde{v} = v_0 + mv_1$ $(m > p)$. Then, for every $u \in \tau^\perp \cap \sigma^\vee \cap M$ it follows that $\langle \tilde{v}, u\rangle = \langle v_0, u\rangle = \langle v, u\rangle$, while for every $u \in (\sigma^\vee \setminus \tau^\perp) \cap M$ it follows that $\langle \tilde{v}, u\rangle > p$. Therefore

$$\min_{x^u \in \mathfrak{a}} \langle \tilde{v}, u\rangle = \min_{x^u \in \mathfrak{a}} \langle v, u\rangle = p.$$

Hence $T_\infty(\tilde{v}) \subset \mathrm{Cont}^p(\mathfrak{a})$.

On the other hand, $\rho(\tilde{v}) = v$ yields $\overline{T_\infty(\tilde{v})} \supset T_\infty(v)$, by Proposition 4.11. □

By these lemmas, we obtain that an irreducible component of $\mathrm{Cont}^p(\mathfrak{a})$ is the closure of $T_\infty(v)$ for some $v \in \sigma \cap N$ such that $\min_{x^u \in \mathfrak{a}} \langle v, u\rangle = p$. Here, by Proposition 4.8 and Proposition 5.7, we obtain the answer to the embedded version of Nash problem.

Theorem 5.11. *Let $\mathfrak{a}$ be an invariant ideal on an affine toric variety X defined by a cone σ. Then, an irreducible component of* $\mathrm{Cont}^p(\mathfrak{a})$ *is the closure of $T_\infty(v)$ for an element v minimal in $V(\mathfrak{a}, p) = \{v' \in \sigma \cap N \mid \min_{x^u \in \mathfrak{a}} \langle v', u\rangle = p\}$ with respect to the order $\leqslant_\sigma$. Therefore the valuations $\{val_v \mid v \in \sigma \cap N$ minimal in $V(\mathfrak{a}, p)\}$ correspond bijectively to the irreducible components of* $\mathrm{Cont}^p(\mathfrak{a})$.

Remark 5.12. Let $G(\mathfrak{a}) \subset M_\mathbb{R}$ be the Newton polytope of $\mathfrak{a}$ as in Fig. 1 and $\Delta(\mathfrak{a})$ the dual fan of $G(\mathfrak{a})$. The dual fan is the subdivision of σ. Then, the function $g(v) := \min_{u \in G(\mathfrak{a})} \langle v, u\rangle$ $(v \in \sigma)$ is a strongly convex piecewise linear function with respect to the fan $\Delta(\mathfrak{a})$. Therefore the subset $g^{-1}(p) = \{v \in \sigma \mid g(v) = p\}$ is the boundary of some convex polytope as in the Fig. 2. The minimal elements of $V(\mathfrak{a}, p)$ are on this boundary. It

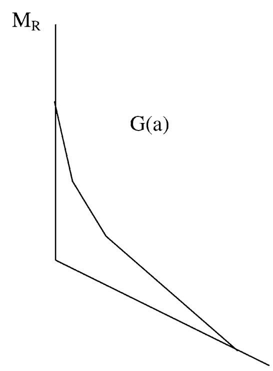

Fig. 1.

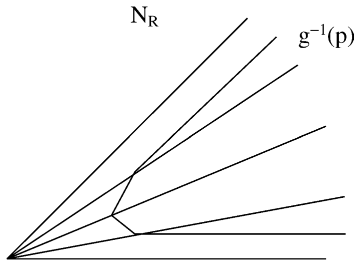

Fig. 2.

is clear that this convex polytope is $pG(\mathfrak{a})^\circ$, where $G(\mathfrak{a})^\circ$ is the polar polytope defined as $\{v \in \sigma \mid g(v) \geqslant 1\}$.

We can see that a lattice point of a compact face of $g^{-1}(p)$ is always a minimal element of $V(\mathfrak{a}, p)$, therefore it gives a valuation corresponding to an irreducible component of $\mathrm{Cont}^p(\mathfrak{a})$. If p is divisible enough so that every vertex of $pG(\mathfrak{a})^\circ$ is in N, then the minimal elements in $V(\mathfrak{a}, p)$ coincide with the lattice points on the compact faces of $g^{-1}(p)$.

Remark 5.13. The referee kindly informed the following to the author: For $u \in \sigma^\vee \cap M$, the log canonical threshold $\mathrm{lc}(X, V(\mathfrak{a}), V(x^u))$ turns out to be the maximal value λ such that $x^u \notin \mathcal{I}(X, \mathfrak{a}^\lambda)$ by [2], where $\mathcal{I}(X, \mathfrak{a}^\lambda)$ is a multiplier ideal for $\mathfrak{a}$. Some multiple of the primitive vector $v \in \sigma \cap N$ corresponding to a divisor which computes $\mathrm{lc}(X, V(\mathfrak{a}), V(x^u))$ lies on a compact face of $g^{-1}(p)$ for some p. Conversely, for some multiple of a primitive vector $v \in \sigma \cap N$ on a compact face of $g^{-1}(p)$, there exists $u \in \sigma^\vee \cap M$ such that the divisor corresponding to v computes the log canonical threshold $\mathrm{lc}(X, V(\mathfrak{a}), V(x^u))$.

Example 5.14. Let X be an affine toric variety defined by a cone σ. Then the components in $\pi^{-1}(\mathrm{Sing}\, X)$ are $\overline{T_\infty(v)}$'s, where v's are the minimal elements in $\bigcup_{\tau < \sigma:\mathrm{singular}} \tau^o \cap N$ with respect to the order $\leqslant_\sigma$. Here, τ^o is the relative interior of τ. This is proved as follows: Let $\mathfrak{a}$ be the ideal of $\mathrm{Sing}\, X$, then it is an invariant ideal. As $\pi^{-1}(\mathrm{Sing}\, X) = \bigcup_{p \geqslant 1} \mathrm{Cont}^p(\mathfrak{a})$, it follows that an irreducible component of $\pi^{-1}(\mathrm{Sing}\, X)$ is $\overline{T_\infty(v)}$, where v is minimal among v''s such that $v' \in \sigma \cap N$ and $\min_{x^u \in \mathfrak{a}} \langle v', u \rangle \geqslant 1$ by Theorem 5.11. Here, $\min_{x^u \in \mathfrak{a}} \langle v', u \rangle \geqslant 1$ if and only if $\alpha(0) \in \mathrm{Sing}\, X$ for α with $v_\alpha = v'$, which is equivalent to the fact that $v' \in \tau^o$ for a singular face $\tau < \sigma$ by [8, Proposition 3.9].

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