ADVANCES IN MATHEMATICS 20, 43–56 (1976)

On the de Rham Theory of Certain Classifying Spaces*

R. BOTT

Harvard University, Cambridge, Massachusetts 02138

H. SHULMAN

University of Pennsylvania, Philadelphia, Pennsylvania 19174

AND

J. STASHEFF

Temple University, Philadelphia, Pennsylvania 19122

In this thesis, H. Shulman proved the vanishing phenomena for characteristic classes of foliations without the usual geometric constructions of connections, curvature, etc. In this note, we present a setting for this, which was noted independently by the senior authors about two years ago.

Since that time, this subject has grown considerably, so that our account here is to a large extent anachronistic. In particular, the work of Kamber and Tondeur, who combine simplicial and curvature techniques but avoid classifying spaces, as well as the work of Vey, Bott, Haefliger and others has progressed far beyond the results outlined here. Nevertheless, our ideas are on the one hand very simple, but on the other, involve technicalities that have been resolved only recently, that a short account at this time still seems worthwhile.

The basic concept, which has been in the air for quite some time, and notably in the work of Deligne on mixed Hodge structures, is that the de Rham theory can be profitably employed as a tool for studying certain nonmanifolds, namely, those that are obtained as the geometric realization of a *simplicial manifold.*

* The authors gratefully acknowledge support from the NSF under grants GP 43613X, GP 38590, and GP 23976.

Indeed, if

$$M\colon M_0 \leftleftarrows M_1 \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; M_2 \cdots$$

is such an object (that is, M is a contravariant functor from the category of finite nonempty ordered sets to C^∞-manifolds), then applying the de Rham functor Ω, produces a cosimplicial module

$$\Omega M\colon \Omega M_0 \rightrightarrows \Omega M_1 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \Omega M_2 \cdots,$$

whose associated double complex, also denoted by ΩM, should be thought of as the *de Rham complex* of the *simplicial manifold* M. On the other hand, the object M also has a natural space associated to it, called its geometric realization and denoted by $| M |$, and a folk theorem has it that

EXTENDED DE RHAM THEOREM.

$$H^*(| M |; \mathbb{R}) \simeq H^*(\Omega M).$$

provided de Rham is valid on each of the constituent manifolds M_i *of* M.

Now, this very plausible extension of de Rham can be immediately applied for instance to the study of $H^*(BG)$, where G is a Lie group and BG its classifying space. Indeed, as is pointed out most succinctly in Segal's paper [11], BG can be taken to be the geometric realization of a semisimplicial manifold

$$NG\colon * \leftleftarrows G \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; G \times G \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow}\; G \times G \times G \cdots,$$

which is really the manifold version of the old Eilenberg–McLane construction.

Thus, applying our Folk theorem in this instance yields a double complex for the computation of $H^*(BG)$, which is of the form

$$\Omega NG\colon * \rightrightarrows \Omega G \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \Omega(G \times G) \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \Omega(G \times G \times G),$$

and so can be interpreted as a *completed version* of the usual Bar construction. (The classical Bar construction, of course, fails for de Rham, because the multiplication $G \times G \to G$ does not induce a coalgebra structure on ΩG.)

In view of the generalized de Rham theorem, one can seek representatives for the characteristic ring of G in the complex ΩNG, and this was first done by Shulman in his thesis, where he extended and reinterpreted a construction given in [14] for GL_q.

In particular, Shulman obtained the following positioning of the real characteristic classes of G in the de Rham complex ΩNG:

THEOREM II (Shulman). *A real characteristic class Φ associated to an invariant polynomial of degree q on the Lie-algebra of G, has a representative in ΩNG that involves only forms of degree $\geqslant q$. Thus:*

$$\Phi = \Phi_{q-1} + \Phi_{q-1} + \cdots + \Phi_0 ,$$

where $\Phi_i \in \Omega^{q+i} N_{q-i} G$.

This result also follows from the work of Kamber and Tondeur [6], and can also be deduced from a theorem of Bott–Hochshild [2], which computes the simplicial homology of each $\Omega^q NG$ in terms of the continuous cohomology of G acting on the qth symmetric power of the dual of the Lie algebra $\mathfrak{g}$ of G. The actual result is:

THEOREM (Bott, Hochshild).

$$H_\delta{}^p(\Omega^q NG) \simeq H^{p-q}_{\mathrm{cont}}(G; S^q \mathfrak{g}^*).$$

(Here, H_{cont} denotes continuous Eilenberg–McLane cohomology of G.) The earlier statement now follows from the fact that $H^0_{\mathrm{cont}}(G; S^q\mathfrak{g}^*) = \mathrm{Inv}_G \cdot (S^q\mathfrak{g}^*)$. Thus, these invariants begin a cocycle Φ_0 in ΩNG, which can be completed by adding correction terms Φ_i, $i > 0$, because there, the δ-cohomology vanishes.

Note that a result corresponding to Theorem II would not hold if Ω is replaced, say, by the integral singular theory. Thus, here, the very special nature of the de Rham theory comes to the fore.

We come now to an account of the vanishing and exotic class phenomena in this setting.

Our first observation is that the construction of ΩNG goes through essentially word for word for the "smooth categories" Γ_q encountered in the theory of foliations à la Haefliger. That is, the classifying space $B\Gamma_q$ for Haefliger structures is again the geometric realization of a semisimplicial *manifold* $N\Gamma_q$ (see Section 2 for details).

$$N\Gamma_q : \mathbb{R}^q \leftleftarrows m_1(\Gamma_q) \;\overset{\longleftarrow}{\leftleftarrows}\; m_2(\Gamma_q) \cdots.$$

Thus, one may speak of the de Rham complex $\Omega N\Gamma_q$, and use it to study the behavior of the usual characteristic classes of the normal bundle to a foliation. Indeed, in our frame work, this amounts to studying the map that the functor

$$\Gamma_q \xrightarrow{\nu} GL_q(\mathbb{R}),$$

associating to a germ of a diffeomorphism $\gamma \in m(\Gamma_q)$ its differential, induces in de Rham:

$$\Omega N\Gamma_q \xleftarrow{\Omega(\nu)} \Omega NG, \qquad G = GL(q, \mathbb{R}).$$

Now, however, the striking feature of $N\Gamma_q$ is, that *for every* r, *the manifolds* $m_r(\Gamma_q)$ *of* $N\Gamma_q$ *are* q*-dimensional.* As a consequence, $\Omega(\nu)$ annihilates all forms of dim $> q$ for trivial reasons, and therefore, by Theorem II, ν^* annihilates all real characteristic classes of dim $> 2q$.

In this way then, we have a very transparent proof of the following

VANISHING THEOREM (Bott). *Let*

$$B\nu\colon B\Gamma_q \to BGL_q$$

be the map induced by the Jacobian map $\nu\colon \Gamma_q \to GL_q$. *Then, over the reals,* $(B\nu)$ *induces the zero homomorphism in* dim $> 2q$.

Actually, this argument also immediately suggests how to construct potentially new characteristic classes in $B\Gamma_q$. Namely, let $\mathscr{F}_q\Omega NGL_q$ denote the complex of ΩNGL_q consisting of forms of degree $> q$. Then, because dim $N\Gamma_q = q$, $\Omega\nu^*$ annihilates this submodule and so induces a map

$$\bar{\Omega}\nu^*\colon \Omega NGL_q/\mathscr{F}_q\,\Omega GL_q \to \Omega N\Gamma_q.$$

Hence, $H^*(\Omega NGL_q/\mathscr{F}_q\Omega GL_q)$ has a natural map to $H^*(B\Gamma_q)$. On the other hand, this cohomology is relatively computable and leads to all the potential exotic classes in $B\Gamma_q$, as determined, say, in [3] or [6].

This essentially surveys the content of this note and in the various sections, we will be concerned mainly with the technicalities that arise when one seeks to carry out the above program.

1. The Generalized de Rham Theorem

A simplicial space

$$X: X_0 \leftleftarrows X_1 \overleftarrow{\leftleftarrows} X_2 \cdots \tag{1.1}$$

has several geometric realizations associated to it, and they are discussed and compared in detail by Segal in [12].

On the one hand, one has the fat realization $\| X \|$ of X, obtained from the disjoint union

$$\coprod X_i \times \Delta^i, \qquad \Delta^i \text{ the } i \text{ simplex,}$$

by identifying only with respect to the "boundaries" in X. On the other hand, one has the more geometrically intuitive lean realization $| X |$, which is obtained from $\| X \|$ by also identifying with respect to the degeneracies in X. The natural map

$$\| X \| \to | X |$$

is, in general, not a weak homotopy equivalence, but is one if all the degeneracies in X are cofibrations. Furthermore, it is only the fat realization that behaves relative to maps. More precisely, it has the property:

(*) *If $X_i \to Y_i$ is a map of simplicial spaces that is a weak homotopy equivalence for each i, then so is the induced map*

$$\| X_i \| \to \| Y_i \|.$$

This assertion is essentially proved by Segal [12, Appendix]. However, he there states the result for homotopy equivalences, and alas, operates generally in the category of compactly generated Hausdorf spaces. On the other hand, the argument he outlines also goes through in the present context.

We turn now to a singular analog of our Folk theorem.

THEOREM (Folk). *Let S_* denote the functor of singular chains, so that S_* applied to the simplicial space X, gives rise to a simplicial chain complex $i \to S_*(X_i)$, whose associated double complex will be denoted by $S_*(X)$. With this understood, one has a functorial isomorphism*

$$H\{S_*(X)\} = H(S_* \| X \|). \tag{1.2}$$

Remarks. (1) When X is a simplicial set, the formula (1.2) reduces to the well-known fact that the singular theory of the realization is computable combinatorially. (2) The usual spectral sequences of a double complex of course now yield spectral sequences converging to $H(S_* \| X \|)$. In particular, one has an E_2 term of the form

$$E_2^{p,q} = H_p(H_q(X)),$$

(where the internal H is singular and the external one is derived from the simplicial structure) which is familiar in the literature. (3) A special instance of (1.2) occurs in the work of Kozul [7, 8] on fiber spaces, and there are hints at (1.2) in many contexts, e.g., Stasheff [15], etc. But we have not been able to find an explicit reference in the literature, and therefore, will outline a proof of (1.2) along lines suggested by Segal. First, consider the singular functors $\mathscr{S}_q$ associating to a space its singular q simplexes. Applying $\mathscr{S}_q$ to the X_i of X, produces a doubly simplicial set $(i, j) \to S_j(X_i)$, which we will denote by $X_{i,j}$. Now, any double simplicial set $Y_{i,j}$ has fat and lean geometric realizations $\| Y_{i,j} \|$ and $| Y_{i,j} |$ obtained from

$$\coprod X_{i,j} \times \Delta^i \times \Delta^j$$

in the evident manner, and as the cofibration conditions are here eminently satisfied, the two realizations have the same weak homotopy type. Furthermore, $H_*(\mathscr{S} \| Y_{i,j} \|)$ can be computed combinatorially, that is, from the free chain complex $\mathscr{F} Y_{i,j}$ generated by the $Y_{i,j}$: Thus

$$H(\mathscr{F} Y_{*,*}) \simeq H(\| Y_{i,j} \|).$$

Note now that in our situation, i.e., $X_{i,j} = \mathscr{S}_j(X_i)$, the left-hand side is precisely $H\{S_*(X)\}$. Hence, it suffices to show that $\| X_{i,j} \|$ has the same weak homotopy type as $\| X \|$. Now the realization of $\| X_{i,j} \|$ can be factored through realizing in the vertical direction (i.e., along j, and then in the horizontal direction.) On the other hand, the vertical realization $\| X_{i,j} \|_j$ creates the fat geometric realization of the singular complex of X_i. Thus, we have a natural map

$$\| X_{i,j} \|_j \to X_i ,$$

which is a weak homotopy equivalence for each i.

Hence, the induced map in the fat realization

$$\| X_{i,j} \| = \| \, \| X_{i,j} \|_j \, \|_i \to \| X \|$$

is also a weak homotopy equivalence. Q.E.D.

We are now ready to consider the de Rham version of (1.2). For this purpose, recall first of all that the de Rham complex ΩM of a smooth manifold is mapped into the singular $\mathbb{R}$-cochains $S^*(M)$ on M, only via the subcomplex $\tilde{S}_*(M)$, generated by the smooth singular simplexes in $S_*(M)$. Precisely, one has the natural inclusion:

$$i_\# : \tilde{S}_*(M) \hookrightarrow S_*(M),$$

which by a smoothing argument is shown to be a chain equivalence, so that the obvious map

$$\Omega^* M \to \tilde{S}^*(M)$$

given by integration, composed with $(i_\#)^{-1}$, yields the de Rham homomorphism

$$H^*(\Omega^* M) \to H^*(SM) = H^*(M).$$

Now, the classical smoothing argument for the chain equivalence of $i_\#$ is by and large only carried out for Hausdorf manifolds in the literature (see [17] for instance) and for certain models of the Haefliger classifying spaces it would be convenient to have it in general.

One technique for extending α to the non-Hausdorf situation is to use the strong excision property of the singular theory, which allows one to compute $H\{S_*(M)\}$ by means of the subcomplex $S_*{}^{\mathscr{U}}(M)$ generated by $\mathscr{U}$-small simplexes, with $\mathscr{U}$ any open cover of M. On the other hand, M is *locally Hausdorf* so that $\tilde{S}^{\mathscr{U}}(M) \subset S^{\mathscr{U}}(M)$ is a chain equivalence by the classical argument for a cover consisting of Hausdorf sets. In detail, this argument takes the following form:

Let $\mathscr{U} = \{\mathscr{U}_\alpha\}_{\alpha \in J}$ be an open cover of a space X, which for simplicity we assume indexed by an ordered index set J.

This situation naturally defines a *simplicial space* $X_{\mathscr{U}}$, whose ith components consist of the disjoint union of the $(i+1)$-fold intersections of the $\mathscr{U}_\alpha$. Thus

$$X_{\mathscr{U}} : \coprod \mathscr{U}_\alpha \leftleftarrows \coprod_{\alpha<\beta} \mathscr{U}_\alpha \cap \mathscr{U}_\beta \leftleftarrows\!\!\leftarrow, \qquad \text{etc.} \tag{1.3}$$

Further, the inclusions $\mathscr{U}_\alpha \subset X_J$ combine to define a natural map ϵ, of X_J into the trivial semisimplicial object defined by X (and again denoted by X) which plays the role of an augmentation. In fact

$$| X | \xleftarrow{|\epsilon|} | X_{\mathscr{U}} | \tag{1.4}$$

is a homotopy equivalent if X is paracompact as is pointed out in Segal [11].

Now, let $\mathscr{S}_q$ be the qth singular functor. Then, $\mathscr{S}_q$ applied to (1.3) yields a homotopy equivalence:

$$| \mathscr{S}_q X | \xleftarrow{\mathscr{S}_q \epsilon} | \mathscr{S}_q X_{\mathscr{U}} |.$$

Applying the free-abelian group functor, this translates into the following well-known (Folk) resolution for the $\mathscr{U}$-small-simplexes of X, which is valid without any conditions on the space X:

THE SINGULAR MEYER VIETORIS RESOLUTION. *Let $S_*^{\mathscr{U}}(X)$ denote the group of singular $\mathscr{U}$-small chains on X. Then, ϵ induces a resolution of $S_*^{\mathscr{U}}$ in the sense that for each q:*

$$0 \longleftarrow S_q^{\mathscr{U}}(X) \xleftarrow{\epsilon^*} S_q\left(\coprod \mathscr{U}_\alpha\right) \longleftarrow S_q\left(\coprod_{\alpha<\beta} \mathscr{U}_\alpha \cap \mathscr{U}_\beta\right) \longleftarrow \tag{1.5}$$

is an exact sequence.

In short then, ϵ induces chain equivalences of the total complexes

$$S_*^{u} \xleftarrow[\epsilon_*]{} S_*(X_u),$$

and by the combinatorial nature of the argument, also of the differentiable singular chains when X is a manifold M.

Thus, in that case, one has a natural diagram:

$$\begin{array}{ccccc} S_*(M) & \xleftarrow{i_*} & S_*^{u}(M) & \xleftarrow{\epsilon} & S_*(M_J) \\ & & \uparrow{\scriptstyle s_*'} & & \uparrow{\scriptstyle s_*} \\ & & \tilde{S}_*^{u}(M) & \xleftarrow{\tilde{\epsilon}} & \tilde{S}_*(M_J). \end{array} \tag{1.6}$$

Now, s_* induces a chain equivalence if we apply the smoothing to a Hausdorf cover $\{u_\alpha\}_{\alpha\in J}$ of M. Hence, s_*' is an equivalence, and of course, i_* is one by the excission property of S_*.

At this stage, we have the following straightforward corollary of the Vietoris sequence:

COROLLARY. *There is a natural de Rham map*

$$H^*(\Omega M) \longrightarrow H^*(M; \mathbb{R}), \tag{1.7}$$

whether M is Hausdorf or not. Further, if $\{\mathscr{U}_\alpha\}_{\alpha \in J}$ is any open cover of M by paracompact sets, then

$$H^*(\Omega M_{\mathscr{U}}) \simeq H^*(M; \mathbb{R}). \tag{1.8}$$

In short, the de Rham–Čech construction always gives the singular theory provided only that the cover is by paracompacts.

On the other hand, the map

$$H^*(\Omega M) \xrightarrow{\epsilon^*} H^*(\Omega M_{\mathscr{U}})$$

will in general not be an isomorphism. Of course, if M is paracompact itself, a smooth partition of one can be used in the usual way to prove that ϵ^* is an isomorphism.

If we now combine (1.2) with (1.7), we obtain the sought after

GENERALIZED DE RHAM THEOREM. *For a simplicial manifold*

$$M: n \to M_n,$$

there is a functorial map of

$$H^*(\Omega M) \to H^*(\| M \|). \tag{1.9}$$

Furthermore, if each M_i is paracompact, then (1.9) *is an isomorphism.*

2. THE DE RHAM THEORY OF $B\Gamma_q$

To apply our de Rham theorem, we have to recall the basic steps in Haefliger's classification theory for foliations. In this theory, which proceeds in close analogy to Lie group Bundle theory, the structure group is replaced by a C^∞ category Γ_q defined as follows:

The objects $o(\Gamma_q)$ consist of $\mathbb{R}^q$ in its usual topology and C^∞ structure.

The morphisms $m(\Gamma_q)$ are the germs of diffeomorphisms of $\mathbb{R}^q$, in the sheaf topology over $\mathbb{R}^q$. Thus, under the source and target maps

$$\mathbb{R}^q \underset{t}{\overset{s}{\leftleftarrows}} m(\Gamma_q), \tag{2.1}$$

$m(\Gamma_q)$ is a sheaf over $\mathbb{R}^q$, and as such, inherits a C^∞ structure of a q-manifold from $\mathbb{R}^q$. The resulting manifold is, of course, highly non-Hausdorf.

Now, the classification theory of Haefliger starts with numerable Γ_q-cocycles $\{\gamma_{\alpha\beta}\}$ over a space X, defines equivalences of such cocycles to obtain the notion of a Haefliger structure on X, and then shows that the *concordance classes of such structures* are classified by a classifying space. The most universally applicable model for such a classifying space seems to be the "Milnor construction applied to Γ_q" and in Segal's terminology, this is equivalent to applying our realization functor to the nerve $N\hat{\Gamma}_q$ of the unwinding of Γ_q over $\mathbb{Z}^+$. We recall this terminology: First, for any category $\mathscr{C}$, the nerve of $\mathscr{C}$ is a simplicial object:

$$N\mathscr{C}: o(\mathscr{C}) \leftleftarrows m_1(\mathscr{C}) \overset{\longleftarrow}{\underset{\longleftarrow}{\leftleftarrows}} m_2(\mathscr{C}) \cdots, \tag{2.2}$$

where $o(\mathscr{C})$ are the objects and $m_i(\mathscr{C})$ denotes diagrams

$$\cdot \xleftarrow{\gamma_1} \cdot \xleftarrow{\gamma_2} \cdots \xleftarrow{\gamma_i}$$

of morphisms in $m(\mathscr{C})$, the structure maps being attained by dropping the end arrows and composing adjacent ones. If $\mathscr{C}$ carries a topology (C^∞ structure, etc.) $N(\mathscr{C})$, then $N(\mathscr{C})$ clearly carries a topology (C^∞ structure, etc.). Second, we recall that if $\mathbb{Z}^+$ denotes the category with objects the positive integers and one morphism for every $i \leqslant j$, then the unwinding of $\mathscr{C}$ over $\mathbb{Z}^+$ is defined by Segal to be the subcategory $\hat{\mathscr{C}}$ of the product category $\mathscr{C} \times \mathbb{Z}^+$, obtained by deleting all morphisms $(j, i \leqslant i)$ with j not an identity. The essential virtue of this construction is that whereas in $N\mathscr{C}$, the degeneracies well may not be cofibrations, they always will be in $N\hat{\mathscr{C}}$.

Furthermore, the natural map

$$\hat{\mathscr{C}} \to \mathbb{Z}^+$$

induces a map

$$| N\hat{\mathscr{C}} | \to | N\mathbb{Z}^+ |,$$

and as the right-hand space is the infinite simplex, the star cover of $N\mathbb{Z}^+$ pulls back naturally to $| N\hat{\mathscr{C}} |$, and it is this cover that plays an essential role in the classification theory.

In any case, it is then seen rather easily that $\mathscr{C}$-valued cocyle classes on open numerable covers on a space X and are classified by maps into

$| N\hat{\mathscr{C}} |$, see [4, 5, 10] for details. In the situation at hand, this classification implies the following

THEOREM (Buffet–Lor). *The concordance classes of Γ_q-structures are classified by the space $| N\hat{\Gamma}_q |$.*

Remarks. (1) Actually the use of $\hat{\Gamma}_q$ rather than Γ_q is only a matter of convenience as tom Dieck [16] has shown recently that for any topological category, $\mathscr{C}$ the unwound lean realization is of the same homotopy type as the fat realization of $\mathscr{C}$:

THEOREM (tom Dieck). *The natural map $\hat{\mathscr{C}} \to \mathscr{C}$ induces a homotopy equivalence*

$$| N\hat{\mathscr{C}} | \sim \| N\mathscr{C} \|.$$

(2) Quite recently, Segal has suggested other models for the classifying space of foliations, which work well on paracompact spaces. In particular, he shows that the realization of the topological category $\mathscr{E}_q$ described below is such a classifying space, and as will be immediately apparent, $N\mathscr{E}_q$ is definitely a Hausdorff simplicial manifold with paracompact constituents, whose degeneracies are cofibrations.

Segal's result is the following:

THEOREM (Segal). *Let $\mathscr{E}_q$ be the topological category whose objects are all pairs (x, U), whose U is an open subset of $\mathbb{R}^q$ and $x \in U$, and whose morphisms $(x, U) \xrightarrow{f} (x', U')$ are all smooth embeddings $f\colon U \to U'$ such that $f(x) = x'$. Then, the map sending f to its germ at x defines a natural map $| \mathscr{E}_q | \to | N(\Gamma_q)|$, which is a paracompact equivalence.*

In short then, any one of the three spaces $\| N\Gamma_q \|$, $| N\hat{\Gamma}_q |$, $| N\mathscr{E}_q |$ can be considered as a classifying space for foliations, and they all have the following nearly selfevident property.

PROPOSITION. *The simplicial spaces $N\Gamma_q$, $N\hat{\Gamma}_q$, and $N\mathscr{E}_q$ are all q-dimensional.*

To explain this phenomenon, consider the case $N\Gamma_q$, and in particular, of $m_2(\Gamma_q)$. One has the diagram:

$$\begin{array}{ccc} m_2(\Gamma_q) & \longrightarrow & m(\Gamma_q) \\ {\scriptstyle s^{-1}(t)}\downarrow & & \downarrow{\scriptstyle t} \\ m(\Gamma_q) & \xrightarrow{s} & o(\Gamma_q) = \mathbb{R}^q, \end{array}$$

which exhibits $m_2(\Gamma_q)$ as the pull-back of t by s. Both s and t are local homeomorphisms. Hence, $s^{-1}(t)$, and the composition $s \circ s^{-1}(t)$ will also be local homeomorphisms. This shows that $m_2(\Gamma_q)$ is a C^∞ q-manifold.

Finally, because the target of a composition of two diffeomorphisms varies smoothly with the source of the first one, it follows that composition $m_2(\Gamma_q) \to m_1(\Gamma_q)$ is smooth, and also a local homeomorphism. The general case is quite analogous. The argument of course works equally for $\hat{\Gamma}_q$ because $\mathbb{Z}^+$ is a discrete category. In $\mathscr{E}_q$, the remark is even more immediate.

The Vanishing Theorem of the Introduction is now also immediate for any one of our three models. For example, the composition

$$\nu: \hat{\Gamma}_q \to \Gamma_q \to GL_q,$$

of the forgetful functor, together with sending a germ into its Jacobian, induces a map

$$N\nu: N\hat{\Gamma}_q \to NGL_q,$$

of smooth simplicial manifolds, and the induced maps of the de Rham complex

$$(N\nu)^*: \Omega^*NGL_q \to \Omega^*N\hat{\Gamma}_q,$$

clearly annihilates the subcomplex $\mathscr{F}_q\Omega^*NGL_q$ of forms of degree $> q$. The de Rham map, therefore, factors through the quotient to yield the diagram:

$$\begin{array}{ccccc} H(\Omega^*NGL_q) & \xrightarrow{r^*} & H(\Omega^*NGL_q/\mathscr{F}_q\Omega^*NGL_q) & \xrightarrow{\pi^*} & H(\Omega^*N\hat{\Gamma}_q) \\ \downarrow \wr & & & & \downarrow \\ H(|NGL_q|) & & \xrightarrow{|\nu|^*} & & H^*(|N\hat{\Gamma}_q|), \end{array} \tag{2.3}$$

with the vertical maps indicating our generalized de Rham homomorphism. On the left-hand side, this is an isomorphism because we are dealing with paracompact constituents in $N(GL_q)$. (In fact, here in the group case, m_r is simply isomorphic to r copies of the group under consideration.)

From the position of the characteristic ring in Ω^*NGL_q, it is now clear that already r^* annihilates all characteristic classes of dim $2q$, so that (2.3) is a natural strengthening of the vanishing phenomena with the algebra $H(\Omega^*NGL_q/\mathscr{F}_q\Omega^*NGL_q)$ playing the role of potentially new characteristic classes. This algebra will be computed in a subsequent note to show that it agrees with the exotic classes found by a variety of people; in particular, one finds here all the smooth classes of $| N\hat{\Gamma}_q |$ in the sense of Bott–Haefliger [3].

We close this section with a few additional comments concerning this whole argument.

(1) Clearly, the procedure we outlined here is valid whenever one deals with a category Γ of "Lie-type." Rather than defining these, let us just consider two examples:

EXAMPLE 1. Let $\Gamma = \Gamma_q\mathbb{C} \subset \Gamma_{2q}$ consisting of germs preserving a holomorphic structure.

EXAMPLE 2. Let $\Gamma = \Gamma_q S \subset \Gamma_{2q}$ consist of the germs preserving a simplectic form on $\mathbb{R}^{2q}$.

In these cases, $N\hat{\Gamma}$ will not only be a simplicial manifold, but will be a complex one in Example 1 and a symplectic one in Example 2. We will write G for the image group of the jacobian map ν for Γ. Thus, in our first case, $G = GL_q(\mathbb{C}) \subset GL_{2q}$; in the second, $G = Spl(q) \subset GL_{2q}$. In each of these cases, therefore, the normal map

$$\nu: \Gamma \to G$$

can be studied by de Rham methods, and in particular, we get a natural map of

$$H(\Omega^*NG/\mathscr{F}_r\Omega^*NG) \to H^*(| N\hat{\Gamma} |),$$

where r is the dimension of the real representation of Γ involved ($2q$ in both our examples). On the other hand, it will not be true that this procedure always yields the "continuous cohomology of $N\hat{\Gamma}$." For instance, it fails in the symplectic case. However, it is likely that this procedure would work in the limit as $\nu: \Gamma \to G$ is replaced by its kth prolongations: $\nu^{(k)}: \Gamma \to G^{(k)}$.

REFERENCES

1. R. BOTT, Lectures on characteristic classes and foliations (Notes by Lawrence Conlon), *in* "Lecture Notes in Mathematics," No. 279, pp. 1–94. Springer–Verlag, New York, 1972.
2. R. BOTT, On the Chern–Weil homomorphism and the continuous cohomology of Lie groups, *Advances in Math.* 11 (1973), 289–303.

3. R. Bott and A. Haefliger, On characteristic classes of Γ-foliations, *Bull. Amer. Math. Soc.* **78** (1972), 1039–1044.
4. A. Haefliger, Feuilletages sur les variétés ouvertes, *Topology* **9** (1970), 183–194.
5. A. Haefliger, Homotopy and integrability, *in* "Lecture Notes in Mathematics," No. 197, Springer–Verlag, New York, 1970.
6. F. Kamber and P. Tondeur, Characteristic invariants of foliated bundles, *Manuscripta Math.* **11** (1974), 51–89.
7. J. L. Koszul, Multiplicateurs et classes caractéristiques, *Trans. Amer. Math. Soc.* **89** (1958), 256–266.
8. J. L. Koszul, Espaces fibrés associés et pré-associés, *Nagoya Math. J.* **15** (1959), 155–169.
9. J. P. May, "Simplicial Objects in Algebraic Topology," Van Nostrand, Princeton, 1967.
10. M. Mostow, J. Perchik, and R. Bott, Notes on Gel'fand Fuks cohomology and characteristic classes, Eleventh Holiday Symposium, New Mexico State University, Las Cruces, New Mexico, 1973.
11. G. Segal, Classifying spaces and spectral sequences, *Inst. Hautes Études Sci., Publ. Math.* **34** (1968), 105–112.
12. G. Segal, Categories and cohomology theories, *Topology* **13** (1974), 293–312.
13. G. Segal, Classifying spaces for foliations, to appear.
14. H. Shulman, On characteristic classes, Thesis, Univ. of California, Berkeley, 1972.
15. J. Stasheff, Homotopy associativity of H-spaces II, *Trans. Amer. Math. Soc.* **108** (1963), 293–312.
16. T. tom Dieck, On the homotopy type of classifying spaces, *Manuscripta Math.* **11** (1974), 41–49.
17. F. W. Warner, "Foundations of Differential Manifolds and Lie Groups," Scott Forsman, Glenview, Illinois, 1971.