Graham’s pebbling conjecture on products of many cycles

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Abstract

A pebbling move on a connected graph $G$ consists of removing two pebbles from some vertex and adding one pebble to an adjacent vertex. We define $f_t(G)$ as the smallest number such that whenever $f_t(G)$ pebbles are on $G$, we can move $t$ pebbles to any specified, but arbitrary vertex. Graham conjectured that $f_1(G \times H) \leq f_1(G)f_1(H)$ for any connected $G$ and $H$. We define the $\alpha$-pebbling number $\alpha(G)$ and prove that $\alpha(C_p \times \cdots \times C_p \times C_p \times G) \leq \alpha(C_p) \cdots \alpha(C_p)\alpha(G)$ when none of the cycles is $C_5$, and $G$ satisfies one more criterion. We also apply this result with $G = C_5 \times C_5$ by showing that $C_5 \times C_5$ satisfies Chung’s two-pebbling property, and establishing bounds for $f_t(C_5 \times C_5)$.

Keywords: Pebbling; Graham’s conjecture; Cartesian product; Cycle

1. Background

We show that the pebbling number of the Cartesian product of arbitrarily many cycles with a connected graph $G$ satisfies Graham’s conjecture as long as none of the cycles are $C_5$ and $G$ satisfies Wang’s odd two-pebbling property [10] and one more numerical criterion; that is, in this case, the pebbling number of the product is at most the product of the pebbling numbers of the individual cycles. We do this by defining the alpha-pebbling number of a graph, which is similar to notions used by Moews [7], and is based on the odd two-pebbling property. We also show that $C_5 \times C_5$ satisfies the necessary conditions. In this paper, we assume that all graphs are connected.

Chung [1] defined a distribution on a graph as a placement of pebbles on the vertices of the graph. A pebbling move then consists of removing two pebbles from one vertex, and adding one pebble to an adjacent vertex. Then the pebbling number of a vertex $v$ in $G$ is the smallest number $f(G, v)$ such that from every placement of $f(G, v)$ pebbles, it is possible to move a pebble to $v$ by a sequence of pebbling moves. She also defined the t-pebbling number of $v$ in $G$ as the smallest number $f_t(G, v)$ such that from every placement of $f_t(G, v)$ pebbles, it is possible to move $t$ pebbles to $v$. Then the pebbling number of $G$ and the t-pebbling number of $G$ are the smallest numbers, $f(G)$ and $f_t(G)$, such that from any placement of $f(G)$ pebbles or $f_t(G)$ pebbles, respectively, it is possible to move one or $t$ pebbles, respectively, to any specified target vertex by a sequence of pebbling moves. Thus, $f(G)$ and $f_t(G)$ are the maximum values of $f(G, v)$ and $f_t(G, v)$ over all vertices $v$.

Chung also defined the two-pebbling property of a graph, and Wang [10] extended her definition to the odd two-pebbling property as follows.

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**Theorem 4.** Theorem 4 would imply Graham’s conjecture for all graphs with the odd two-pebbling property.

To prove Graham’s conjecture for products of graphs, Chung used the notion of the **alpha conjecture**. It is a natural analog of Graham’s conjecture. If \( f \) is the two-pebbling property, then we can move two pebbles to any specified target vertex whenever \( p \) and \( q \) satisfy the inequality \( p + q > 2f(G) \) (respectively, whenever \( p \) and \( r \) satisfy \( p + r > 2f(G) \)). Clearly, any graph with the two-pebbling property also satisfies the odd two-pebbling property. It is not known whether any graph satisfies the odd two-pebbling property, but not the two-pebbling property.

**Definition 2** is motivated by Moews’ notion [7] of \( \alpha \)-pebbling.

**Definition 2.** The **alpha-pebbling number of** \( v \) in \( G \) is the smallest number \( \alpha(G, v) \) such that:

1. \( f(G, v) \leq \alpha(G, v) \), and
2. From every placement of \( p \) pebbles on \( G \) in which \( r \) vertices are occupied by an odd number of pebbles, if \( p + r > 2\alpha(G, v) \), then we can move two pebbles to \( v \).

The **alpha-pebbling number of** \( G \), \( \alpha(G) \), is the maximum of \( \alpha(G, v) \) over all vertices \( v \). Thus, \( f(G) \leq \alpha(G) \), and the equality holds if and only if \( G \) satisfies the odd two-pebbling property. When we discuss graphs whose pebbling numbers are the same for every vertex (in particular, complete graphs, cycles, and products of these graphs), we generally do not specify the target vertex.

Chung attributed Conjecture 3 to Graham, and proved Theorem 4 (Theorem 5), but the proof can be adapted to show Theorem 5, so we also attribute this result to Chung.

**Conjecture 3** (Graham). For any graphs \( G \) and \( H \), if \( G \times H \) represents the Cartesian product of \( G \) and \( H \), then

\[
f(G \times H) \leq f(G)f(H).
\]

**Theorem 4** (Chung). Suppose \( G \) is a graph which satisfies the two-pebbling property, and let \( K_t \) denote the complete graph on \( t \) vertices. Then:

1. \( f(K_t \times G) \leq tf(G) \), and
2. If \( f(K_t \times G) = tf(G) \), then \( K_t \times G \) satisfies the two-pebbling property.

**Theorem 5** (Chung). For any graph \( G \), \( \alpha(K_t \times G) \leq \alpha(K_t)\alpha(G) = t\alpha(G) \).

Chung used Theorem 4 to prove Graham’s conjecture for products of \( K_t \)’s, including hypercubes (products of \( K_2 \)’s). She also proved that trees satisfy the two-pebbling property (and therefore, the odd two-pebbling property). Moews [7] proved Theorem 6, though with slightly different notation.

**Theorem 6** (Moews). For any tree \( T, \alpha(T \times G, (x, y)) \leq \alpha(T, x)\alpha(G, y) \). In particular, if \( P_m \) is the path with \( m \) vertices, then \( \alpha(P_m \times G, (x, y)) \leq \alpha(P_m)\alpha(G, y) = 2^{m-1}\alpha(G, y) \).

We call Conjecture 7 the alpha conjecture. It is a natural analog of Graham’s conjecture.

**Conjecture 7** (The Alpha Conjecture). For any graphs \( G \) and \( H \), and any vertex \( (x, y) \) in \( G \times H \), we have \( \alpha(G \times H) \leq \alpha(G)\alpha(H) \), and \( \alpha(G \times H, (x, y)) \leq \alpha(G, x)\alpha(H, y) \).

Conjecture 7 would imply Graham’s conjecture for all graphs with the odd two-pebbling property.

There are several advantages to the notation of \( \alpha(G) \). Much of the literature on pebbling [1,2,4,9,10] proves theorems of the form: “If \( G \) is a certain type of graph, and \( H \) satisfies either the two-pebbling property or the odd two-pebbling property, then \( f(G \times H) \leq f(G)f(H) \).” The proofs of the theorems which require \( H \) to satisfy the two-pebbling property apply with obvious modification even when \( H \) only satisfies the odd two-pebbling property. If we rewrite the conclusion to read “\( f(G \times H) \leq f(G)\alpha(H) \)” (or if \( G \) satisfies the odd two-pebbling property we could write “\( f(G \times H) \leq \alpha(G)\alpha(H) \”), our result applies to all graphs \( H \); if \( H \) does not satisfy the odd two-pebbling property, \( G \times H \) may not obey Graham’s conjecture, but the result is interesting nonetheless. Also, if we can prove
the stronger result that \( \alpha(G \times H) \leq \alpha(G)\alpha(H) \), we can chain together as many copies of the relevant graphs as we choose, as both Chung and Moews did in their work.

We could similarly define \( \beta(G, v) \) and \( \beta(G) \) by replacing \( r \) (the number of vertices with an odd number of pebbles) with \( q \) (the total number of occupied vertices) in Definition 2. Then \( f(G) = \beta(G) \) if and only if \( G \) satisfies the two-pebbling property (instead of the odd two-pebbling property). Therefore, a brief discussion of the benefits of each of these pebbling numbers along with a justification of our focus on alpha-pebbling numbers is in order. We use the two-pebbling properties when we transfer pebbles from \( \{x_i\} \times H \) to \( \{x_j\} \times H \) in \( G \times H \) for some edge \((x_i, x_j)\) in \( G \). These transfers do not change the value of \( r \) in \( \{x_i\} \times H \), so by using alpha-pebbling numbers, we can more easily ensure that we can put two pebbles on some vertex \((x_i, y)\). On the other hand, these transfers might reduce \( r \) in \( \{x_j\} \times H \), whereas they cannot reduce \( q \), so using beta-pebbling numbers would allow us more easily to put two pebbles on a vertex \((x_j, y)\). Because we generally make transfers toward a target vertex \((x, y)\), we usually want to put two pebbles on a vertex in the source of the transfers, as this gives us another way to direct pebbles toward the target. Thus, alpha-pebbling numbers appear more useful than beta-pebbling numbers.

Pachter, Snevily and Voxman [8] found the pebbling numbers of cycles. Snevily and Foster [9] gave an upper bound for the \( t \)-pebbling numbers of odd cycles. This bound was shown to be exact in [2], which gave the \( t \)-pebbling number of even cycles as well. These numbers are given in Theorem 8.

**Theorem 8** (Pachter/Snevily/Voxman, Snevily/Foster, Herscovici). The \( t \)-pebbling number of the cycles \( C_{2k} \) and \( C_{2k+1} \) are

\[
\begin{align*}
    f_t(C_{2k}) &= 2^k \cdot t \\
    f_t(C_{2k+1}) &= \frac{2^{k+2} - (-1)^k}{3} + 2^k(t - 1).
\end{align*}
\]

In particular, \( f(C_{2k}) = 2^k \) and \( f(C_{2k+1}) = \frac{2^{k+2} - (-1)^k}{3} \).

Snevily and Foster also showed that cycles satisfy the two-pebbling property (so \( f(C_n) = \alpha(C_n) \)). Another proof was given in [2], where it was also shown that products of two cycles satisfy Graham’s conjecture. The proofs in [2] could be extended to obtain Theorems 9 and 10:

**Theorem 9.** For any graph \( G \), \( f(C_{2k} \times G) \leq \alpha(C_{2k})\alpha(G) = 2^k\alpha(G) \).

**Theorem 10.** Suppose \( G \) has \( n \geq 5 \) vertices and satisfies the inequality

\[
2f_4(G) + n - 5 < 7X
\]

for some \( X \geq \alpha(G) \). Then \( f(C_{2k+1} \times G) \leq \alpha(C_{2k+1})X \) whenever \( 2k + 1 \geq 7 \). Furthermore, all odd cycles satisfy (1) with \( X = \alpha(G) \).

We use Proposition 11 to eliminate the requirement in Theorem 10 that \( G \) has at least five vertices.

**Proposition 11.** If \( G \) is a connected graph with at most four vertices, then \( \alpha(G \times H) \leq \alpha(G)\alpha(H) \) for every graph \( H \).

**Proof.** Since \( G \) is connected, either it is a tree or it contains a copy of \( C_3 \) or \( C_4 \). If \( G \) is a tree, \( \alpha(G \times H) \leq \alpha(G)\alpha(H) \) by Theorem 6. Similarly, if \( G \cong C_3 \cong K_3 \), the same conclusion follows from Theorem 5. The only other graph which contains a three-cycle but not a four-cycle is a triangle with an extra vertex and edge attached. This graph can be viewed as the star \( K_{1,3} \) (which is a tree) with an extra edge, and it has \( \alpha(G) = \alpha(K_{1,3}) = 5 \). Thus, \( \alpha(G \times H) \leq \alpha(K_{1,3} \times H) \leq \alpha(K_{1,3})\alpha(H) = \alpha(G)\alpha(H) \). Finally, if \( G \) contains a copy of \( C_4 \) (and has no other vertices), then \( \alpha(G \times H) \leq \alpha(C_4 \times H) \leq 4\alpha(H) = \alpha(G)\alpha(H) \). □

Combining Proposition 11 and Theorem 10 gives Theorem 12.

**Theorem 12.** If \( G \) satisfies the inequality (1) for some \( X \geq \alpha(G) \), and if \( 2k + 1 \geq 7 \), then \( f(C_{2k+1} \times G) \leq \alpha(C_{2k+1})X \). □
In this paper, we strengthen Theorems 9 and 12 to show that \( \alpha(C_m \times G) \leq \alpha(C_n) \alpha(G) \) as long as \( m \) is even or \( m \neq 5 \) and \( G \) satisfies (1) with \( X = \alpha(G) \) (we use Theorem 5 for products involving \( C_3 \cong K_3 \)). Herscovici and Higgins [3] proved that \( f(C_5 \times C_3) = f(C_5) \cdot f(C_3) = 25 \), and we show that \( C_5 \times C_5 \) satisfies the two-pebbling property. Thus, \( \alpha(C_m \times C_n) \leq \alpha(C_m) \alpha(C_n) \) for any cycles \( C_m \) and \( C_n \). We also show that \( C_5 \times C_5 \) satisfies (1) by finding bounds on \( f(C_5 \times C_5) \).

By proving the results involving \( \alpha \)-pebbling numbers, we aim to chain together the products of cycles. Toward that end, this paper shows that, under certain circumstances, products of arbitrarily many odd cycles together with an arbitrary graph satisfy the inequality (1), and this allows us to conclude that \( \alpha(C_p \times \cdots \times C_p \times C_p \times G) \leq \alpha(C_p) \cdots \alpha(C_p) \alpha(C_p) X \), as long as \( G \) satisfies (1) and none of the cycles is \( C_5 \).

2. Alpha-pebbling products involving cycles

In this section, we show that the upper bounds on the pebbling numbers of \( C_n \times G \) obtained in Theorems 9 and 12 are also upper bounds on \( \alpha(C_n \times G) \). Throughout this section, we use the following notation from [2] (see Fig. 1).

**Notation:** Let the vertices of \( C_n \) be \( \{x_0, x_1, \ldots, x_{n-1}\} \) in order. Without loss of generality, assume that \( x_0 \) is the target vertex in \( C_n \). Let \( k = \left\lfloor \frac{n}{2} \right\rfloor \), and define the vertex sets \( A \) and \( B \) by

\[
A = \{x_1, x_2, \ldots, x_k\}, \\
B = \{x_{n-k}, x_{n-k-2}, \ldots, x_{n-k+1}\}.
\]

By abuse of notation, we also write \( A \) and \( B \) for the subgraphs induced by the vertices of \( A \) and \( B \), respectively.

Given a distribution of pebbles on \( C_n \times G \), let \( p_i, q_i, \) and \( r_i \) represent the number of pebbles on \( \{x_i\} \times G \), the number of occupied vertices in \( \{x_i\} \times G \), and the number of vertices of \( \{x_i\} \times G \) with an odd number of pebbles, respectively. Also let \( p_A, p_B, r_A \), and \( r_B \) represent the number of pebbles on \( A \times G \) and \( B \times G \), and the number of vertices in \( A \times G \) and \( B \times G \) with an odd number of pebbles, respectively. Thus, the number of pebbles in a distribution on \( C_{2k} \times G \) is given by \( p_0 + p_A + p_B + p_k \), and the number of pebbles in a distribution on \( C_{2k+1} \times G \) is \( p_0 + p_A + p_B + p_k + p_{k+1} \).

Similarly, let the vertices of the path \( P_{k+1} \) be \( x_1, x_2, \ldots, x_k, x_{k+1} \), in that order. In Lemma 13, we use \( p_A, p_k, r_A \) and \( r_k \) as above, but to denote numbers of pebbles and vertices in subgraphs of \( P_{k+1} \times G \); here, we again take \( A = \{x_1, \ldots, x_k\} \).

**Lemma 13.** Suppose \( p_A + p_k + r_A + r_k > 2^k \alpha(G) \) in \( P_{k+1} \times G \), and let \( y \) be a vertex of \( G \). Then either

1. We can transfer \( \frac{p_A + p_k + r_A + r_k - 2^k \alpha(G) - 2}{2} \) pebbles from \( \{x_k\} \times G \) to \( \{x_{k+1}\} \times G \) and still place two pebbles on \( (x_1, y) \), or
2. \( p_A + r_A > 2^{k-1} \alpha(G) \), so we can move two pebbles to the vertex \( (x_1, y) \) without using the pebbles on \( \{x_k\} \times G \).
Proof. We first note that transferring pebbles from \( \{x_k\} \times G \) to \( \{x_{k+1}\} \times G \) does not change the number of vertices in \( \{x_k\} \times G \) with an odd number of pebbles. Therefore, if we reduce the number of pebbles on \( \{x_k\} \times G \) to

\[
p_k' = 2^k \alpha(G) + 2 - p_A - r_A + r_k,
\]

the pebbles on \( (A \cup \{x_k\}) \times G \) still suffice to put two pebbles on \( (x_1, y) \), since \( p_A + p_k' + r_A + r_k > 2^k \alpha(G) \). If \( r_k \leq p_k' \) this strategy allows us to transfer

\[
\frac{p_k - p_k'}{2} = \frac{p_A + p_k + r_A + r_k - 2^k \alpha(G) - 2}{2}
\]
pebbles to \( \{x_{k+1}\} \times G \). On the other hand, if \( r_k > p_k' \), then from the definition of \( p_k' \) we have

\[
p_A + r_A > 2^k \alpha(G) + 2 - 2r_k \geq 2^k \alpha(G) + 2 - 2|V(G)| > (2^k - 2) \alpha(G).
\]

Therefore, \( p_A + r_A \) is strictly larger than \( 2\alpha(P_{k-1}) \alpha(G) = 2^{k-1} \alpha(G) \), as required by Definition 2, and since \( A \) is isomorphic to \( P_{k-1} \), the pebbles on \( A \times G \) are sufficient to put two pebbles on \( (x_1, y) \) by Theorem 6. \( \square \)

We now consider products involving even cycles and those involving odd cycles separately. We begin with the even cycles.

**Theorem 14.** For any graph \( G \), \( \alpha(C_{2k} \times G) \leq \alpha(C_{2k}) \alpha(G) = 2^k \alpha(G) \).

Proof. By Theorem 9, we only have to show that we can move two pebbles onto \( (x_0, y) \) in \( C_{2k} \times G \) from any distribution in which

\[
p_0 + p_A + p_B + p_k + r_0 + r_A + r_B + r_k = p + r > 2^{k+1} \alpha(G).
\]

Furthermore, if \( p_0 + p_A + r_0 + r_A > 2^k \alpha(G) \), we are done by Theorem 6, since \( (A \cup \{x_0\}) \times G \) is isomorphic to \( P_k \times G \). Hence, we may assume that \( p_B + p_k + r_B + r_k > 2^k \alpha(G) \), and similarly, \( p_A + p_k + r_A + r_k > 2^k \alpha(G) \). Now the pebbles on \( (B \cup \{x_k\}) \times G \) are sufficient to put two pebbles on \( (x_n-1, y) \), so we may assume that the pebbles on \( A \times G \) are not sufficient to put two pebbles on \( (x_1, y) \) without using those on \( \{x_k\} \times G \). Therefore, by Lemma 13, we may transfer

\[
\frac{p_A + p_k + r_A + r_k - 2^k \alpha(G) - 2}{2}
\]
pieces from \( \{x_k\} \times G \) to \( \{x_{k+1}\} \times G \), and still put a pebble on \( (x_0, y) \) from \( (x_1, y) \). We could then put a second pebble on \( (x_0, y) \) from \( (B \cup \{x_0\}) \times G \) unless

\[
p_0 + p_A + p_A + p_k + r_A + r_k - 2^k \alpha(G) - 2 \leq 2^{k-1} \alpha(G) - 1,
\]
or equivalently,

\[
p_0 + p_B + \frac{p_A + p_k + r_A + r_k}{2} \leq 2^k \alpha(G).
\]

We would likewise succeed unless

\[
p_0 + p_A + p_B + p_k + r_B + r_k \leq 2^k \alpha(G).
\]

Now adding together (3) and (4) and subtracting (2) gives

\[
p_0 + \frac{p_A + p_B}{2} < r_0 + \frac{r_A + r_B}{2}.
\]

But this is impossible, as each \( p \) is at least as large as the corresponding \( r \). \( \square \)

We now consider products of the form \( C_{2k+1} \times G \) with \( 2k + 1 \geq 7 \). By Theorem 12, if \( X \geq \alpha(G) \) and \( G \) satisfies (1) then \( \alpha(C_{2k+1})X \) satisfies the first condition in Definition 2, i.e. \( f(C_{2k+1} \times G) \leq \alpha(C_{2k+1})X \). In Theorem 16, we show that \( \alpha(C_{2k+1}) \alpha(G) \) always satisfies the second condition whether \( G \) satisfies (1) or not. Thus, \( \alpha(C_{2k+1} \times G) \leq \alpha(C_{2k+1}) \alpha(G) \) whenever \( f(C_{2k+1} \times G) \leq \alpha(C_{2k+1}) \alpha(G) \), and in particular, whenever we can apply Theorem 12 with \( X = \alpha(G) \).

**Lemma 15** (Corollary 2.1 in [2]) gives a useful bound for working with odd cycles.

**Lemma 15.** If \( 2k + 1 \geq 7 \), then \( \alpha(C_{2k+1}) \geq 2^k + 2^{k-2} + 1 \).

**Proof.** We observe that \( \alpha(C_{2k+1}) - 2^k - 2^{k-2} = \frac{2^{k+2} - (-1)^k}{3} - 2^k - 2^{k-2} = \frac{2^{k-2} - (-1)^k}{3} \), and this last fraction is equal to 1 if \( k \) is 3 or 4, and greater than 1 for larger \( k \). \( \square \)
Theorem 16. Suppose that $2k + 1 \geq 7$ and $p + r > 2\alpha(C_{2k+1})\alpha(G)$. If $p$ pebbles are placed on $C_{2k+1} \times G$, and $r$ vertices have an odd number of pebbles, then we can move two pebbles to any target vertex in $C_{2k+1} \times G$.

Proof. Let $y$ be an arbitrary vertex of $G$. Consider a distribution of pebbles on $C_{2k+1} \times G$ from which two pebbles cannot be moved to $(x_0, y)$. We show that $p + r \leq (2^{k+1} + 2^{k-1} + 2)\alpha(G) \leq 2\alpha(C_{2k+1})\alpha(G)$. Since two pebbles cannot be moved to $(x_0, y)$, we have

$$p_0 + r_0 \leq 2\alpha(G).$$

(5)

Now suppose both of the following inequalities hold:

$$p_A + p_k + r_A + r_k \leq 2^k\alpha(G),$$
$$p_B + p_{k+1} + r_B + r_{k+1} \leq 2^k\alpha(G).$$

(6)

(7)

In this case, adding them together with (5) gives $p + r \leq (2^{k+1} + 2\alpha(G)) \leq 2\alpha(C_{2k+1})\alpha(G)$, as desired.

Otherwise, we may assume without loss of generality that $p_A + p_k + r_A + r_k > 2^k\alpha(G)$. Thus, we can put one pebble on $(x_0, y)$ by putting two on $(x_1, y)$. Our first attempt to put two pebbles on $(x_0, y)$ involves transferring pebbles from $\{x_k\} \times G$ to $\{x_{k-1}\} \times G$. These transfers might reduce $r_{k-1}$, but they would not change any other $r_i$’s. This strategy succeeds unless

$$p_0 + p_A + \frac{p_k - r_k}{2} + r_0 + (r_A - r_{k-1}) \leq 2^k\alpha(G).$$

(6)

Alternatively, we keep enough pebbles on $\{x_k\} \times G$ to put two pebbles on $(x_1, y)$ and transfer the remaining pebbles onto $\{x_{k+1}\} \times G$. These transfers might reduce $r_{k+1}$, but they would not change any other $r_i$’s. Using Lemma 13, we consider two cases.

Case 1: If we do not need the pebbles on $\{x_k\} \times G$ to put two pebbles on $(x_1, y)$, we transfer $\frac{p_k - r_k}{2}$ pebbles to $\{x_{k+1}\} \times G$. We could now put a second pebble on $(x_0, y)$ via $(x_{2k}, y)$ unless

$$p_B + p_{k+1} + \frac{p_k - r_k}{2} + r_B \leq 2^k\alpha(G).$$

(7)

Adding (6) and (7) together with the inequality $r_{k-1} + 2r_k + r_{k+1} \leq 4|V(G)| \leq 4\alpha(G)$ gives

$$p + r \leq (2^{k+1} + 4)\alpha(G) \leq (2^{k+1} + 2^{k-1} + 2)\alpha(G).$$

Case 2: If some of the pebbles on $\{x_k\} \times G$ are needed to move two pebbles onto $(x_1, y)$, we can still transfer $\frac{p_A + p_k + r_A + r_k - 2^k\alpha(G) - 2}{2}$ pebbles from $\{x_k\} \times G$ to $\{x_{k+1}\} \times G$. In this case, we can put a second pebble on $(x_0, y)$ via $(x_{2k}, y)$ unless

$$p_B + p_{k+1} + \frac{p_A + p_k + r_A + r_k - 2^k\alpha(G) - 2}{2} + r_B \leq 2^k\alpha(G),$$

or equivalently,

$$p_B + p_{k+1} + \frac{p_A + p_k + r_A + r_k}{2} + r_B \leq (2^k + 2^{k-1})\alpha(G) + 1.$$  

(8)

But now, adding (6) and (8) along with the inequalities $r_{k-1} \leq p_A \leq \frac{p_A + r_A}{2}$ and $r_k + r_{k+1} \leq 2|V(G)| \leq 2\alpha(G)$, we find

$$p + r \leq (2^{k+1} + 2^{k-1} + 2)\alpha(G) + 1 \leq 2\alpha(C_{2k+1})\alpha(G) + 1.$$

Since $p$ and $r$ have the same parity, $p + r$ is even. Thus $p + r \leq 2\alpha(C_{2k+1})\alpha(G)$, as desired. \square

Theorem 17 follows directly from Theorems 12 and 16.

Theorem 17. Suppose $G$ satisfies inequality (1) for some $X \geq \alpha(G)$. Then $\alpha(C_{2k+1} \times G) \leq \alpha(C_{2k+1})X$ whenever $2k + 1 \neq 5$.

Proof. Theorem 5 applies to $C_3 \times G$. For larger odd cycles, Theorem 12 shows that $\alpha(C_{2k+1})X$ satisfies the first condition in Definition 2, and Theorem 16 shows that this number satisfies the second condition. \square
3. The \( t \)-pebling number of iterated products involving odd cycles

In this section, we show that if \( 2k + 1 \geq 7 \) and if \( C_{2k+1} \times G \) satisfies Graham’s conjecture, then it also satisfies the inequality (1) with \( X = \alpha(C_{2k+1})\alpha(G) \). We can therefore apply Theorem 17 inductively. This result does not require \( G \) to satisfy (1). We first find an upper bound on \( f_t(C_n \times G) \).

**Theorem 18.** Suppose \( f_t(C_n \times G) \leq \alpha(C_n)X \) for some \( X \geq \alpha(G) \), and let \( k = \lfloor \frac{t}{2} \rfloor \). Then

\[
f_t(C_n \times G) \leq \alpha(C_n)X + 2^k \cdot (t - 1)\alpha(G) \leq f_t(C_n)X.
\]

In particular, if \( n \) is even or if \( n \neq 5 \) and \( G \) satisfies (1) with \( X = \alpha(G) \) then \( f_t(C_n \times G) \leq f_t(C_n)\alpha(G) \).

**Proof.** The proof is by induction on \( t \), where the base case \( (t = 1) \) is given. For \( t \geq 2 \), suppose that we have \( \alpha(C_n)X + 2^k \cdot (t - 1)\alpha(G) \) pebbles on \( C_n \times G \). Then there are \( 2^k\alpha(G) \) pebbles either on \( (A \cup \{x_0, x_k\}) \times G \) or on \( (B \cup \{x_0, x_{n-k}\}) \times G \). Therefore, we can put one pebble on \( (x_0, y) \) at a cost of at most \( 2^k\alpha(G) \) (since \( \alpha(P_{k+1} \times G) \leq 2^k\alpha(G) \) by Theorem 6), and we may inductively use the remaining \( \alpha(C_n)X + 2^k \cdot (t - 2)\alpha(G) \) pebbles to put \( t - 1 \) additional pebbles on \( (x_0, y) \). \( \square \)

Lourdusamy [5] proposed Conjecture 19 and proved it when \( G \) is an even cycle and \( H \) satisfies a variation of the two-pebbling property.

**Conjecture 19** (Lourdusamy). For any connected graphs \( G \) and \( H \), we have \( f_t(G \times H) \leq f(G)f_t(H) \).

Theorem 18 verifies this conjecture when \( H \) is a cycle as long as \( G \) satisfies the odd two-pebbling property and the product satisfies Graham’s conjecture, or equivalently, as long as Conjecture 19 holds for \( G \times H \) with \( t = 1 \). Conjecture 20 is a symmetric version of Conjecture 19.

**Conjecture 20.** For any connected graphs \( G \) and \( H \), we have \( f_{st}(G \times H) \leq f_s(G)f_t(H) \).

Proposition 21 shows that odd cycles with seven or more vertices satisfy a slightly stronger inequality than (1). We combine it with Theorem 18, to prove a variation of Graham’s conjecture on products involving many odd cycles.

**Proposition 21.** If \( 2k + 1 \geq 7 \) then

\[
2f_4(C_{2k+1}) + (2k + 1) \leq 7\alpha(C_{2k+1}). \tag{9}
\]

**Proof.** We need to show that \( 2(\alpha(C_{2k+1}) + 3 \cdot 2^k) + (2k + 1) \leq 7\alpha(C_{2k+1}) \), or equivalently, that

\[
6 \cdot 2^k + 2k + 1 \leq 5\alpha(C_{2k+1}) = 5 \left( \frac{2^{k+2} - (-1)^k}{3} \right).
\]

Multiplying both sides by 3 and using some algebra, we find that this is equivalent to

\[
6k + 3 \leq 2 \cdot 2^k - 5(-1)^k.
\]

Equality holds when \( k = 3 \) and \( k = 4 \), and it is straightforward to verify that \( 6k + 8 < 2^{k+1} \) when \( k \geq 5 \). \( \square \)

Corollary 22 now follows from Theorem 18 and Proposition 21.

**Corollary 22.** If \( 2k + 1 \geq 7 \) and \( f(C_{2k+1} \times G) \leq \alpha(C_{2k+1})Y \) for some \( Y \geq \alpha(G) \), then

\[
2f_4(C_{2k+1} \times G) + |V(C_{2k+1} \times G)| \leq 7\alpha(C_{2k+1})Y. \tag{10}
\]

In particular, if \( G \) satisfies (1) with \( X = Y \) then \( C_{2k+1} \times G \) satisfies (10) and therefore, it also satisfies (1) with \( X = \alpha(C_{2k+1})Y \).

**Proof.** Applying Theorem 18 with \( X = Y \) gives

\[
2f_4(C_{2k+1} \times G) + |V(C_{2k+1} \times G)| \leq 2f_4(C_{2k+1})Y + (2k + 1)|V(G)|.
\]

Now \( |V(G)| \leq f(G) \leq \alpha(G) \leq Y \), so multiplying inequality (9) by \( Y \) and applying Proposition 21 gives the desired result. \( \square \)
Corollary 23. If $G$ satisfies (1) with $X = Y$ for some $Y \geq \alpha(G)$ then
\[ \alpha(C_{p_1}) \times \cdots \times C_{p_2} \times C_{p_1} \times G) \leq \alpha(C_{p_1}) \cdots \alpha(C_{p_2}) \alpha(C_{p_1}) Y \]
as long as each $p_j$ is an odd number larger than five. Furthermore, the graph $C_{p_1} \times \cdots \times C_{p_2} \times C_{p_1} \times G$ satisfies (1) with $X = \alpha(C_{p_1}) \cdots \alpha(C_{p_2}) \alpha(C_{p_1}) Y$.

Proof. We use induction on $j$. The basis $j = 0$ is given as the hypothesis. Theorem 17 shows that if $C_{p_1} \times \cdots \times C_{p_2} \times C_{p_1} \times G$ satisfies (1) with $X = \alpha(C_{p_1}) \cdots \alpha(C_{p_2}) \alpha(C_{p_1}) Y$, then $\alpha(C_{p_1+1}) \times C_{p_1} \times \cdots \times C_{p_2} \times C_{p_1} \times G) \leq \alpha(C_{p_1+1}) \alpha(C_{p_1}) \cdots \alpha(C_{p_2}) \alpha(C_{p_1}) Y$, and Corollary 22 shows that $C_{p_1+1} \times C_{p_1} \times \cdots \times C_{p_2} \times C_{p_1} \times G$ satisfies (1) with $X = \alpha(C_{p_1+1}) \alpha(C_{p_1}) \cdots \alpha(C_{p_2}) \alpha(C_{p_1}) Y$, allowing us to continue the induction. \[ \square \]

Theorem 24. If $G$ satisfies (1) for some $X \geq \alpha(G)$ then
\[ \alpha(C_{p_1}) \times \cdots \times C_{p_2} \times C_{p_1} \times G) \leq \alpha(C_{p_1}) \cdots \alpha(C_{p_2}) \alpha(C_{p_1}) X \]
as long as each $p_i \neq 5$.

Proof. We may assume that the odd cycles with at least seven vertices are the first cycles multiplied by $G$ and then use induction on the number of other cycles (i.e., even cycles and copies of $C_3$) in the product. If there are no other cycles, the theorem follows from Corollary 23, and the result still holds when we add even cycles and copies of $C_3$ to our Cartesian product by Theorem 14 and Theorem 5, respectively. \[ \square \]

4. $C_5 \times C_5$

Herscovici and Higgins [3] proved that $f(C_5 \times C_5) = 25$, satisfying Graham’s conjecture. In this section, we show that $C_5 \times C_5$ satisfies Chung’s two-pebbling property, and therefore also satisfies Wang’s odd two-pebbling property. We also establish bounds for the $t$-pebbling number of $C_5 \times C_5$. We assume without loss of generality that the target vertex is $(x_0, x_0)$. We keep the notation of the previous section, but we also introduce new notation for pebbles on individual vertices of $C_5 \times C_5$.

Notation: For any distribution of pebbles on $C_5 \times C_5$, let $p_{ij}$ denote the number of pebbles on the vertex $(x_i, x_j)$, and let $q_{ij}$ be 1 or 0, depending on whether $(x_i, x_j)$ is occupied or not.

The following fact is a consequence of Theorem 4, since $C_5$ satisfies the two-pebbling property. Lemmas 26 and 27 give us conditions under which we can move two pebbles to $(x_0, x_0)$ in $C_5 \times C_5$.

Fact 25. If $p + q > 10$ in $C_5$, or if $p + q > 20$ in $K_2 \times C_5$, we can move two pebbles to any vertex of the graph. \[ \square \]

Lemma 26. Two pebbles can be moved to $(x_0, x_0)$ in $C_5 \times C_5$ if $p_0 \leq 5$ and $4p_0 + p_1 + p_2 + q_1 + q_2 \geq 41$.

Proof. We have $p_1 + p_2 + q_1 + q_2 \geq 25$ as long as $p_0 \leq 4$. Let $s_j = p_1 + p_2 + q_1 + q_2$, for $0 \leq j \leq 4$. If each $s_j \geq 5$, we can put two pebbles on each of the vertices $(x_1, x_1), (x_1, x_0)$, and $(x_1, x_4)$. These pebbles are then sufficient to move two pebbles onto $(x_0, x_0)$. Otherwise, we have some $s_j \geq 6$. In this case, we can move a pebble onto $(x_0, x_j)$ while reducing $p_1 + p_2 + q_1 + q_2$ by at most 4. This does not decrease $4p_0 + p_1 + p_2 + q_1 + q_2$, so we repeat this process until $p_0 = 5$. At this point (or if we started with $p_0 = 5$), we have $p_1 + p_2 + q_1 + q_2 \geq 21 > 4f(C_5)$, so the pebbles on $\{x_0\} \times C_5$ are sufficient to put one pebble on the target, and those on $\{x_1, x_2\} \times C_5$ are sufficient to put on a second. \[ \square \]

Lemma 27. We can move two pebbles onto $(x_0, x_0)$ in $C_5 \times C_5$ if all of the following inequalities hold:

\[ p + q \geq 51, \tag{11} \]
\[ p_1 + q_1 + \frac{p_2 - r_2}{2} \geq 11, \tag{12} \]
\[ p_1 + q_1 > q_2 + r_2. \tag{13} \]
We first note that we can put two pebbles directly on \((x_0, x_0)\) unless
\[ p_0 + p_1 + q_0 + q_1 + \frac{p_2 - r_2}{2} \leq 20. \]  
(14)

Now if \(p_1 + q_1 \geq 11\), we can already put two pebbles on \((x_1, x_0)\), so we try to put two pebbles on \((x_4, x_0)\). We can do this unless
\[ p_3 + p_4 + q_3 + q_4 + \frac{p_2 - r_2}{2} \leq 20. \]  
(15)

But adding (14) and (15) along with the inequality \(q_2 + r_2 \leq 10\) contradicts (11).

If \(p_1 + q_1 \leq 10\), we use (12) to transfer \(11 - p_1 - q_1\) pebbles from \(\{x_2\} \times C_5\) to \(\{x_1\} \times C_5\). This costs \(22 - 2p_1 - 2q_1\) pebbles from \(\{x_2\} \times G\) and does not reduce \(q_1\) or alter \(r_2\). We can then move one pebble to \((x_0, x_0)\) via \((x_1, x_0)\). This time, we can put two pebbles on \((x_4, x_0)\) unless
\[ p_3 + p_4 + q_3 + q_4 + \frac{p_2 - (22 - 2p_1 - 2q_1) - r_2}{2} \leq 20, \]
or equivalently,
\[ p_1 + q_1 + p_3 + p_4 + q_3 + q_4 + \frac{p_2 - r_2}{2} \leq 31. \]  
(16)

But now, adding (14) and (16) and adding \(q_2 + r_2\) to both sides, we obtain \(p + q + p_1 + q_1 \leq 51 + q_2 + r_2\), which is impossible if both (11) and (13) hold. Therefore, if (11) and (12), and (13) all hold, two pebbles can be moved to \((x_0, x_0)\). □

**Theorem 28.** \(C_5 \times C_5\) satisfies the two-pebbling property. Therefore, \(\alpha(C_5 \times C_5) = 25\).

**Proof.** We know \(f(C_5 \times C_5) = 25\) (see [3]), so suppose \(p\) pebbles occupy \(q\) vertices of \(C_5 \times C_5\). We show that we can move two pebbles onto \((x_0, x_0)\) whenever \(p + q \geq 10f(C_5) + 1 = 51\). We make the following assumptions:

1. \(p_0 + q_0 \leq 10\); otherwise, the pebbles on \(\{x_0\} \times C_5\) are enough to two-pebble \((x_0, x_0)\).
2. \(p_1 + p_2 + q_1 + q_2 \geq 21\). Since \(p + q \geq 51\) and \(p_0 + q_0 \leq 10\), either \(p_1 + p_2 + q_1 + q_2 \geq 21\) or \(p_3 + p_4 + q_3 + q_4 \geq 21\).

Without loss of generality, we may assume that \(p_1 + p_2 + q_1 + q_2 \geq 21\). Thus, we can put one pebble on \((x_0, x_0)\) using those on \(\{x_1, x_2\} \times C_5\).
3. \(p_0 \leq 4\), since \(p_0 \geq 5\) would allow us to move a pebble to \((x_0, x_0)\) in addition to the one that we can move there from \(\{x_1, x_2\} \times C_5\).
4. \(4p_0 + p_1 + p_2 + q_1 + q_2 \leq 40\), or equivalently,
\[ 2p_0 + \frac{p_1 + p_2 + q_1 + q_2}{2} \leq 20, \]  
(17)

since otherwise, we could put two pebbles on the target by **Lemma 26**, since \(p_0 \leq 4\) by assumption 3.

We now explore transferring pebbles from \(\{x_2\} \times C_5\) to \(\{x_3\} \times C_5\). There are two cases: either we can keep at least one pebble on each occupied vertex while reducing the number of pebbles on \(\{x_2\} \times C_5\) to at most \(22 - p_1 - q_1 - q_2\), or there would be at least \(23 - p_1 - q_1 - q_2\) pebbles on \(\{x_2\} \times C_5\) even when each vertex has at most two pebbles (so another transfer would clear some vertex). We consider these cases separately.

**Case 1:** Suppose we can reduce the number of pebbles on \(\{x_2\} \times C_5\) to at most \(22 - p_1 - q_1 - q_2\) pebbles, while keeping one pebble on each occupied vertex. After these transfers, there would still be enough pebbles on \(\{x_1, x_2\} \times C_5\) to put two pebbles on \((x_1, x_0)\). We could also put two pebbles on \((x_4, x_0)\) unless
\[ p_3 + p_4 + q_3 + q_4 + \frac{p_2 - (22 - p_1 - q_1 - q_2)}{2} \leq 20, \]
or equivalently,
\[ p_3 + p_4 + q_3 + q_4 + \frac{p_1 + p_2 + q_1 + q_2}{2} \leq 31. \]  
(18)

Adding (17) and (18) gives \((p + q) + (p_0 - q_0) \leq 51\). But \(p + q \geq 51\), so we may assume that \(p + q = 51\) and \(p_0 = q_0\). In particular, equality holds in both (17) and (18). Since equality holds in (17), \(\frac{p_1 + p_2 + q_1 + q_2}{2} \) is an even integer. Since
it also holds in (18), \( p_3 + p_4 + q_3 + q_4 \) is odd. Therefore, \( p_3 + p_4 \neq q_3 + q_4 \), so some vertex of \( \{x_3, x_4\} \times C_5 \) has at least two pebbles. We can now transfer a pebble from \( \{x_3, x_4\} \times C_5 \) to \( \{x_0, x_1, x_2\} \times C_5 \). After this transfer, the inequality (17) no longer holds, and we still have \( p_0 \leq 5 \), so by Lemma 26, we can put two pebbles on the target.

Case 2: Suppose removing pebbles in pairs from the vertices of \( \{x_2\} \times C_5 \) creates a situation where no \( (x_2, x_i) \) has more than two pebbles, but there are still \( p_2^* \geq 23 - p_1 - q_1 - q_2 \) pebbles on \( \{x_2\} \times C_5 \). In this case, we claim that we can apply Lemma 27 to the original distribution. If \( p_1 + q_1 \geq 11 > q_2 + r_2 \), Lemma 27 clearly applies. Otherwise, if \( p_1 + q_1 \leq 10 \), we consider the situation after the pebbles have been removed. At this point, there are at most 10 pebbles on \( \{x_2\} \times C_5 \), so \( 23 - p_1 - q_1 - q_2 \leq p_2^* \leq 10 \), and \( p_1 + q_1 \geq 13 - q_2 \geq 8 \). We also have \( p_2^* + q_2 \geq 13 \). Since a single pebble on some vertex contributes at most two to \( p \), the pebbles on \( \{x_2\} \times C_5 \) must be left with two pebbles. Therefore, (12) holds, since \( p_1 + q_1 + \frac{p_2^* + q_2}{2} \geq 8 + 3 = 11 \). Furthermore, \( r_2 \leq 2 \), so \( q_2 + r_2 \leq 7 < p_1 + q_1 \) in the original distribution, and we can put two pebbles on \( (x_0, x_0) \) by Lemma 27.

We now establish an upper bound on \( f_t(C_5 \times C_5) \). We begin with one lemma.

**Lemma 29.** Suppose fourteen pebbles are placed on \( K_2 \times C_5 \). Then one pebble may be moved to any vertex at a cost of at most eight pebbles. Furthermore, \( f_2(K_2 \times C_5) = 18 \).

**Proof.** Let the vertex set of \( K_2 \) be \( \{y_1, y_2\} \), and without loss of generality, we assume that the target is \( (y_1, x_0) \). If there are eight pebbles either on \( K_2 \times \{x_0, x_1, x_2\} \) or on \( K_2 \times \{x_3, x_4, x_0\} \), these pebbles are sufficient to reach the target, since \( \alpha(K_2 \times P_3) = 8 \). Otherwise, there must be seven pebbles each on \( K_2 \times \{x_1, x_2\} \) and \( K_2 \times \{x_3, x_4\} \), and none on \( K_2 \times \{x_0\} \). If there are four pebbles on \( \{y_1\} \times \{x_1, x_2\} \), these are enough to reach \( (y_1, x_0) \) at a cost of at most four pebbles. Otherwise, we can reach \( (y_2, x_0) \) at a cost of at most four pebbles. Similarly, we can reach either \( (y_1, x_0) \) or \( (y_2, x_0) \) at a cost of at most four pebbles from \( K_2 \times \{x_3, x_4\} \). If neither side can reach \( (y_1, x_0) \), they both can reach \( (y_2, x_0) \) at a total cost of at most eight pebbles, so together they can reach \( (y_1, x_0) \).

In particular, if eighteen pebbles are on \( K_2 \times C_5 \), we can put one pebble on any target at a cost of at most eight pebbles, and then the remaining ten are sufficient to put a second pebble on the target. On the other hand, if we place fifteen pebbles on \( (y_2, x_2) \) and one pebble each on \( (y_2, x_3) \) and \( (y_1, x_3) \), we see that seventeen pebbles are not sufficient to put two pebbles on the target.

**Theorem 30.** We have \( f_t(C_5 \times C_5) \leq 16t + 9 \). In particular, \( f_4(C_5 \times C_5) \leq 73 \), so \( C_5 \times C_5 \) satisfies the inequality (1) with \( X = 25 \).

We proceed by induction on \( t \) with two basis cases. If \( t = 1 \) the upper bound is sharp, as shown in [3]. If \( t = 2 \), we suppose that 41 pebbles are placed on \( C_5 \times C_5 \). We assume that the target is \( (x_0, x_0) \), and \( p_0 \leq 4 \), since otherwise, we can put a pebble on \( (x_0, x_0) \) at a cost of at most five pebbles, and the remaining 36 pebbles are sufficient to put a second pebble on the target. We also assume without loss of generality that \( p_1 + p_2 \geq 19 \), so, by Lemma 29, there are enough pebbles on \( \{x_1, x_2\} \times C_5 \) to reach \( (x_0, x_0) \) via \( (x_1, x_0) \). Since we start with a total of 41 pebbles, we either have \( p_1 + p_2 \geq 22 \) or \( p_3 + p_4 + p_0 \geq 20 \). In the first case, we apply Lemma 29 twice to put two pebbles on \( (x_1, x_0) \) at a cost of at most sixteen pebbles, and then use the remaining 25 pebbles to put a pebble on \( (x_0, x_0) \). In the second case, the pebbles on \( \{x_3, x_4, x_0\} \times C_5 \) are sufficient to put a second pebble on \( (x_0, x_0) \), since \( \alpha(P_3 \times C_5) \leq \alpha(P_3)\alpha(C_5) = 20 \).

Now if \( t \geq 3 \), we start with \( 16t + 9 \geq 57 \) pebbles on \( C_5 \times C_5 \). In particular, either \( p_0 \geq 5 \) or \( p_1 + p_2 \geq 27 \) or \( p_3 + p_4 \geq 27 \). If \( p_0 \geq 5 \) we can put a pebble on \( (x_0, x_0) \) at a cost of at most five pebbles, and in the other cases, we again apply Lemma 29 twice to put a pebble on the target at a cost of at most sixteen pebbles. In either case, we have at least \( 16(t - 1) + 9 \) pebbles remaining, so by induction, we can put \( t - 1 \) additional pebbles on \( (x_0, x_0) \).

**Theorem 31.** We have
\[
\alpha(C_{p_j} \times \cdots \times C_{p_2} \times C_{p_1}) \leq \alpha(C_{p_j}) \cdots \alpha(C_{p_2})\alpha(C_{p_1})
\]
for all \( \alpha \) as long as at most two of the \( p_k \)’s are equal to five.

**Proof.** We apply Theorem 24 with \( G \) as either the trivial graph, \( G = C_5 \), or \( G = C_5 \times C_5 \), depending on the number of copies of \( C_5 \) in the product.
Theorem 30 gives us an upper bound on $f_t(C_5 \times C_5)$. Theorem 34 gives a lower bound. We adapt a technique used by Moews in [7].

**Definition 32.** Let $v$ be a vertex in a graph $G$. We define the weight of a vertex $v'$ with respect to $v$ in $G$ and the weight of a pebble on $v'$ by $w_v(v') = 2^{-d(v, v')}$, where $d(v, v')$ represents the distance between $v'$ and $v$.

We also define the total weight with respect to $v$ of a distribution on $G$ as the sum of the weight of each pebble in that distribution.

Observe that any distribution with $t$ pebbles on $v$ automatically has weight at least $t$ with respect to $v$. Note also that pebbling moves can never increase this weight, since the weight of a new pebble is at most twice the weight of the two removed pebbles. Therefore, any distribution from which $t$ pebbles can be moved to $v$ also has weight at least $t$ with respect to $v$.

One more definition simplifies our work:

**Definition 33.** Given a graph $G$, the reflection of the vertex $(x, y)$ in $G \times G$ is the vertex $(y, x)$. The reflection of the pebbling move from $(x, y)$ to $(x', y')$ in $G \times G$ is the pebbling move from $(y, x)$ to $(y', x')$ (when such a move is possible). The reflection of a sequence of pebbling moves in $G \times G$ is the sequence of reflections of the individual pebbling moves (again when such a sequence is possible).

Note that we can always reflect a sequence of pebbling moves when all the pebbles involved start on vertices of the form $(x, x)$.

**Theorem 34.** We have $16t + 7 \leq f_t(C_5 \times C_5) \leq 16t + 9$.

**Proof.** By Theorem 30, we only need to establish the lower bound. Toward that end, let $D$ be the distribution in $C_5 \times C_5$ with $16t - 5$ pebbles on $(x_2, x_2)$ and $11$ pebbles on $(x_3, x_3)$. Note that the total weight of this distribution with respect to $(x_0, x_0)$ is $t + \frac{3}{8}$. We show that $t$ pebbles cannot be moved to $(x_0, x_0)$ starting from $D$.

Suppose that we are given a sequence of pebbling moves which starts from $D$ and ends with $t$ or more pebbles on $(x_0, x_0)$. Using the methods from Section 2 of [6], we may, perhaps after omitting certain moves from this sequence, reorder the sequence so that it consists of a concatenation of $t$ subsequences, each of which starts with a sequence of moves which moves two pebbles to a vertex adjacent to $(x_0, x_0)$ and finishes by moving one pebble from this vertex to $(x_0, x_0)$, leaving no pebbles anywhere other than $(x_0, x_0)$, $(x_2, x_2)$, and $(x_3, x_3)$.

We may assume without loss of generality that all pebbles that reach $(x_0, x_0)$ get there from either $(x_1, x_0)$ or $(x_4, x_0)$, since we may reflect a subsequence of pebbling moves which go through $(x_0, x_1)$ or $(x_0, x_4)$. We first show that $2t$ pebbles cannot be moved onto $(x_1, x_0)$.

Since an odd number of pebbles start on $(x_2, x_2)$, there are two possibilities for any sequence of pebbling moves: either an additional pebble is moved onto $(x_2, x_2)$, or at least one pebble is left behind on this vertex. Moving a pebble onto $(x_2, x_2)$ uses four pebbles from $(x_3, x_3)$ (clearly it is not helpful to consume pebbles on $(x_2, x_2)$ for the purpose of adding a pebble onto that vertex). After these moves, the weight of the resulting distribution with respect to $(x_1, x_0)$ is $\frac{16t - 4}{8} + \frac{7}{16} = 2t - \frac{1}{16} \leq 2t$. Therefore, it would be impossible to put $2t$ pebbles on $(x_1, x_0)$. On the other hand, if we leave a pebble on $(x_2, x_2)$ and try to put $2t$ pebbles onto $(x_1, x_0)$, we are aiming for a distribution whose total weight with respect to $(x_1, x_0)$ is at least $2t + \frac{1}{8}$. Such a distribution is unreachable from $D$, since the total weight of $D$ with respect to $(x_1, x_0)$ is only $\frac{16t - 5}{8} + \frac{11}{16} = 2t + \frac{1}{16}$.

Alternatively, we try to put a pebble on $(x_0, x_0)$ by going through $(x_4, x_0)$. Let $u_2$ and $u_3$ be the number of pebbles used from $(x_2, x_2)$ and $(x_3, x_3)$, respectively. In order to put two pebbles on $(x_4, x_0)$, we need $\frac{u_2}{16} + \frac{u_3}{8} \geq 2$, or equivalently,

$$u_2 + 2u_3 \geq 32.$$

On the other hand, using 23 pebbles would leave a distribution whose total weight with respect to $(x_0, x_0)$ is less than $t$ (including the pebble that was moved to $(x_0, x_0)$). Therefore, we also require

$$u_2 + u_3 \leq 22.$$

We also have $u_3 \leq 11$, since $(x_3, x_3)$ starts with only eleven pebbles. Satisfying these inequalities requires $u_3 = 10$ or $u_3 = 11$. 

If \( u_3 = 10 \) we have \( u_2 = 12 \). If ten pebbles are used from \((x_3, x_3)\) and twelve pebbles are used from \((x_2, x_2)\), we are left with \(16t - 17\) pebbles on \((x_2, x_2)\) and a lone pebble on \((x_3, x_3)\). The weight of this distribution with respect to \((x_0, x_0)\) is \(t\) (again, including the pebble already on \((x_0, x_0)\)), so every pebble remaining must be used, and no more weight can be lost. However, the pebble on \((x_3, x_3)\) can only be used if another pebble is moved onto that vertex. Such a move requires a loss of weight, since the weight of \((x_3, x_3)\) is minimal. Therefore, \(t\) pebbles cannot be moved to \((x_0, x_0)\) by this strategy. On the other hand, if \( u_3 = 11 \) and \( u_2 \leq 11 \), the only way to use an eleventh pebble on \((x_3, x_3)\) is by adding a twelfth pebble to that vertex at the cost of four pebbles from \((x_2, x_2)\). This leaves us with at most seven more pebbles that may be used from \((x_2, x_2)\), but it is impossible to put two pebbles on \((x_4, x_0)\) using twelve pebbles on \((x_3, x_3)\) and seven pebbles on \((x_2, x_2)\), since the total weight of such pebbles with respect to \((x_4, x_0)\) is \( \frac{31}{16} < 2 \).

Therefore, any sequence of pebbling moves that allows us to reach \((x_0, x_0)\) through \((x_4, x_0)\) (or its reflection) costs enough pebbles to make it impossible to put \( t - 1 \) additional pebbles on \((x_0, x_0)\). As we have also shown that it is impossible to put \( t \) pebbles on \((x_0, x_0)\) by only going through \((x_1, x_0)\) or its reflection, we may conclude that \( f_t(C_5 \times C_5) \geq 16t + 7 \), as required. \( \Box \)

It was shown in [3] in the last paragraph of the proof that \( f(C_5 \times C_5) = 25 \) that one pebble can be moved to a target vertex from any placement of 23 pebbles on the outermost two copies of \( C_5 \). Although 25 pebbles are required to ensure that one pebble can reach any target, it is possible that 39 pebbles are sufficient to put two pebbles on \((x_0, x_0)\) (or any target). Indeed, an argument similar to the one given in [3] shows that some of the pebbles could not be on the outermost copies \( C_5 \times \{x_2, x_3\} \) and some pebbles could likewise not be on \( \{x_2, x_3\} \times C_5 \). It might be possible to show that when some pebbles begin on the inner copies, the first pebble on \((x_0, x_0)\) costs fewer than sixteen pebbles. We therefore conjecture that the lower bound in Theorem 34 is tight.

**Conjecture 35.** If \( t \geq 2 \), then \( f_t(C_5 \times C_5) = 16t + 7 \).

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**References**

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